

Experimentation in Networks^{*}

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Abstract

We introduce a model of strategic experimentation on social networks in which forward-looking agents learn from their own and neighbors' successes. In equilibrium, private discovery is followed by social diffusion. Social learning crowds out own experimentation, so total information decreases with network density; we determine density thresholds below which agents' asymptotic learning is perfect. In contrast, agent welfare is single-peaked in network density and achieves a second-best benchmark level at intermediate levels that achieve a balance between discovery and diffusion. We also show how learning and welfare differ across directed, undirected and clustered networks.

1 Introduction

The discovery and diffusion of innovations are key drivers of long-term economic growth. This is illustrated by the seminal papers of Griliches (1957) and Coleman, Katz, and Menzel (1957) that document the spread of new technologies by farmers and doctors. From the perspective of societal welfare, discovery and diffusion are complements: Mokyr (1992) argues that both are required for sustained economic progress. From an individual strategic perspective, they are substitutes: Grossman and Stiglitz (1980) famously point out that if prices

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aggregate information efficiently, then individual agents have no incentive to privately generate such information. Economic theory has made large strides in understanding information acquisition and aggregation in centralized settings such as financial markets, auctions, and collective experimentation. These incentives are less well understood in decentralized settings, where information slowly diffuses through society. This paper seeks to reconcile these forces in a parsimonious equilibrium model of experimentation on networks.

The classic paper on this topic, Bala and Goyal (1998), considers myopic non-Bayesian agents who ignore the network. This shortcuts strategic considerations and allows one to solve the model as a sequence of static decision problems. In contrast, our agents are forward-looking and Bayesian, so both past and future social learning crowds out private experimentation. To maintain tractability, we impose structure on the network and assume agents learn via perfect good news events; this reduces each agent’s problem to choosing a deterministic cutoff time, with social learning described by simple ordinary differential equations, opening the gate to a myriad of questions about experimentation on networks.

We use this new approach to study how asymptotic information and welfare depend on network density, as measured by either the degree in regular random networks or by the size of the core in core-periphery networks. For either measure we show that agents’ asymptotic information decreases monotonically in network density and they learn the truth when the network is sufficiently sparse. In contrast, welfare is single-peaked in network density and attains a second-best welfare benchmark when density is intermediate; such networks both encourage generation of information and quickly diffuse the discoveries. Finally, we provide a tight comparison between directed, undirected and clustered networks. Collectively, these results paint a clear picture about learning dynamics, information aggregation, and welfare in networks of forward-looking, Bayesian agents.

In the model, I agents (Iris, John, Kata...) are connected by an exogenous network (e.g. clique, tree, core-periphery). They can each experiment with a new technology whose state is high or low; experimentation generates successes at random times iff the state is high. Agents learn from own and neighbors’ successes but do not observe neighbors’ actions. This simple model captures a number of applications: Consider farmers learning about the success of a new crop from neighbors, doctors learning about a new drug from colleagues, or landowners learning about the presence of oil from nearby frackers.

In Section 3, we first characterize Iris’s best-response to arbitrary strategies of other agents. Observing a success perfectly reveals the high state and essentially ends the game for her. Before this time, Iris’s experimentation decision is based on her *social learning curve*, i.e. the expected experimentation of her neighbors. We show that Iris’s dynamic experimentation problem is solved by a simple *cutoff strategy*: In the absence of success, Iris

stops experimenting at some cutoff time τ_i . An increase in social information crowds out Iris’s private experimentation, lowering her cutoff time: Unsuccessful past social learning makes Iris pessimistic, while future social information lowers the information value of her own experimentation.

Next, we illustrate in examples how to generate Iris’s social learning curve from others’ cutoff times. In the clique network, social learning is fast but shallow: The agents collectively experiment as much as a single agent would by herself. Adding agents speeds up learning but does not raise aggregate information because the density of the network chokes off experimentation prematurely. In the line network, social learning is slow but deep: The agents collectively experiment an infinite amount. Eventually they learn the state perfectly, but the sparsity of the network constrains the speed of learning.

In Section 4, we study the effect of network density on asymptotic information and welfare. Specifically, we consider two canonical types of networks (regular random networks and core-periphery networks) as $I \rightarrow \infty$. To study aggregate information, define the *asymptotic information* to be the total information created by society; there is *asymptotic learning* if asymptotic information is unbounded, meaning that the agents eventually learn the state. To study welfare, we propose a second-best benchmark that provides an upper bound on equilibrium utility (of the worst-off agent) across all networks. The clique does not attain this benchmark because agents do not aggregate enough information; the line does not attain it either because learning is too slow. But we show that both large random networks and large core-periphery networks do attain the benchmark when network density is intermediate.

We first study large regular random networks with degree n^I . This model encompasses sparse trees, where $n^I \equiv n$, and dense cliques, where $n^I/I \rightarrow 1$. Theorem 1 completely characterizes asymptotic information and welfare as functions of network density. Asymptotic information falls in network density and asymptotic learning obtains if density is below a threshold. Specifically the agents fully learn if the *time-diameter* (the typical time for information to travel between two agents) exceeds a threshold σ^* .¹ Welfare is single-peaked in network density and attains the second-best benchmark if $n^I \rightarrow \infty$ and $n^I/I \rightarrow 0$. Intuitively, asymptotic learning requires sparsity to sustain private experimentation incentives; high welfare additionally requires density to promptly diffuse news across society.

To study the role of network position on experimentation incentives we next turn to core-periphery networks, where K^I *core* agents are connected to everyone while $I - K^I$ *peripheral* agents are only connected to core agents. In equilibrium, core agents have more social information than peripherals, so experiment less and have higher utility. While core agents

¹The threshold σ^* renders agents indifferent about experimentation at $t = 0$ when they expect to learn the state perfectly at $t = \sigma^*$.

experiment little themselves (if at all), they serve an important role as information brokers connecting the peripherals. As $I \rightarrow \infty$, asymptotic learning and welfare exhibit similar properties to large random networks, with core size substituting for the degree. Theorem 2 shows that asymptotic information decreases in network density, and asymptotic learning obtains if K^I remains below a threshold κ^* .² Welfare is single-peaked in network density and attains the second-best benchmark if K^I exceeds κ^* and $K^I/I \rightarrow 0$.

Our analysis of large random networks and core-periphery networks points to a fundamental tradeoff between social learning and welfare. These goals are often thought to be aligned: Hayek (1945) famously emphasizes the importance of information aggregation for allocative efficiency. However, in our model agents must be incentivized to acquire information, so the fast diffusion required for second-best welfare can lower total information. Indeed, for core-periphery networks the two goals are mutually exclusive.

Our two families of networks differ in their network structure and thus exhibit different social learning dynamics. In large random networks, the typical pair of agents has distance $\log I / \log n^I$; as the degree grows, social learning occurs in a single burst at a fixed time σ . In contrast, in core-periphery networks, all peripheral agents are two links apart; social learning occurs as the information generated by peripherals filters through the core. The resulting cumulative social learning curves are thus convex for large random networks but concave for core-periphery networks.

In Section 5, we study different types of links in the context of regular tree networks, including directed links (e.g. Twitter), undirected links (e.g. LinkedIn), and triangles of links that capture clustering (e.g. Facebook). Trees approximate large random networks, and are highly tractable because neighbors' behavior is independent; this allows us to characterize social learning in the contagion phase by simple ordinary differential equations. For example, in a directed line, social information arrives at a constant rate, whereas in a directed tree with degree $n \geq 2$, the arrival rate rises over time. Theorem 3 provides tight bounds on the utility of agents across different networks. The utility of an agent in an undirected tree with degree n is sandwiched between her value in directed trees with degree $n - 1$ and n . Thus, agents prefer directed to undirected links, but even more strongly prefer to be in a tree with one more neighbor. Similarly, the utility of an agent in a triangle tree with degree n is sandwiched between her value in undirected trees with degree $n - 1$ and n . Link types are thus of second-order importance for dense networks, where the distinction between, say, 74 friends and 78 friends becomes blurry. Collectively, these results demonstrate the qualitative and quantitative importance of network structure.

²The threshold κ^* renders peripherals indifferent about experimentation at $t = 0$ when κ^* core agents work forever.

1.1 Literature

At the core of the paper is a “perfect good news” model of strategic experimentation with unobserved actions and private payoffs. In the context of a clique, Keller, Rady, and Cripps (2005) study a good-news model with observed actions and private payoffs, Bonatti and Hörner (2011) consider a good-news model with unobserved actions and public payoffs, and Bonatti and Hörner (2017) consider a bad-news model with unobserved actions and private payoffs. In all of these papers, agents use mixed strategies. Specifically, in the first two papers, agents gradually phase out their experimentation as the public belief approaches the exit threshold. In our model, agents use simple cutoff strategies; this allows us to go beyond the clique and solve for equilibria in rich classes of networks. We also think that the assumptions of unobserved actions and private payoffs is a natural way to model a network of farmers, doctors or oil frackers whose externalities are purely informational.

Observational learning on networks was pioneered by Bala and Goyal (1998) who study myopic, non-Bayesian agents and provide conditions on the network under which (i) agents reach a consensus and (ii) the agents learn the state.³ Subsequent work has generalized these two limit results in models with forward-looking, Bayesian agents who incorporate the future value of information when choosing to experiment. Rosenberg, Solan, and Vieille (2009) consider a very general model that encompasses strategic experimentation on networks, and shows that all agents eventually play the same action. Camargo (2014) considers a continuum-agent model with “random sampling”, and shows that information aggregates if each action is myopically optimal for a positive measure of agents’ heterogeneous priors. By focusing on good news learning, we can characterize learning dynamics at each point in time, rather than restricting attention to long-run behavior. This is important because agents care about when innovations diffuse and not just if they diffuse; indeed, this consideration underlies the contrast between sparse networks that aggregate information and the denser networks that maximize welfare.⁴

Most closely related to our model, Salish (2015) embeds a discrete-time version of Keller, Rady, and Cripps’s (2005) strategic experimentation model in a network. Neighbors observe each others’ actions, which thus signal successes of second neighbors; Salish side-steps such

³Sadler (2020b) characterizes outcomes more completely in Bala-Goyal’s model with Brownian learning.

⁴A parallel literature considers dynamic learning games where private information is initially endowed to agents, instead of being learned over time. Gale and Kariv (2003) show that consensus must emerge when agents are Bayesian and myopic. Mossel, Sly, and Tamuz (2015) extend this result to forward-looking agents, and also show that agents eventually learn the state if the network is not too connected (e.g. the network is undirected with bounded degree). Another classic literature considers agents who move in sequence, learning from (a subset of) prior agents. Acemoglu et al. (2011) show that society learns the state if signals are unbounded and agents (indirectly) observe an unbounded number of agents. Mossel et al. (2020) unify many of the results in these literatures by looking at steady-state asymptotic behavior.

signaling by introducing an additional learning channel, whereby successes are automatically transmitted across the network, one link per period. The paper shows that experimentation tends to phase out over time, and that ring and star networks aggregate more information than the clique. In contrast, our best-responses are determined by simple cutoffs, allowing us to characterize aggregate information and welfare as functions of network density.

The complexity of Bayesian updating has led some authors to consider reduced-form models of information acquisition and aggregation. For example, Bramoullé and Kranton (2007) and Galeotti and Goyal (2010) consider a local public goods game where each agent chooses a contribution level, and benefits from her neighbors’ contributions. Since our agents’ optimally choose a deterministic stopping time, we recover the tractability of the reduced-form models of experimentation in a model of Bayesian learning.

In seeking to characterize learning dynamics on networks, the paper is related to Board and Meyer-ter-Vehn (2021). In that paper, myopic agents sequentially choose to acquire information at a single point in time. Here, forward-looking agents simultaneously choose to acquire information at every point in time. The different models give rise to different economic forces: The forward-looking agents in this paper anticipate the arrival of future social information which crowds out their private experimentation, and the repeated choices give rise to the clean distinction between an experimentation phase and a contagion phase. This paper also focuses on a different question: How does aggregate information and welfare change with network density? The results in Section 5 correspond most closely to our prior paper, where we studied different types of links in the configuration model.⁵

The paper also complements a growing empirical literature that studies how people learn about innovations from their neighbors. Conley and Udry (2010), Banerjee et al. (2013), BenYishay and Mobarak (2019) and Beaman et al. (2021) study the spread of new production techniques and financial innovations in developing countries. Fetter et al. (2018) and Hodgson (2021) study the diffusion of fracking and oil exploration decisions. And Moretti (2011) and Finkelstein et al. (2021) explore the adoption of new products. Such empirical analysis lacks a simple framework with forward-looking Bayesian agents that can be estimated and used for counterfactuals. This paper proposes such a framework.

⁵More specifically, Board and Meyer-ter-Vehn (2021) considered a more general configuration model with multiple types. It showed that social learning was greater in directed networks than comparable undirected networks, and greater in tree networks than comparable triangle networks. In comparison, Section 5 only considers regular networks but provides a tighter characterization of the value of link type.

2 Model

Network. Agents $\{1, \dots, I\}$ are connected by a network $g \subseteq \{1, \dots, I\}^2$ that represents who observes whom. If i (Iris) observes j (John), we write $i \rightarrow j$ or $(i, j) \in g$, and call j a *neighbor* of i . The set of Iris's neighbors is $N_i(g)$. The network may be directed or undirected. It may be deterministic or random; denote the random network by G with realization g .

Game. The agents seek to learn about the effectiveness of a new technology as captured by a state $\theta \in \{L, H\}$. Time is continuous, $t \in [0, \infty)$. At time $t = 0$, agents share a common prior $\Pr(\theta = H) = p_0$. At each time t , agent i privately chooses effort $A_{i,t} \in [0, 1]$ at flow cost c . This effort results in successes at random arrival times (T_i^1, T_i^2, \dots) with arrival rate $A_{i,t} \mathbb{I}_{\{\theta=H\}}$. Agent i observes her own and her neighbors' past successes, but not others' actions. If the network is random, she knows G but nothing about the realization g , not even her own degree.⁶

Payoffs. Agents receive payoff $x > c$ from their own successes. Payoffs are discounted at rate $r > 0$, so Iris's expected discounted value equals

$$V_i = \max_{\{A_{i,t}\}_{t \geq 0}} E \left[x \sum_{\iota=1}^{\infty} e^{-rT_i^\iota} - c \int_0^{\infty} e^{-rt} A_{i,t} dt \right] \quad (1)$$

where the expectation is taken over quality θ , network G , and arrival times $\{T_i^\iota\}$. We solve for weak perfect Bayesian equilibria, where agents who have observed a success infer that $\theta = H$.

Remarks. Agents observe neighbors' successes but not their actions. This makes the model tractable by focusing on a single mechanism of information transmission. We think this is reasonable in many applications (e.g. a farmer doesn't know if her neighbor experimented with a new crop, but observes the results of a successful harvest). Agents also cannot communicate with each other directly. Indeed, an agent has little incentive to reveal her failures which tend to make her neighbors more pessimistic and lower their experimentation.

Our model is equivalent to a model where each agent can only succeed once for a payoff of $x + (x - c)/r$. Agents do not get to observe their neighbors' repeated successes in this model variant, but this does not matter since the first observed success reveals $\theta = H$ perfectly.

⁶The main role of this assumption is to ensure our agents are symmetric in our large regular random networks in Section 4.2. It has no impact on the analysis in Sections 3 and 4.3.

3 Preliminary Analysis

3.1 Best-Responses: Cutoff Strategies

In this section, we characterize the best response of a generic agent, Iris, given arbitrary strategies of other agents.

As a benchmark, consider the single-agent experimentation problem, or equivalently Iris's problem when she has no neighbors. After her first success, she sets $A_{i,t} = 1$ and obtains continuation value $y := (x - c)/r$. Before that, her posterior belief evolves according to

$$p_t = P^\emptyset(t) := \frac{p_0 e^{-t}}{p_0 e^{-t} + (1 - p_0)}.$$

Iris thus experiments until time $\bar{\tau}$ when her belief hits the *single-agent threshold belief* $p_{\bar{\tau}} = \underline{p} := c/(x + y)$. It is also useful to define the *myopic threshold belief* $\bar{p} := c/x$, where Iris would stop if she ignored the future benefit of success, y .

Now, consider the general problem where Iris learns from her neighbors $N_i(G)$. Write $T_i = T_i^1$ for Iris's first success time, and $S_i := \min_{j \in N_i(G)} T_j$ for her neighbors' first success time. After Iris observes a success at $\min\{T_i, S_i\}$, she chooses maximal effort and receives continuation value y . We can thus restrict attention to earlier times, and write $\{a_{i,t}^\emptyset\}_{t \geq 0}$ for her *experimentation*, i.e. her effort before $\min\{T_i, S_i\}$. Also write

$$b_{i,t} := E^H \left[\sum_{j \in N_i(G)} A_{j,t} \middle| t < T_i, S_i \right] \quad (2)$$

for Iris's rate of social learning, where the expectation is taken over the random network G and success times $\{T_j\}$, conditional on $\theta = H$.⁷ We also define Iris's cumulative social learning $B_{i,t} := \int_0^t b_{i,s} ds$, and abuse terminology by referring to both $\{b_{i,t}\}$ and $\{B_{i,t}\}$ as Iris's *social learning curve*. Since Iris's experimentation is unobservable to others and her own success effectively ends the game for her, Iris takes $\{b_{i,t}\}$ as given. We thus study the best response $\{a_{i,t}^\emptyset\}$ to $\{b_{i,t}\}$, and drop the i subscript for the rest of the section.⁸

When $\theta = H$, the random time $\min\{T, S\}$ has hazard rate $a_t^\emptyset + b_t$, and so the chance of not observing a success before t equals $\exp(-\int_0^t (a_s^\emptyset + b_s) ds)$. Bayes's rule then implies the

⁷Iris additionally knows her past actions $\{A_{i,s}\}_{s < t}$, but since these are deterministic given $t < T_i, S_i$ there is no need to include them in the conditional expectation.

⁸Since social learning is fully captured by the curve $\{b_{i,t}\}$, the analysis in this section immediately applies to more general networks. For example, the network may vary over time, or Iris may have private information about her degree $|N_i(g)|$.

posterior belief

$$p_t = P^\emptyset \left(\int_0^t (a_s^\emptyset + b_s) ds \right).$$

Truncating Iris's objective function (1) at the first observed success reduces it to a deterministic control problem

$$V = \max_{\{a_t^\emptyset\}_{t \geq 0}} \int_0^\infty e^{-rt} \left(p_0 e^{-\int_0^t (a_s^\emptyset + b_s) ds} + (1 - p_0) \right) \left((a_t^\emptyset(x + y) + b_t y) p_t - a_t^\emptyset c \right) dt. \quad (3)$$

Intuitively, Iris receives $x + y$ when she succeeds, y when a neighbor succeeds, and incurs effort cost of c when she works. The chance of no success by time- t is $e^{-\int_0^t (a_s^\emptyset + b_s) ds}$ when $\theta = H$, and one when $\theta = L$.

Clearly, Iris experiments above the myopic threshold, $p_t \geq \bar{p}$. Conversely, equation (3) implies that Iris stops experimenting below the single-agent threshold, $p_t \leq \underline{p}$. For beliefs $p_t \in [\underline{p}, \bar{p}]$, her choice depends on her social learning. To avoid trivialities, we restrict $p_0 \in (\underline{p}, 1)$. We say Iris's prior is *optimistic* if $p_0 > \bar{p}$ and *pessimistic* if $p_0 < \bar{p}$.⁹ An optimistic agent always engages in some experimentation, no matter her social learning curve.

We first claim that Iris uses a *cutoff strategy* in that she experiments maximally until some cutoff time τ and then stops, $a_t^\emptyset = \mathbb{I}_{\{t \leq \tau\}}$.¹⁰ Intuitively, it makes no sense to stop experimenting at some τ' but then resume it after neighbors' lack of success over $[\tau', \tau'']$. For a more rigorous argument, suppose Iris shirks at time t but works at time $t + \delta$, and consider the effect of front-loading effort ϵ from $t + \delta$ to t . This has two consequences. First, if the effort pays off, i now gets to enjoy the success earlier, raising her value by $r\delta(p_t(x + y) - c)\epsilon$, which is positive in the relevant range of posteriors $p_t > \underline{p}$. Second, if one of her neighbors succeeds over $[t, t + \delta]$, she ends up working at both t and $t + \delta$, raising her value by $p_t b_t \delta \epsilon (x - c) > 0$. Thus, Iris always prefers to front-load experimentation, giving rise to a cutoff time τ with cutoff belief $p_\tau \in [\underline{p}, \bar{p}]$.

To characterize the optimal cutoff τ , define Iris's *experimentation incentives* at time- t ,

$$\psi_t := p_t \left(x + ry \int_t^\infty e^{-\int_t^s (r + b_u) du} ds \right) - c. \quad (4)$$

To understand (4), suppose that successes from Iris's neighbors arrive at constant rate b , so (4) simplifies to $p_t(x + \frac{r}{r+b}y) - c$. If she raises the cutoff from t to $t + \delta$, she gains the expected payoff from a success $p_t(x + y)\delta$, forgoes the expected benefit of future social learning $p_t(\frac{b}{r+b}y)\delta$, and incurs marginal effort cost $c\delta$. The experimentation incentives are

⁹We are opportunistic about calling the boundary case $p_0 = \bar{p}$ optimistic or pessimistic.

¹⁰Of course, "stopping" is provisional in the sense that Iris starts to work again when she observes one of her neighbors succeed at some $t > \tau$.

the sum of these three effects. We summarize this discussion as follows:

Proposition 1. *Given social information $\{b_t\}$, the agent’s optimal experimentation is given by the cutoff strategy $a_t^\emptyset = \mathbb{I}_{\{t \leq \tau\}}$, where the cutoff time $\tau \in (0, \bar{\tau}]$ uniquely solves $\psi_\tau = 0$ if $\psi_0 > 0$, and $\tau = 0$ if $\psi_0 \leq 0$.*

Proof. The proof in Appendix A.1 formalizes the frontloading argument and shows that the marginal payoff from experimentation at the cutoff is proportional to ψ_t , which in turn single-crosses from above in t . \square

Proposition 1 reduces the potentially complicated dynamic experimentation problem of a forward-looking, Bayesian agent to choosing one number, τ , which is characterized by setting (4) to zero. This tractability allows us to characterize equilibria for rich classes of networks. In contrast to Proposition 1, the seminal papers on strategic experimentation in the clique network, Keller, Rady, and Cripps (2005) and Bonatti and Hörner (2011), both find agents gradually phase out effort in equilibrium. This difference arises because free-riding incentives are greater in their models: In Keller, Rady, and Cripps (2005), actions are observable, so Iris’s neighbors get pessimistic when her experimentation fails; in Bonatti and Hörner (2011), payoffs are public, so Iris does not want to exert effort if others are about to succeed.

3.2 Best-Responses: Comparative Statics

This section derives two useful comparative statics on Iris’s value and her optimal cutoff as a function of social learning. The first result shows that higher social information, $B'_t \geq B_t$ for all $t \geq 0$, raises Iris’s value and leads her to stop earlier. Thus, this is a game of strategic substitutes. The optimal cutoff τ is maximized in the absence of social learning, $B_t \equiv 0$, where it coincides with the single-agent solution, $\tau = \bar{\tau}$.

Lemma 1. *Higher social learning $\{B_t\}_{t \geq 0}$ raises value V and lowers the cutoff τ .*

Proof. A rise in $\{B_t\}$ constitutes Blackwell-superior information, which raises V . Experimentation incentives (4) fall both in pre-cutoff learning B_τ which lowers the cutoff belief $p_\tau = P^\emptyset(\tau + B_\tau)$ and in future learning $\{b_t\}_{t \geq \tau}$. To show that ψ_τ falls in cumulative learning $\{B_t\}$, we need to compare the impact of “early” and “late” increases in b_t . Specifically, differentiating time- τ experimentation incentives (4) with respect to time- t social learning, we get¹¹

¹¹Formally, define $\frac{\partial \psi_\tau}{\partial b_t} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (\psi_\tau(\{b_s^{t,\epsilon}\}_{s \geq 0}) - \psi_\tau(\{b_s\}_{s \geq 0}))$ where $b_s^{t,\epsilon} := b_s + \mathbb{I}_{\{s \in [t-\epsilon, t]\}}$.

$$-\frac{\partial \psi_\tau}{\partial b_t} = \begin{cases} p_\tau (ry \int_\tau^\infty e^{-\int_\tau^s (r+b_u) du} ds + x - c) & \text{for } t < \tau \\ p_\tau ry \int_t^\infty e^{-\int_\tau^s (r+b_u) du} ds & \text{for } t > \tau, \end{cases} \quad (5)$$

where the case $t < \tau$ uses $\frac{\partial p_\tau}{\partial b_t} = -p_\tau(1 - p_\tau)$ and $(1 - p_\tau)(x + ry \int_\tau^\infty e^{-\int_\tau^s (r+b_u) du} ds) = x + ry \int_\tau^\infty e^{-\int_\tau^s (r+b_u) du} ds - (\psi_\tau + c)$. Clearly, (5) is positive and falls in t , weakly for $t < \tau$ and discontinuously at $t = \tau$.

Thus, incentives ψ_τ fall as a function of $\{B_t\}$. Since ψ_t strictly single-crosses from above (by the proof of Proposition 1), the solution τ of $\psi_\tau = 0$ falls in $\{B_t\}$. \square

Equation (5) tells us that pre-cutoff learning B_τ crowds-out the agent's experimentation more than post-cutoff learning $\{b_t\}_{t \geq \tau}$. After the cutoff, it crowds out the option value of own experimentation $ry \int_\tau^\infty e^{-\int_\tau^s (r+b_u) du} ds$, as seen in the second line of (5) for $t = \tau$. Before the cutoff, the additional term $x - c$ in the first line of (5) represents the reduced opportunity of achieving a first success at τ , conditional on $\theta = H$.¹²

Our second result provides a tool for rich comparisons of equilibrium values. Lemma 1 is of limited value for such comparative statics because the order on social learning $\{B_t\}$ is highly incomplete. To obtain a sharper tool, the proof of Lemma 2 shows that by truncating the integral expression for an agent's value (3) at τ we can write the agent's value

$$V = \frac{p_0 x - c}{r} + e^{-r\tau} \left(p_0 e^{-B_\tau - \tau} (x - c) - (1 - p_0) c \frac{r - 1}{r} \right) =: \mathcal{V}(\tau, B_\tau) \quad (6)$$

as a function of only two variables: the cutoff τ , and pre-cutoff social learning B_τ . Before τ , Iris exerts effort anyway, so does not care about the timing of social learning $\{B_t\}_{t \leq \tau}$. After τ , learning $\{B_t\}_{t \geq \tau}$ matters only via the continuation value $V_\tau = p_\tau y \int_\tau^\infty b_s e^{-\int_\tau^s (r+b_u) du} ds = p_\tau(x + y) - c$,¹³ which is a function of (τ, B_τ) since $p_\tau = P^\emptyset(\tau + B_\tau)$.

Lemma 2. *For any social learning curve $\{B_t\}$ with $\psi_0 \geq 0$ and optimal cutoff τ , the agent's value is given by (6). The function $\mathcal{V}(\tau, B_\tau)$ falls in both arguments.*

Proof. See Appendix A.2. \square

The fact that $\mathcal{V}(\tau, B_\tau)$ falls in B_τ may sound counterintuitive. It arises because we fix the optimal stopping time τ , as characterized by $\psi_\tau = 0$. A rise in pre-cutoff learning B_τ must be compensated by a fall in post-cutoff learning $\{b_t\}_{t \geq \tau}$ in order to keep τ constant. More

¹²For optimistic agents, $p_0 > \bar{p}$, this asymmetry is stark. A finite amount $B_t = \bar{\tau}$ of pre-cutoff learning fully crowds out incentives by inducing $p_t < \underline{p}$ and so $\psi_t < 0$. In contrast, no amount of post-cutoff learning fully crowds out incentives since $\psi_0 > p_0 x - c > 0$ for any $\{B_t\}$.

¹³The first equality leverages the fact that all learning after τ is social $\{b_t\}_{t \geq \tau}$, and the second leverages the indifference condition $\psi_\tau = 0$.

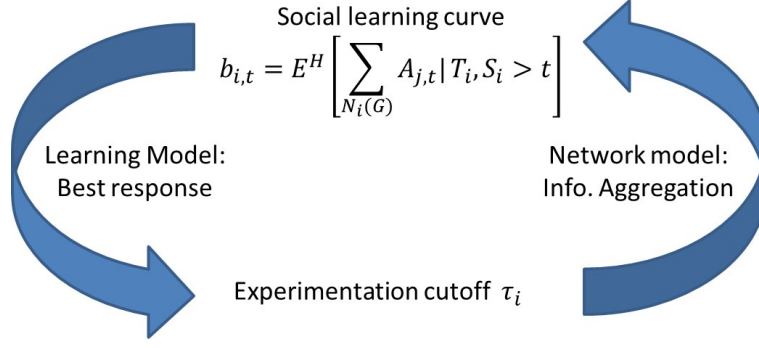


Figure 1: **Equilibrium Analysis.**

strongly, since pre-cutoff learning has a discontinuously larger effect on ψ_τ than post-cutoff learning by (5), we must reduce the latter by a larger amount to compensate. In contrast to (5), the effect of social learning on value, $\partial V / \partial b_t$, is continuous in t , so the combination of a small raise of $b_{\tau-\epsilon}$ and a large drop of $b_{\tau+\epsilon}$ decreases value.

Lemma 2 is the key tool to compare equilibrium welfare across agents and networks since τ and B_τ are easily characterized in equilibrium. For example, suppose one agent optimally shirks $\tau = 0$, while another optimal works $\tau' > 0$. Since $B_\tau = 0$ and $B_{\tau'} \geq 0$, the shirker has higher utility than the worker, $\mathcal{V}(0, 0) > \mathcal{V}(\tau', B_{\tau'})$.

Lemma 2 assumes $\psi_0 \geq 0$, so her stopping time is characterized by $\psi_\tau = 0$.¹⁴ This assumption is satisfied in the random networks in Section 4.2 where all agents exert some effort, and for peripheral agents in the core-periphery networks in Section 4.3.

3.3 Equilibrium: Examples

In the prior sections, we studied Iris's best response τ_i as a function of social learning $\{b_{i,t}\}$. To close the model in equilibrium we must study how individual cutoffs $\{\tau_j\}$ aggregate into social learning curves $\{b_{i,t}\}$, as illustrated in Figure 1. Here we demonstrate this aggregation in three canonical example networks, foreshadowing the more general analysis in Section 4.

Example 1 (Clique). Assume that all I agents observe each other. We claim there is a unique equilibrium in which all agents equally divide the single-agent experimentation between them. That is, each agent i uses cutoff $\tau_i = \bar{\tau}/I$, recalling the single-agent experimentation cutoff $\bar{\tau}$ that solves $P^\emptyset(\bar{\tau}) = \underline{p}$. The resulting social learning curve is shown in Figure 2(a). As I rises, aggregate information is constant while welfare rises as learning accelerates and agents share the cost of experimentation.

¹⁴Otherwise, if the agent is receiving too much social information, her value equals $V = V_0 = p_0 y \int_0^\infty b_s e^{-\int_0^s (r+b_u) du} ds = p_0(x+y) - c - \psi_0 = \mathcal{V}(0, 0) - \psi_0 > \mathcal{V}(0, 0)$.

We prove our claim in two steps. First, the agents collectively experiment as much as a single agent would by herself $\sum_i \tau_i = \bar{\tau}$. This is because any agent who experiments the longest expects no social information after her cutoff, $b_{i,s} = 0$ for $s > \tau_i$. Hence she faces the first-order condition of the single-agent problem, $P^\emptyset(\sum_j \tau_j)(x + y) - c = 0 = \underline{p}(x + y) - c$. Second, the agents must split total experimentation evenly, $\tau_j = \bar{\tau}/I$. This follows because all agents are indifferent when p_t reaches \underline{p} and prefer to front-load experimentation, so they all experiment until $\bar{\tau}/I$. We prove these two steps in Proposition 3.¹⁵ \triangle

Example 2 (Infinite Directed Line). Consider the following network:

$$\dots \rightarrow i \rightarrow j \rightarrow k \rightarrow \dots$$

In the unique symmetric equilibrium, private discovery in an initial *experimentation phase* of length τ is followed by social diffusion in a *contagion phase*.¹⁶ For example, suppose Kata succeeds in the experimentation phase, while Iris and John do not. After τ , Kata's success means that Kata and John continue to work while Iris shirks. Eventually John also succeeds and all three work thereafter.

To solve for the equilibrium cutoff τ , we calculate Iris's learning in the contagion phase,

$$b_{i,t} = E^H [A_{j,t} | t < T_i, S_i] = E^H [A_{j,t} | t < T_j] = \Pr^H (T_k < t | t < T_j) = 1 - e^{-\tau}. \quad (7)$$

The second equality uses that John is Iris's only neighbor, so $S_i = T_j$, and that his effort $A_{j,t}$ is independent of Iris's lack of success, $t < T_i$. The third equality relies on the observation that after τ , John works iff Kata has succeeded. The last equality uses Bayes' rule,

$$\Pr^H (t < T_k | t < T_j) = \frac{\Pr^H (t < T_j | t < T_k) \Pr^H (t < T_k)}{\Pr^H (t < T_j)} = \Pr^H (t < T_j | t < T_k) = e^{-\tau}.$$

Thus, social information arrives at constant rate $b_{i,t} \equiv 1 - e^{-\tau}$, as illustrated in Figure 2(b). While the unconditional probability that Kata has succeeded, and hence John works, rises over time, this positive effect is exactly cancelled by conditioning on the bad news event that

¹⁵The uniqueness of equilibrium is notable. Public good problems with linear costs feature a continuum of equilibria; Bramoullé and Kranton (2007) select via a stability criterion while Galeotti and Goyal (2010) select via a network formation game. We resolve this indeterminacy through impatience. In experimentation papers there are also asymmetric equilibria (e.g. Keller, Rady, and Cripps (2005), Bonatti and Hörner (2011)). As discussed after Proposition 1, free-riding incentives are weaker in our paper, leading putative asymmetric equilibria to unravel.

¹⁶In Section 5 we approximate this infinite network with a sequence of finite random networks, that generate circles of random, exploding length. Corollary 1 shows that the unique equilibrium of each finite random networks is symmetric and Proposition 4 shows that these equilibria converge to the symmetric equilibrium described here.

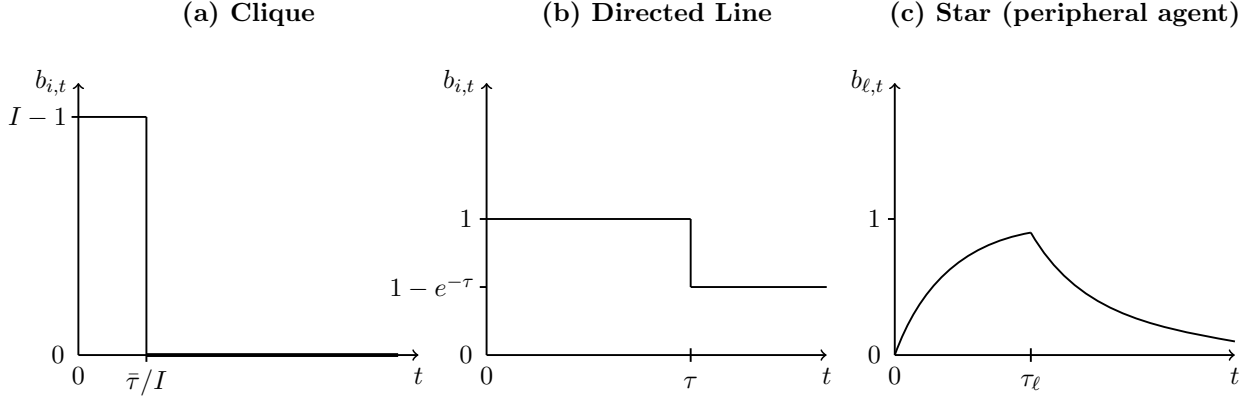


Figure 2: **Social Learning Curves.** This picture illustrates the rate of social learning b_{it} as defined in equation (2) for Examples 1-3, as described in the text.

John has not succeeded yet, $t < T_j$.

Using equation (4), the equilibrium stopping time τ solves

$$\psi_{\tau} = P^{\emptyset}(2\tau) \left(x + \frac{r}{r + (1 - e^{-\tau})} y \right) - c = 0. \quad (8)$$

Example 3 (Star). The star consists of one core agent (Kata, k) and L peripheral agents (Lili, ℓ), and undirected links between k and each ℓ . In any equilibrium, peripherals use a common cutoff τ_{ℓ} (by Proposition 3). Moreover, Kata learns faster than the peripherals and so experiments less herself, $\tau_k < \tau_{\ell}$ (by Lemma 5). Indeed, for $p_0 < \bar{p}$ and large L , Kata does not experiment herself, $\tau_k = 0$.

Information is thus primarily generated by peripherals, but flows via Kata who serves as an information broker. If a peripheral succeeds before τ_{ℓ} , Kata sees this and starts to work; her eventual success then triggers all other peripherals to work. The resulting social learning curve for peripheral agent Lili $b_{\ell,t}$ undergoes two phases, illustrated in Figure 2(c). Up to time τ_{ℓ} , it increases because of other peripherals' experimentation. After τ_{ℓ} , no more additional information is created, and $b_{\ell,t}$ falls as the information filters through Kata and Lili becomes pessimistic about Kata having seen a success. These dynamics are analogous to water that flows into a reservoir while the peripherals experiment, and slowly drains out through a bottleneck as Kata conveys the information. \triangle

3.4 Equilibrium: Existence and Uniqueness

We round off our preliminary analysis by establishing equilibrium existence and presenting a limited uniqueness result.

Proposition 2. *Equilibrium exists.*

Proof. We argue that the best-response mapping in cutoff vectors $\{\tau_j^*\} : [0, \bar{\tau}]^I \rightarrow [0, \bar{\tau}]^I$ is continuous, which implies equilibrium existence by Brouwer’s fixed point theorem. First note that i ’s social learning curve $\{b_{i,t}\}_{t \geq 0}$, defined in equation (2), is pointwise continuous in $\{\tau_j\}_{j \neq i}$ for all $t \neq \tau_j$. Then, Lebesgue’s dominated convergence theorem implies that incentives $\psi_{i,t}$ in (4) are also continuous in $\{\tau_j\}_{j \neq i}$ for all t . Finally, since $\psi_{i,t}$ strictly single-crosses in t (see the proof of Proposition 1), its root $\tau_i^*(\{\tau_j\}_{j \neq i})$ is also continuous. \square

Uniqueness is more difficult. We have not proved equilibrium uniqueness in general, but can show uniqueness for “strongly symmetric” networks. For a deterministic network g and agents $i \neq j$, define $g_{i \leftrightarrow j}$ to be the same network when switching i and j .¹⁷ For a random network G , define $G_{i \leftrightarrow j}$ by $\Pr^{G_{i \leftrightarrow j}}(g) = \Pr^G(g_{i \leftrightarrow j})$ for all g .

Proposition 3. *If i, j are symmetric in G , i.e. $G_{i \leftrightarrow j} = G$, then in any equilibrium $\tau_i = \tau_j$.*

Proof. See Appendix A.3. \square

For an intuition, consider a deterministic, undirected network where i and j are not connected. By contradiction assume $\tau_j < \tau_i$. Since i ’s additional learning over $[\tau_j, \tau_i]$ is more immediate to i than to j , who only benefits indirectly via some other agent k , we can argue that $\min\{T_i, S_i\}$ is smaller than $\min\{T_j, S_j\}$. This greater chance of learning the state depresses i ’s experimentation incentives below j ’s, leading to the contradiction that $\tau_j > \tau_i$.

Say that network G is *strongly symmetric* if $G_{i \leftrightarrow j} = G$ for any pair of agents i, j .

Corollary 1. *Strongly symmetric networks have a unique equilibrium, characterized by a cutoff $\tau \in (0, \bar{\tau})$, such that $\tau_i = \tau$ for all i .*

Proof. Proposition 3 implies that all agents must share the same cutoff τ . Uniqueness of the cutoff follows from strategic substitutes. Consider two cutoffs $\tau < \tau'$ where the former constitutes a symmetric equilibrium: $\psi_{i,\tau} = 0$ when $\tau_j = \tau$ for all $j \neq i$. When others $j \neq i$ use the higher cutoff $\tau_j = \tau'$, Iris learns more $B'_{i,t} \geq B_{i,t}$ for all t , and thus chooses to stop at $\tau'' < \tau < \tau'$ by Lemma 1. Hence τ' does not constitute a symmetric equilibrium. \square

Our notion of symmetry is so strong that only two deterministic networks satisfy it: the clique and the empty network. Indeed, the infinite directed line (Example 2) or a finite directed circle violate it since only agent i observes $i + 1$, so $g \neq g_{i \leftrightarrow j}$ for any $j \neq i$. With

¹⁷Formally, given g , we can define $g_{i \leftrightarrow j}$ by three types of links. First, links involving i and j : $(i, j) \in g_{i \leftrightarrow j}$ iff $(j, i) \in g$, and analogously, $(j, i) \in g_{i \leftrightarrow j}$ iff $(i, j) \in g$. Second, links involving one third party: $(i, k) \in g_{i \leftrightarrow j}$ iff $(j, k) \in g$ and three analogous conditions, replacing i and j and switching the direction of these links. Third, links involving two third parties: $(k, \ell) \in g_{i \leftrightarrow j}$ iff $(k, \ell) \in g$.

this said, many natural classes of random networks, such as the random networks studied in Section 4.2, do satisfy strong symmetry and Corollary 1 applies. Moreover, Proposition 3 is useful beyond strongly symmetric networks; for instance, equilibria in core-periphery networks in Section 4.3 are characterized by one cutoff τ_k for all core-agents and another cutoff τ_ℓ for all peripherals.

4 Density of Links

We now turn to the main question of the paper: How do learning and welfare depend on network density? Section 4.1 introduces some terminology and a second-best benchmark for welfare. Section 4.2 studies large random networks. Section 4.3 studies large core-periphery networks. The results for asymptotic learning and welfare in these two sections run parallel to one another, but the learning dynamics differ.

4.1 Bounds on Learning and Welfare

First consider aggregate information. We study large networks via sequences of networks $\{G^I\}_{I \in \mathbb{N}}$ and write $B^I := \min_j B_{j,\infty}^I$ for the social information of the least-informed agent (in expectation over the realized network g). If G^I admits multiple equilibria, we consider the infimum values of B^I . We define *asymptotic information* as $B = \liminf_{I \rightarrow \infty} B^I$. There is *asymptotic learning* if $B = \infty$, so all agents eventually learn the state.

Next consider welfare. Iris's value is trivially bounded above by the value of learning the state perfectly immediately, $V_i < p_0 y$. Another, less obvious, upper bound on agents' value comes from the fact that for i to socially learn, some other agent $j \neq i$ must generate that social information. By Lemma 2 and equation (6), this implies that $\min_j V_j < \mathcal{V}(0, 0) = p_0(x + y) - c$.¹⁸ This motivates the following Rawlsian welfare upper bound

$$V^* := \min\{p_0 y, p_0(x + y) - c\},$$

illustrated in Figure 3 as a function of p_0 . Given a sequence of networks $\{G^I\}_{I \in \mathbb{N}}$, let $V^I := \min_j V_j^I$ be the expected welfare of the worst-off agent. If G^I admits multiple equilibria, we consider the infimum values of V^I . In the limit, as $I \rightarrow \infty$, define *asymptotic welfare* as $V = \liminf V^I$.

¹⁸This upper bound relies on agents using equilibrium strategies. Consider a sequence of clique networks in which agents use symmetric cutoffs τ^I that vanish individually $\lim \tau^I = 0$ but explode in aggregate $\lim I \tau^I = 0$. Agents' payoffs approach $p_0 y$, which exceeds $p_0(x + y) - c$ for pessimistic priors $p_0 < \bar{p}$. However, in equilibrium, such agents have a strict incentive to shirk for large I (see Example 1).

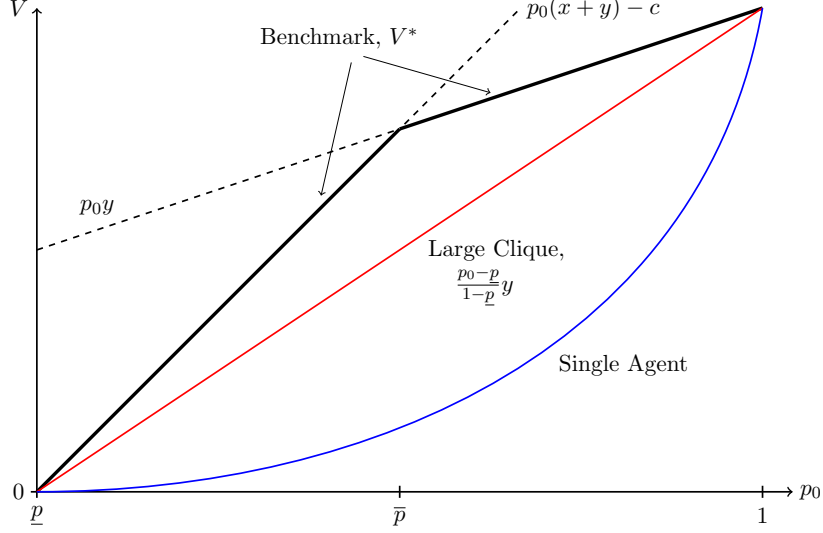


Figure 3: **Value Bounds.** The figure shows the bound on the value V^* as a function of the agent's prior, p_0 . This upper bound is piece-wise linear, with a downward kink at the myopic cutoff \bar{p} . The figure also illustrates agents' values in a large clique and when by themselves.

While asymptotic learning and the welfare benchmark are both driven by social learning $\{B_t\}$, asymptotic learning focuses on the long run, while welfare incorporates discounting. For optimistic priors $p_0 \geq \bar{p}$, the welfare bound requires agents learn the state immediately, so clearly they also learn asymptotically. For pessimistic priors $p_0 < \bar{p}$, asymptotic learning and the welfare benchmark are opposing goals. Recall that the value function $\mathcal{V}(\tau, B)$ from Lemma 2 falls in τ . Thus, the welfare benchmark $V^* = \mathcal{V}(0, 0)$ requires individual experimentation to vanish $\max_{1 \leq i \leq I} \tau_i^I \rightarrow 0$, while asymptotic learning requires aggregate information to diverge, $\sum_{i=1}^I \tau_i^I \rightarrow \infty$. For core-periphery networks, we will see that these two conditions are mutually exclusive.

Our main results, Theorem 1 and 2, show that sequences of random networks and core-periphery networks can attain the asymptotic learning benchmark $B = \infty$ when network density is small, and the welfare benchmark V^* when network density is intermediate.¹⁹

To illustrate the benchmarks, we return to the three examples in Section 3.3.

Example 1, Continued (Clique). In a finite network, the agents share the single-agent experimentation, $\tau_i = \bar{\tau}/I$. As $I \rightarrow \infty$, individual experimentation vanishes and all learning

¹⁹In both cases, the proof of the Theorem characterizes unique limit points for $\{B^I\}$ and $\{V^I\}$, so the \liminf equals the ordinary limit. In the large random networks equilibria are unique, so taking the infimum over equilibrium values is moot. In finite core-periphery networks, we do not know whether equilibrium is unique, but the unique characterization of the limit points does not rely on taking the infimum over equilibria and rather applies for any equilibrium selection.

is social with asymptotic information $B = \bar{\tau}$. Agents' receive all their social information before stopping, $B_\tau = \bar{\tau}$, so asymptotic welfare equals $\mathcal{V}(0, \bar{\tau})$. More concretely, agents' beliefs instantly jump to 1 (if there is a success) or drop to \underline{p} (if there is no success). The payoff to the former is y , so the equilibrium values converge to $\mathcal{V}(0, \bar{\tau}) = (p_0 - \underline{p})y/(1 - \underline{p}) < V^*$, as illustrated in Figure 3. The speed of diffusion in the clique chokes off discovery and means that agents neither asymptotically learn nor obtain the welfare benchmark. \triangle

Example 2, Continued (Infinite Directed Line). In this infinite network, each agent experiments for time $\tau > 0$, where τ solves (8). Asymptotic learning obtains since social learning $B_t = \tau + (1 - e^{-\tau})(t - \tau)$ is unbounded. However, agents learn too slowly and they do not attain the welfare benchmark. Specifically, each agent experiments for τ and learns an additional τ from her neighbor before stopping, so $B_\tau = \tau$ and welfare equals $\mathcal{V}(\tau, \tau) < V^*$. \triangle

Example 3, Continued (Star). When there is a large number of peripherals, the core agent Kata shirks, $\tau_k = 0$. The peripherals thus do all the experimentation and have the lowest information and welfare, so we focus on them. We show in Section 4.3 that agents asymptotically learn iff $p_0 \geq p^s$. This threshold is defined so that a peripheral agent will just work at $t = 0$ if she thinks Kata will instantly learn the state and choose $b_{k,t} \equiv 1$ thereafter,

$$\psi_{\ell,0} = p^s \left(x + \frac{r}{r+1}y \right) - c = 0.$$

Note that $p^s < \bar{p}$, so agents asymptotically learn if they have an optimistic prior.

The welfare result is exactly the opposite: Agents attain the welfare benchmark iff $p_0 \leq p^s$. For a high prior, $p_0 > p^s$, the peripheral agents experiment in the limit, $\tau_\ell > 0$, meaning Kata instantly learns. Thus a peripheral agent learns $B_{\tau_\ell} = \tau_\ell$ before stopping and has value $\mathcal{V}(\tau_\ell, \tau_\ell) < \mathcal{V}(0, 0) = V^*$.²⁰ For a low prior, $p_0 \leq p^s$, the peripherals stop experimenting in the limit, $\tau_\ell = 0$, and since $B_{\tau_\ell} \leq \tau_\ell$, their value converges to its upper bound $V^* = \mathcal{V}(0, 0)$.²¹ Thus, asymptotic learning and the welfare benchmark are not only distinct concepts, but in fact mutually exclusive (for generic priors $p \neq p^s$). \triangle

4.2 Large Random Networks

We first study large random networks. This is a tractable and canonical class of networks that can capture realistic contagion dynamics. For simplicity we focus on regular networks,

²⁰The latter equality presumes $p_0 \leq \bar{p}$. For $p_0 > \bar{p}$, the welfare bound $V^* = p_0 y$ requires immediate perfect social information, which is clearly impossible with a single neighbor.

²¹It may appear paradoxical that the welfare upper bound V^* is achieved for low, but not high prior beliefs p_0 . This is because V^* itself rises as function of p_0 , and is hence a more demanding benchmark for high p_0 .

where agents all have the same number of neighbors, and comment after our main result which insights generalize to non-degenerate degree distributions. This class is rich enough to encompass the clique and trees, as in Sadler (2020a) and Board and Meyer-ter-Vehn (2021).

We construct a *regular random network* as follows. Each of the I agents has $\hat{n}^I \geq 2$ link stubs. We randomly draw pairs of stubs and connect them into undirected links. We then prune self-links (from i to i), multi-links (from i to j), and if $\hat{n}^I I$ is odd the single leftover stub. We assume that agents observe nothing about the network realization, not even their own degree; omitting such information seems innocuous since agents asymptotically know their degree (see Lemma 3, below).

By construction, the random network is strongly symmetric, so Corollary 1 implies there is a unique equilibrium. Denote the symmetric cutoff by τ^I and agents' value by V^I . We consider sequences of such networks with degrees $\{\hat{n}^I\}$, and assume existence of the limits $\nu := \lim \hat{n}^I$, $\lambda := \lim \hat{n}^I / \log I$ and $\hat{\rho} := \lim \hat{n}^I / I$, possibly equal to ∞ .

Let N^I be the number of realized links of a random agent. Some stubs may fail to form links, so N^I is random with expectation $n^I := E[N^I] < \hat{n}^I$. We now argue that we can ignore this complication as $I \rightarrow \infty$.

Lemma 3. *As the network grows large, $I \rightarrow \infty$,*

- (a) *Realized degree:* $N^I / n^I \xrightarrow{D} 1$.
- (b) *Expected degree:* $n^I / I \rightarrow 1 - e^{-\hat{\rho}}$. If $\hat{\rho} = 0$, then $n^I / \hat{n}^I \rightarrow 1$.
- (c) *Information at the cutoff time:* $\lim B_{\tau^I}^I = \lim n^I \tau^I$.

Proof. See Appendix B.1. □

Part (a) means agents essentially know their realized degree N^I . Part (b) means we can ignore the distinction between stubs and links when $\hat{\rho} = 0$. And part (c) means agents do not update N^I during the experimentation phase, a consequence of part (a).

We next introduce three relevant regions of limit network density:

- (1) *Sparse networks.* Here, agents have a bounded number of links, with $\nu := \lim \hat{n}^I = \lim n^I \in \{2, 3, \dots\}$. Proposition 4 in Section 5 shows that such networks approach trees. The contagion dynamics thus resemble the infinite directed line in Example 2.
- (2) *Intermediate networks.* Here, agents links are of order $\log I$, with $\lambda := \lim \hat{n}^I / \log I = \lim n^I / \log I \in (0, \infty)$. In such networks, information spreads across the network in finite time, as in Milgram (1967)'s six degrees of separation. Indeed, Lemma 4 below shows that the inverse $1/\lambda$ measures the network's *time-diameter*, i.e. the time for information to travel between two random agents in the network.²²

²²This time-diameter $1/\lambda = \lim(\log I / n^I)$ is smaller than the typical diameter estimate for large random

- (3) *Dense networks.* Here, agents are connected to a fixed proportion of other agents $\rho := \lim n^I/I = 1 - e^{-\hat{\rho}} \in [0, 1]$. Agents are at most two links apart and we approximate the clique from Example 1 when $\rho = 1$.

The set of network (limit) densities is the union $\{\nu | \nu \in \mathbb{N}\} \cup \{\lambda \cdot \log I | \lambda \in [0, \infty]\} \cup \{\rho \cdot I | \rho \in [0, 1]\}$, endowed with its natural order, after identifying $\infty \cdot \log I$ with $0 \cdot I$.²³

We now define the threshold density for asymptotic learning. For pessimistic priors, $p_0 < \bar{p}$, let $\sigma^* \in [0, \infty)$ be such that perfectly learning the state at time σ^* renders an agent indifferent about experimenting at $t = 0$. Using (4)

$$\psi_0 = p_0 (x + (1 - e^{-r\sigma^*})y) - c = 0. \quad (9)$$

Here, $e^{-r\sigma^*}y$ is the post-experimentation continuation value. For $p_0 \geq \bar{p}$, set $\sigma^* = 0$.

Theorem 1. *In large random networks $\{\hat{n}^I\}$:*

- (a) *Asymptotic information B is a decreasing function of network density: It attains asymptotic learning $B = \infty$ iff $\lambda \leq 1/\sigma^*$ and $\rho = 0$, and strictly falls when $\lambda \geq 1/\sigma^*$.*
- (b) *Welfare V is a single-peaked function of network density: It strictly rises when $\nu < \infty$, attains the benchmark V^* iff $\nu = \infty$ and $\rho = 0$, and then strictly falls when $\rho > 0$.*

Proof. See Appendix B.2 □

Asymptotic learning requires sparse networks. Intuitively, denser networks accelerate diffusion, crowd out discovery, and undermine learning in the long run. Welfare attains the benchmark when network density is intermediate. Intuitively, welfare discounts the future and so relies on both information generation and its quick dissemination.

Theorem 1 goes beyond traditional threshold theorems (see, e.g. Jackson (2010), Section 4.2.2) for our learning and welfare benchmarks. First, we solve for the exact thresholds within the constant, logarithmic, and proportional ranges, e.g. $\lambda \leq 1/\sigma^*$ for asymptotic learning. Second, we characterize learning and welfare for network densities where the benchmarks are not attained.

Figure 4 illustrates Theorem 1 for $p_0 < \bar{p}$.²⁴ The top and middle panels sketch asymptotic information B and welfare V as functions of network density. The bottom panel illustrates

networks $\lim(\log I / \log n^I)$. The smaller diameter reflects a faster contagion process: contagion in our model does not travel one link in every discrete time period; rather each link transmits continuously with rate one. Much like compound interest, this allows nodes infected at $t' \in [t, t + 1]$ to begin transmitting immediately, instead of having to wait until $t + 1$.

²³This order treats many sequences of networks as equally dense. For instance, $n^I = \log \log I$ or $n^I = (\log I)^{1/2}$ both correspond to $\nu = \infty, \lambda = 0$. Theorem 1 shows that asymptotic information B and welfare V of a sequence of networks $\{n^I\}$ only depends on its limit density (ν, λ, ρ) .

²⁴For $p_0 \geq \bar{p}$, experimentation incentives are higher and asymptotic learning obtains as long as $\rho = 0$.

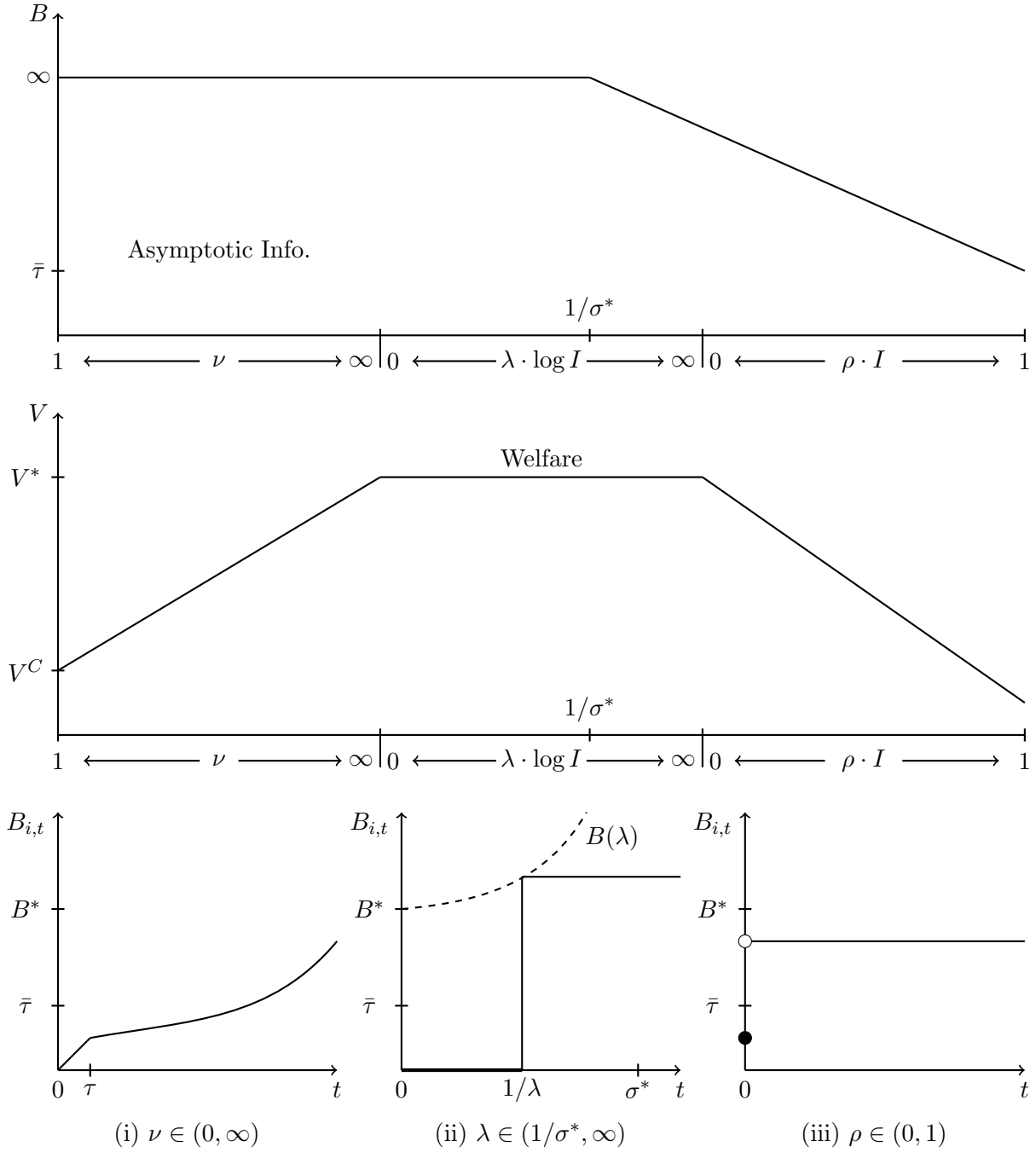


Figure 4: **Large Random Networks for Pessimistic Priors**, $p_0 \leq \bar{p}$. The **top** panel shows asymptotic information B as a function of network density, as described in Theorem 2(a). The **middle** panel shows welfare V as a function of network density, as described in Theorem 2(b). The **bottom** panel shows the cumulative learning curves of a typical agent in three canonical cases, as discussed in the text. In particular, σ^* is defined by (9), $B(\lambda)$ by (11), and B^* by (12).

the underlying cumulative social learning curves $\{B_t\}$ for our three regions of network density.²⁵ We discuss Figure 4 in order of increasing network density.

We begin with sparse networks, $\nu < \infty$. These networks approximate trees, with independent information across Iris’s neighbors. Social learning in the contagion phase $\{B_t\}_{t \geq \tau}$, illustrated in Figure 4(i), is convex with rate b_t described by a first-order ODE (see Section 5). This convexity reflects the fact that an agent has ν first-degree neighbors, $\nu(\nu-1)$ second-degree neighbors, $\nu(\nu-1)^2$ third-degree neighbors etc., so contagion accelerates over time. Each agent experiments for a bounded time $\tau > 0$, which ensures asymptotic learning,²⁶ while welfare falls short of the benchmark, $\mathcal{V}(\tau, \nu\tau) < \mathcal{V}(0, 0)$.

Now consider intermediate or dense networks, $\nu = \infty$. As illustrated in Figure 4(ii) and (iii), the social learning curve $\{B_t\}$ is a step-function with a single step at time σ . That is, agents observe the first success at time σ , or never. In analogy to epidemiological contagion processes, we also say that agents get “exposed” at σ . For intermediate networks, we have $\sigma = \sigma(\lambda) > 0$; for dense networks, we have $\sigma = 0$.

To state the underlying result, consider any sequence of cutoffs $\{\tau^I\}$ (not necessarily equilibrium) with limit $\sigma := \lim_{I \rightarrow \infty} \frac{-\log \tau^I}{n^I} \in [0, \infty]$. Let S^I be the random time at which agent i gets exposed, and S the binary random time with $\Pr(S = \sigma) = 1 - e^{-\lim_{I \rightarrow \infty} I\tau^I}$ and $\Pr(S = \infty) = e^{-\lim_{I \rightarrow \infty} I\tau^I}$.²⁷

Lemma 4. *Assume $\nu = \infty$. As $I \rightarrow \infty$, i gets exposed at time σ or never, $S^I \xrightarrow{D} S$, and learns all generated information, $\lim_{I \rightarrow \infty} I\tau^I = B$.*

Proof. The full proof is in Appendix B.3. For an intuition, suppose Iris’s neighbors are a negligible share of the population, $\rho = 0$. At τ^I , the chance at least one agent has succeeded is $1 - e^{-I\tau^I}$, and there are approximately $n^I I\tau^I$ exposed agents. The contagion then grows geometrically at rate n^I , so there are approximately $n^I I\tau^I e^{n^I t}$ exposed agents at time t and, heuristically, everyone is exposed when $n^I I\tau^I e^{n^I t} = I$, or $t = \frac{-\log(n^I I\tau^I)}{n^I} \rightarrow \sigma$.²⁸ This argument slightly overstates exposures because of double-counting. But this problem scales with the share of exposed agents and we only need the argument as long as this share is negligible; once a fixed share of the population is exposed, all agents are exposed immediately since $n^I \rightarrow \infty$. The proof uses Chernoff bounds to make these arguments rigorous. \square

²⁵The more intuitive rates of social learning $\{b_t\}$ in Figure 2 fail to exist for $\nu = \infty$.

²⁶Recall that an agent eventually learns the successes of all agents in her component. Given $\nu \geq 2$, the component of a typical agent has size proportional to I almost surely.

²⁷Asymptotically the distinction between $\min\{T^I, S^I\}$ and S^I vanishes. If $\tau^I \rightarrow 0$ (as will be the case in equilibrium), i ’s experimentation is certain to fail; if $\lim_{I \rightarrow \infty} \tau^I > 0$, social learning is asymptotically immediate and perfect, $\sigma = 0$ and $\lim_{I \rightarrow \infty} I\tau^I = \infty$, so $S = 0$.

²⁸Recalling footnote 22, here we see the difference between typical discrete-time contagion models where exposed agents grow like $e^{(\log n^I)t}$ and our continuous-time model with the faster rate $e^{n^I t}$.

To sharpen Lemma 4, note that the time-diameter upper-bounds the exposure time

$$\sigma = \lim \frac{-\log \tau^I}{n^I} = \lim \frac{\log I - \log I\tau^I}{n^I} = \frac{1}{\lambda} - \lim \frac{\log I\tau^I}{n^I}. \quad (10)$$

With finite aggregate information, $B = \lim I\tau^I < \infty$, they coincide $\sigma = 1/\lambda$; but with $B = \infty$, a diverging number of agents succeed during experimentation, so exposure can happen earlier, $\sigma \leq 1/\lambda$.

With this characterization of social learning curves for any cutoffs τ^I , we now return to the question of equilibrium. For intermediate networks, the proof of Theorem 1 shows that pre-cutoff learning must vanish, $(n^I + 1)\tau^I \rightarrow 0$.²⁹ Welfare thus converges to the second-best benchmark $\mathcal{V}(\tau^I, n^I\tau^I) \rightarrow \mathcal{V}(0, 0) = V^*$.

Using (4), the indifference condition at $t = 0$ when learning $B(\lambda)$ at time $\sigma(\lambda)$ becomes

$$p_0(x + (1 - e^{-r\sigma(\lambda)})(1 - e^{-B(\lambda)})y) - c = 0. \quad (11)$$

The dashed line in Figure 4(ii) illustrates the solution $(\sigma(\lambda), B(\lambda))$ of (11). As $\sigma(\lambda)$ rises over $[0, \sigma^*]$, the corresponding $B(\lambda)$ rises from $B^* = B(\infty)$ defined by

$$p_0(x + e^{-B^*}y) = c, \quad (12)$$

to $B(\lambda) = \infty$ at $\lambda = 1/\sigma^*$, as captured by (9). Intuitively, the lower incentives due to later learning are compensated by greater information $B(\lambda)$ in order to maintain indifference.

For low-density intermediate networks with $\lambda \in (0, 1/\sigma^*)$, an exposure time equal to the time-diameter $\sigma = 1/\lambda > \sigma^*$ renders experimentation incentives (11) positive for any $B(\lambda)$. That is, when the delay exceeds σ^* , no amount of information can fully crowd out own experimentation. Equilibrium must therefore feature $\sigma(\lambda) = \sigma^* < 1/\lambda$ implying infinite information, $B(\lambda) = \infty$, by (10), so (11) becomes (9). Thus, in this range, the exposure time and information are independent of network density.

For high-density intermediate networks $\lambda \in (1/\sigma^*, \infty)$, perfect learning $B(\lambda) = \infty$ renders incentives (11) negative for any $\sigma(\lambda) \leq 1/\lambda < \sigma^*$. That is, learning is so fast that perfect information would choke off experimentation entirely. Equilibrium must thus feature finite information, $B(\lambda) < \infty$, implying $\sigma(\lambda) = 1/\lambda$ by (10). The resulting $(\sigma(\lambda), B(\lambda)) = (1/\lambda, B(\lambda))$ are described by initial indifference (11) and illustrated in Figure 4(ii).

Finally, consider dense networks, where agents are connected to a fixed proportion $\rho \in (0, 1)$ of others. Learning is immediate as seen in Figure 4(iii). Such networks are analogous

²⁹Intuitively, with $\nu = \infty$ and $\rho = 0$ the ratio of second neighbors to first neighbors diverges, so non-trivial pre-cutoff learning implies immediate and perfect post-cutoff learning, which is inconsistent with equilibrium when $p_0 < \bar{p}$.

to the clique. With total information B , agents learn ρB before stopping and $(1 - \rho)B$ immediately after stopping. The indifference condition

$$P^\emptyset(\rho B) (x + e^{-(1-\rho)B}y) = c$$

then determines total information B . As $\rho \rightarrow 1$, we approach the clique and $B \rightarrow \bar{\tau}$.

Theorem 1 is stated for regular, undirected networks. The analysis immediately extends to regular directed and triangular networks (as described in Section 5). Networks with non-degenerate degree distributions introduce an alternative possibility for asymptotic learning to fail: An agent may be isolated, or more generally the size of her limit component may be finite. This arises with positive probability in Erdos-Renyi networks with bounded expected degree n^I ; asymptotic learning then requires intermediate network density with $\nu = \infty$ and $\lambda \leq 1/\sigma^*$.³⁰ More remarkably, the next section derives a result analogous to Theorem 1 for core-periphery networks.

4.3 Core-Periphery Networks

In this section we study core-periphery networks. Theorem 2 shows that asymptotic information falls with network density while welfare is single-peaked, echoing Theorem 1 for random networks. This analysis serves three purposes. First, core-periphery networks are of intrinsic interest: They are used to describe financial markets (e.g. Li and Schürhoff (2019)) and can arise endogenously in network formation models (Galeotti and Goyal (2010)). Second, core-periphery networks allow us to examine the role of network position for information generation. Third, core-periphery networks have a different neighborhood structure, with relatively few first neighbors in the core slowly transmitting the information generated by the more numerous peripherals. As a result, social learning curves are then concave rather than convex in the contagion phase.

A *core-periphery network* is an undirected, deterministic network that consists of K core agents and $L = I - K$ peripheral agents. The core agents k are connected to everyone. The peripheral agents ℓ are only connected to core agents. See Figure 5 for an illustration. When $K = 1$, we have the star from Example 3.

Lemma 5. *Any equilibrium in a core-periphery network is characterized by two cutoffs, τ_k for all agents in the core, and τ_ℓ for all peripherals. Core agents work less, $\tau_k < \tau_\ell$, and have*

³⁰Networks with non-degenerate degree distributions challenge our assumption that agents do not observe their own degree. The role of this assumption is to guarantee symmetry and the characterization of the unique equilibrium by means of a single cutoff τ by Corollary 1. If instead agents observe their degree, high-degree agents experiment less and equilibrium is characterized by a multi-dimensional fixed point, which may undermine sharp comparative statics.

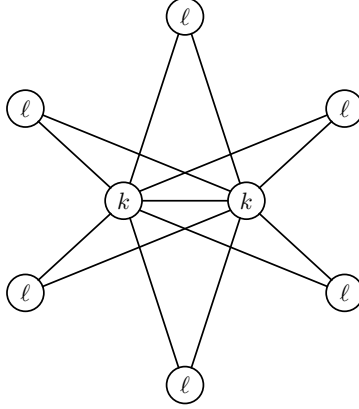


Figure 5: **Core-Periphery Network** with $K = 2$ core agents, and $L = 6$ peripherals.

higher values, $V_k > V_\ell$.

Proof. By symmetry and Proposition 3, equilibrium is characterized by cutoffs (τ_k, τ_ℓ) . Core agents k observe all successes immediately, and so have greater total information than peripherals who observe some successes with delay, $B_{k,t} + \min\{t, \tau_k\} > B_{\ell,t} + \min\{t, \tau_\ell\}$ for all $t > 0$. Lemma 7 in Appendix A.3 implies $\tau_k < \tau_\ell$.³¹ Since peripherals experiment more, core agents have greater social learning, $B_{k,t} > B_{\ell,t}$ for all $t > 0$, and so $V_k > V_\ell$ by Lemma 1. \square

We now characterize equilibrium cutoffs. Core agents k observe all successes immediately, so their social learning follows $b_{k,t} \equiv (K - 1)\mathbb{I}_{\{t \leq \tau_k\}} + L\mathbb{I}_{\{t \leq \tau_\ell\}}$. Experimentation incentives (4) are given by

$$\psi_{k,\tau_k} = P^\emptyset(I\tau_k) \left(x + y \left(1 - (1 - e^{-(r+L)(\tau_\ell - \tau_k)}) \frac{L}{r + L} \right) \right) - c \quad (13)$$

where the opportunity cost is the continuation value from having L peripherals experiment over $[\tau_k, \tau_\ell]$. In equilibrium, $\psi_{k,\tau_k} \leq 0$ with equality if $\tau_k > 0$.

Peripheral agents ℓ only observe the successes of core agents, so their social learning $b_{\ell,t}$ equals K before τ_k and then drops to Ka_t where $a_t := \Pr^H(T_{\ell'} < t \text{ for at least one } \ell' \neq \ell | t < T_\ell, t < T_k \text{ for all } k)$ is the conditional probability that some other peripheral agent has

³¹There is a subtlety here. Lemma 1 tells us that more social learning leads to less experimentation, but this is insufficient to conclude that core agents experiment less. For example, consider the star network and assume peripherals do not experiment; the core agent then has no social information but the same amount of total information as peripherals. Lemma 7 adapts the arguments from Lemma 1 to show that greater total learning (including self-learning) implies less experimentation.

succeeded by t and hence the core agents are working. This follows

$$\frac{\dot{a}}{1-a} = (L-1)\mathbb{I}_{\{t \leq \tau_\ell\}} - Ka = \begin{cases} L-1-Ka & t \in (\tau_k, \tau_\ell) \\ -Ka & t > \tau_\ell \end{cases} \quad (14)$$

with boundary condition $a_{\tau_k} = 1 - e^{-(L-1)\tau_k}$, as shown in Appendix B.4. Before τ_ℓ , social learning a_t rises because of experimentation by the other $L-1$ peripherals, tempered by the lack of success by the K core agents. After τ_ℓ , only the latter effect remains, so learning $b_{\ell,t} = Ka_t$ slows down. Using equation (4), peripherals' cutoff $\tau_\ell > 0$ then solves

$$\psi_{\ell, \tau_\ell} = P^\emptyset \left(K \left(\tau_k + \int_{\tau_k}^{\tau_\ell} a_t dt \right) + \tau_\ell \right) \left(x + ry \int_{\tau_\ell}^{\infty} e^{-\int_{\tau_\ell}^t (r+Ka_s) ds} dt \right) - c = 0.$$

For fixed $I < \infty$, we do not know whether the equilibrium cutoffs (τ_k, τ_ℓ) are unique.

In order to cleanly characterize how social information and welfare depend on the network density, we consider sequences of core-periphery networks which we index by $I \in \mathbb{N}$. Each network is determined by its core size K^I ; the number of peripherals is then $L^I = I - K^I$. We assume the following two limits exist. Define $\kappa := \lim K^I \in \mathbb{N} \cup \{\infty\}$ as the limit of the absolute core size, and $\rho := \lim K^I/I \in [0, 1]$ as the limit of the relative core size, as a proportion of the population. The set of network densities is the union $\{\kappa | \kappa \in \mathbb{N}\} \cup \{\rho \cdot I | \rho \in [0, 1]\}$ endowed with its natural order.

We now define a threshold on core size that is critical for both asymptotic learning and welfare. For pessimistic priors $p_0 < \bar{p}$, define $\kappa^* \in (0, \infty)$ such that learning from κ^* core agents who experiment forever, $b_{\ell,t} \equiv \kappa^*$, renders a peripheral agent indifferent about experimenting at $t = 0$,

$$\psi_{\ell,0} = p_0 \left(x + \frac{r}{r + \kappa^*} y \right) - c = 0. \quad (15)$$

For optimistic priors, $p_0 \geq \bar{p}$, set $\kappa^* = \infty$.

Theorem 2. *In core-periphery networks $\{K^I\}$ and any equilibrium selection $\{\tau_k^I, \tau_\ell^I\}$:*

(a) *Asymptotic information B is a decreasing function of network density: It attains asymptotic learning $B = \infty$ iff $\kappa \leq \kappa^*$ and $\rho = 0$, and strictly falls when $\kappa \geq \kappa^*$.*

(b) *Welfare V is a single-peaked function of network density: It strictly rises when $\kappa \leq \kappa^*$, it attains the benchmark V^* iff $\kappa \in [\kappa^*, \infty]$ and $\rho = 0$, and strictly falls when $\rho > 0$.*

Proof. See Appendix B.5. □

Asymptotic learning is achieved for sufficiently small core size; welfare attains the benchmark for intermediate core size. Figure 6 illustrates Theorem 2 for $p_0 < \bar{p}$. The top and

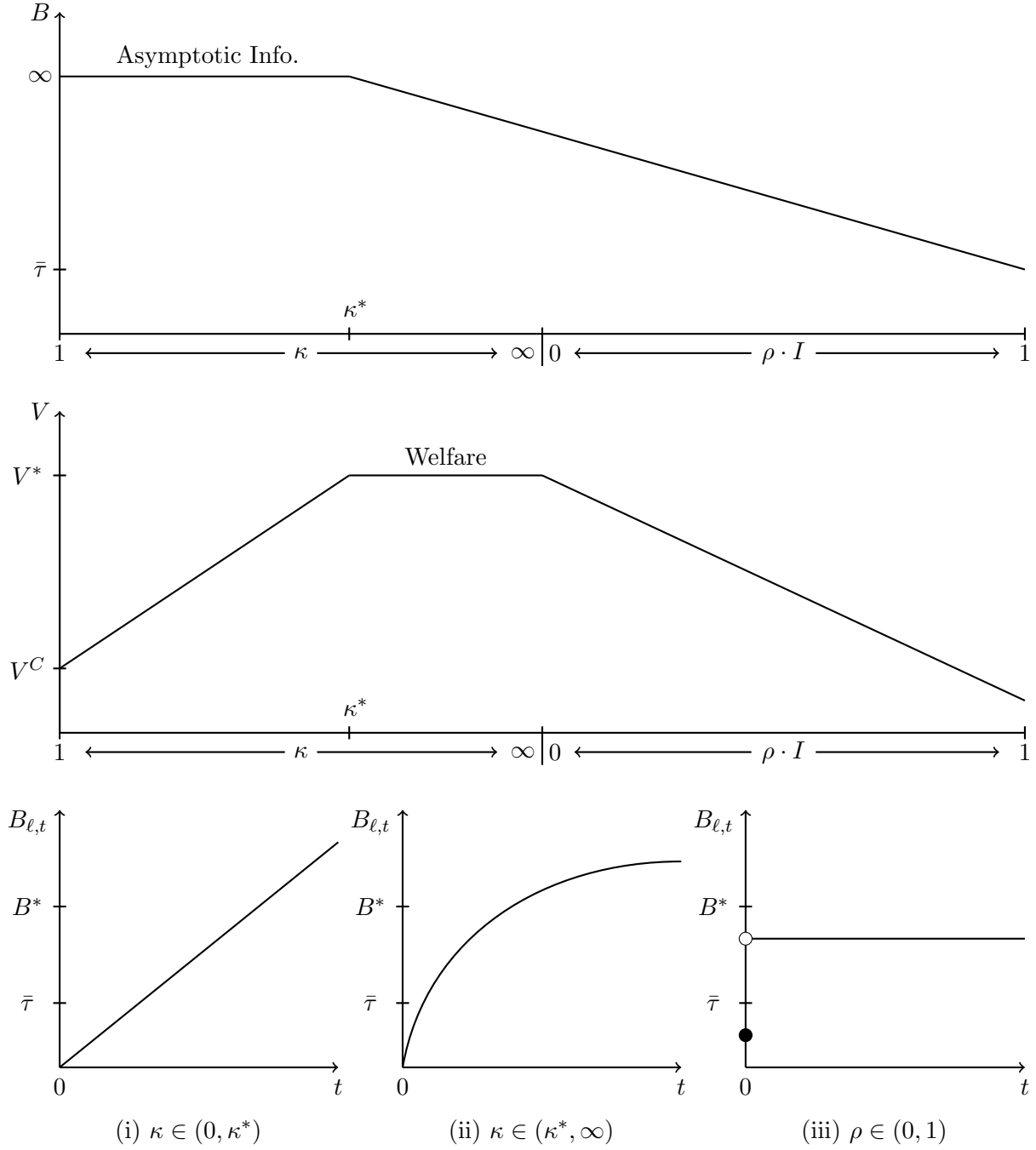


Figure 6: **Core-Periphery for Pessimistic Priors**, $p_0 \leq \bar{p}$. The **top** panel shows asymptotic information B as a function of network density, as described in Theorem 2(a). The **middle** panel shows welfare V as a function of network density, as described in Theorem 2(b). The **bottom** panel shows the learning curves of a peripheral agent in three regions of network density, as discussed in the text. In particular, B^* is defined by (12) and κ^* is defined by (15).

middle panels sketch asymptotic information B and welfare V as functions of core size. The bottom panel illustrates three typical social learning curves $\{B_{\ell,t}\}$. While asymptotic learning and second-best welfare may *a priori* seem to be related goals, Theorem 2 shows that for pessimistic priors they are generically mutually exclusive. Asymptotic learning requires a small core size $\kappa \leq \kappa^*$, while second-best welfare requires a large core size $\kappa \geq \kappa^*$.³²

As with random networks, there are three regions of network density with qualitatively different social learning dynamics. First, consider a small core $\kappa < \kappa^*$, as illustrated in Figure 6(i). The exploding number of peripherals experiment for a bounded time interval, $\tau_\ell > 0$, and collectively create an exploding amount of information in an instant. This crowds out experimentation by core agents. Peripherals choose to experiment since the flow of social information is restricted by the small core size. Formally, $B_{\ell,t} = Kt$ so equation (15) implies $\psi_{\ell,0} > 0$ given than $\kappa < \kappa^*$. Eventually the network aggregates information, but since each peripheral generates a non-vanishing amount of information, their utility falls short of the benchmark $\mathcal{V}(\tau_\ell, \kappa\tau_\ell) < V^*$.

Second, consider an intermediate core $\kappa \in (\kappa^*, \infty)$, as illustrated in Figure 6(ii).³³ With this core size, perfect information from peripherals would crowd out peripherals' experimentation incentives. In equilibrium, peripheral agents lower their cutoffs, limiting their total information $B = \lim L^I \tau_\ell^I < \infty$. The level of B is determined by peripheral agents' indifference condition at $t = 0$,

$$\psi_{\ell,0} = p_0 \left(x + ry \int_0^\infty e^{-\int_0^t (r+b_{\ell,s})ds} dt \right) - c = 0$$

where ℓ 's social learning curve satisfies

$$1 - e^{-B_{\ell,t}} = (1 - e^{-B})(1 - e^{-\kappa t}).$$

Intuitively, ℓ learns the state if some peripheral learned it and a core agent succeeds. As in the star, $b_{\ell,t}$ falls over time as agents grow pessimistic about the chance that one of them succeeded. Asymptotic learning fails, but agents do obtain the welfare benchmark, $\mathcal{V}(0,0)$, as pre-cutoff learning $(\kappa+1)\tau_\ell^I$ vanishes. For large κ , the core transmits information increasingly fast, reinforcing the crowding out and reducing asymptotic information. When $\kappa = \infty$ but $\rho = 0$, $B = B^*$ solves (12), so $B_{\ell,t}$ jumps to B^* and remains constant thereafter.

Third, consider a large core $\rho \in (0,1]$, as illustrated by Figure 6(iii). Now core agents

³²The pessimistic prior assumption is important. For optimistic priors, $p_0 \geq \bar{p}$, it is easier to motivate agents to experiment. Our welfare benchmark requires asymptotic learning and both of these goals are obtained if $\kappa = \infty$ and $\rho = 0$. While this is a single point $0 \cdot I$ in our density order, there are many sequences that satisfy both conditions, e.g. $K^I = \log \log I$, $K^I = \log I$, $K^I = (n^I)^{1/2}$.

³³For optimistic priors, $p_0 \geq \bar{p}$, this region is empty.

generate a non-vanishing share of total information. Social learning is asymptotically immediate, $B_{\ell,t} = B_{k,t} = B$ for all $t > 0$, and core agents' indifference condition becomes

$$P^\emptyset(B_{k,\tau}) (x + ye^{-(B-B_{k,\tau})}) - c = 0$$

with pre-cutoff learning $B_{k,\tau} = \lim I\tau_k^I$. This equation together with the analogous, but more involved expression for peripherals' pre-cutoff learning $\lim B_{\ell,\tau_\ell}^I$, pin down asymptotic information B which falls in ρ , with $B = \bar{\tau}$ for $\rho = 1$ as in the clique.

These results are reassuringly parallel to the ones for random networks in Section 4.2. In both cases, asymptotic information decreases in density, while welfare is single-peaked. However, significant differences arise from the higher ratio of second neighbors to first neighbors. First, the contrast between asymptotic information and welfare is starker: With $p_0 < \bar{p}$, asymptotic learning and second-best welfare are mutually exclusive under core-periphery networks, yet overlap under random networks. Second, social learning slows down over time in core-periphery networks with a finite core, as the information trickles through the core; in contrast, social learning speeds up over time in random networks, as the number of indirect neighbors grows exponentially with path length.

5 Types of Links

In Section 4 we examined the effect of the density of links on experimentation and welfare; we now consider the effect of the type of links. We study this question in the context of regular trees, as illustrated in Figure 7. In a directed tree $\vec{\mathcal{T}}^{(n)}$, there is at most one directed path between any two agents; this resembles users following each other on Twitter. In an undirected tree $\bar{\mathcal{T}}^{(n)}$, there is at most one undirected path between any two agents; this resembles the connections between acquaintances on LinkedIn. And in a triangle tree $\hat{\mathcal{T}}^{(n)}$, agents are connected in triangles; this resembles clusters of friends on Facebook. Trees approximate large random networks; they are tractable because of the independence across neighbors. E.g. in a triangle tree, Iris's neighbors j and k have independent information given Iris has not observed a success.

Proposition 4 shows that the large random networks with finite degree from Section 4.2 converge to trees studied here. So motivated, Theorem 3 shows that equilibrium value in a directed tree with n neighbors is greater than in an undirected tree with n neighbors, but less than an undirected tree with $n + 1$ neighbors. A similar comparison applies between undirected trees and triangle trees. For large n , these tight bounds imply that the type of the link is of minor importance for agents' behavior and utility.

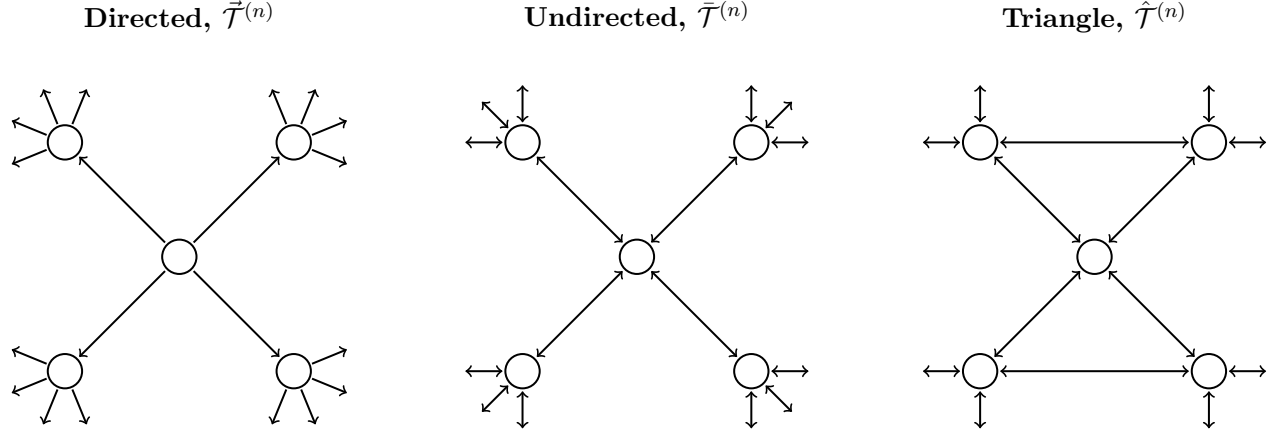


Figure 7: **Regular Trees** with degree $n = 4$.

5.1 Trees

We first consider *directed trees* with degree n , $\vec{\mathcal{T}}^{(n)}$. Assuming symmetric cutoffs τ , let $a_t = E^H[A_{j,t}|t < S_i]$ be Iris's expectation of neighbor John's effort in the contagion phase, $t \geq \tau$, in the absence of a success. Extending equation (7) in Example 2,

$$1 - a_t = \Pr^H(t < T_k, \forall k \in N_j | t < T_j) = \frac{\Pr^H(t < T_j, T_k, \forall k \in N_j)}{\Pr^H(t < T_j)} = \frac{\exp\left(-(n+1)\tau - n \int_\tau^t a_s ds\right)}{\exp\left(-\tau - \int_\tau^t a_s ds\right)}. \quad (16)$$

For the denominator, the hazard rate of John's first success time T_j equals 1 in the experimentation phase $t \leq \tau$ and a_t in the contagion phase $t > \tau$. For the numerator, the hazard rate of the success time $\min\{T_j, T_k, T_{k'}, \dots\}$ equals $n+1$ in the experimentation phase $t \leq \tau$; in the contagion phase $t > \tau$, the lack of success by $\{T_k, T_{k'}, \dots\}$ implies $A_{j,t} = 0$, so the hazard rate drops to a_t for each $k \in N_j$. Differentiating, Iris's belief follows the ODE

$$\dot{a}_t = (n-1)a(1-a) \quad (17)$$

with initial condition $a_\tau = 1 - e^{-n\tau}$ given by the probability that one of John's n neighbors succeeded in the experimentation phase.³⁴ In the case of the directed line in Example 2, $n = 1$, Iris's belief $a_t = 1 - e^{-n\tau}$ is constant over time. If $n \geq 2$, Iris's belief rises over time because the good news from John's expected inflow of information outweighs the bad news from his observed lack of success. The net effect is captured by the factor $(n-1)$ in (17): The more neighbors John has, the faster he observes a success, and the faster Iris's rate of social learning increases.

³⁴Inverting, (17) admits the closed-form solution $a_t = 1/(1 + \exp(-(n-2)(t+c)))$ with constant c .

In order to study undirected and triangle trees we must address “backward” links, where agent i reasons about j who simultaneously learns from i ’s successes (or lack thereof).

Example 4 (Undirected Line). Consider the infinite undirected line

$$\dots \leftrightarrow i \leftrightarrow j \leftrightarrow k \leftrightarrow \dots$$

As in the directed line (Example 2), let a_t be i ’s expectation of j ’s effort at time $t > \tau$ conditional on not seeing a success; this coincides with i ’s expectation that k has succeeded:

$$a_t := E^H [A_{j,t} | t < T_i, S_i] = 1 - \Pr^H (t < T_k | t < T_i, T_j). \quad (18)$$

Calculating this conditional expectation is more subtle than in the directed line, where we could simply drop the “ $t < T_i$ ” term. Now that j also observes i but has not seen i succeed, it is useful to introduce the expectation E^{-i} over others’ success times $\{T_j\}_{j \neq i}$, given a symmetric cutoff τ , no successes of i , and $\theta = H$. Then, $a_t = E^{-i} [A_{j,t} | t < T_j]$ and

$$\Pr^H (t < T_k | t < T_i, T_j) = \Pr^{-i} (t < T_k | t < T_j) = \frac{\Pr^{-i} (t < T_j, T_k)}{\Pr^{-i} (t < T_j)} = \frac{\exp \left(-2\tau - \int_{\tau}^t a_s ds \right)}{\exp \left(-\tau - \int_{\tau}^t a_s ds \right)} = e^{-\tau}. \quad (19)$$

For the denominator, the hazard rate of John’s first success time T_j equals 1 in the experimentation phase $t \leq \tau$ and a_t in the contagion phase $t > \tau$, under expectation E^{-i} . For the numerator, the hazard rate of the success time $\min\{T_j, T_k\}$ equals 2 in the experimentation phase $t \leq \tau$; in the contagion phase $t > \tau$, the lack of success by i, j, k implies $A_{j,t} = 0$, so the hazard rate drops to $E^H [A_{k,t} | t < T_j, S_j] = a_t$. Substituting back into (18) yields $a_t = 1 - e^{-\tau}$, just like in the directed line. Thus, the additional backward link does not affect the information i learns from j . Intuitively, while j has one more link, he is no more informed once we condition on i not having succeeded. That is, j ’s link to i does not help i since she cannot learn from herself.

In contrast to the directed line, i now has two neighbors, so using (4), the equilibrium stopping time τ solves

$$\psi_{\tau} = P^{\emptyset}(3\tau) \left(x + \frac{r}{r + 2(1 - e^{-\tau})} y \right) - c = 0.$$

Comparing this to (8), agents experiment less in the undirected line, which has more sources of social information and hence greater crowding out. \triangle

We now generalize this example and consider an *undirected tree* $\bar{\mathcal{T}}^{(n)}$ in which everyone

has n neighbors. Adapting (16) and (19), Iris's belief of John's effort follows the ODE

$$\dot{a} = (n - 2)a(1 - a) \quad (20)$$

with initial condition $a_\tau = 1 - e^{-(n-1)\tau}$. For the undirected line, $n = 2$, from Example 4 we recover the time-invariant beliefs $a_t = 1 - e^{-\tau}$. When $n \geq 3$, Iris's belief increases over time. Compared to the directed tree, undirected links lower both the initial condition and the rate of increase of John's expected effort by one degree.

Finally, consider a triangle tree, $\hat{\mathcal{T}}^{(n)}$. Adapting (16), Iris's belief of John's effort follows

$$\dot{a} = (n - 3)a(1 - a) \quad (21)$$

with initial condition $a_\tau = 1 - e^{-(n-2)\tau}$. When compared to the undirected tree, this lowers both the initial condition and the rate of increase by an additional neighbor. Intuitively, i knows the triangle links $j \rightarrow i, k$ do not provide information for j , because she conditions on not having seen a success by i or k .

To see how this difference in social learning feeds back into the equilibrium cutoff τ , write experimentation incentives (4) as a function of social learning

$$\psi_\tau(\{na_t\}) = P^\emptyset((n+1)\tau) \left(x + ry \int_{s=\tau}^{\infty} e^{-\int_{t=\tau}^s (r+na_t)dt} ds \right) - c = 0. \quad (22)$$

Substituting the solutions of the ODEs (17), (20) and (21) for $\{a_t\}_{t \geq \tau}$ yields unique equilibrium cutoffs $\vec{\tau}, \bar{\tau}, \hat{\tau}$ with associated social learning curves $\{\vec{a}_t\}, \{\bar{a}_t\}, \{\hat{a}_t\}$.

Theorem 3. *Equilibrium cutoff times for regular trees are ranked as follows:*

$$\hat{\tau}^{(n+2)} < \bar{\tau}^{(n+1)} < \vec{\tau}^{(n)} < \bar{\tau}^{(n)} < \hat{\tau}^{(n)}.$$

Equilibrium values are ranked in the opposite way:

$$\hat{V}^{(n+2)} > \bar{V}^{(n+1)} > \vec{V}^{(n)} > \bar{V}^{(n)} > \hat{V}^{(n)}.$$

Proof. See Appendix C.1. □

This result provides a tight relationship between the value of different network structures and the value of extra neighbors. Intuitively, for fixed τ , the directed network $\vec{\mathcal{T}}^{(n)}$ has the same number of neighbors as the undirected network $\bar{\mathcal{T}}^{(n)}$, but more social information per neighbor since the neighbor's backward link is wasted. This extra social information provides value and crowds out experimentation. Conversely, the undirected network $\bar{\mathcal{T}}^{(n+1)}$ has the same social information per neighbor as the directed network $\vec{\mathcal{T}}^{(n)}$ but more neighbors.

Again, this extra social information provides value and crowds out the agent’s effort.

This result is important for two reasons. First, it provides comparative statics across canonical networks in terms of experimentation, social learning, and welfare. In contrast, the rest of the literature typically focuses on long run considerations (e.g. whether the agents reach consensus or aggregate information). Second, it allows us to quantitatively assess the importance of network structure. This matters for network design (e.g. reducing clustering is useful, but adding connections is even better). It can also be useful for empirical work since we need not worry about exactly specifying the network (at least, within a class) if agents have many neighbors; this allows us to employ well understood models such as dyadic link formation networks (e.g. Graham, 2017).

We end the section by showing that trees are indeed the limit of random networks as $I \rightarrow \infty$. The construction of the sequences of regular, random networks follows Section 4.2. For directed networks, suppose each agent has n stubs that we randomly connect to n distinct other agents; define the equilibrium cutoff $\bar{\tau}^I$, which is unique by Proposition 3, and social learning $\{\bar{b}_t^I\}_t$. For undirected networks, each agent has n stubs that we randomly connect in pairs into undirected links; define the equilibrium cutoff $\bar{\tau}^I$ and social learning $\{\bar{b}_t^I\}_t$. For triangle networks each agent has $n/2$ stub pairs that we randomly connect in triples of stub pairs into triangles; define the equilibrium cutoff $\hat{\tau}^I$ and social learning $\{\hat{b}_t^I\}_t$.

To see how these random networks approximate trees as $I \rightarrow \infty$, fix a “root agent” i and “uncover the network” from this root. That is, first randomly connect i ’s link stubs, then the stubs of i ’s n neighbors, and so on. For fixed n and $I \rightarrow \infty$, i ’s “local network” is almost surely a tree. The next result formalizes the idea that i ’s social learning only depends on her local network, and thus the network equilibria converge to the heuristic tree equilibria. The comparisons in Theorem 3 thus apply equally to large random networks.

Proposition 4. *The equilibria of the large random networks with degree n converge to the equilibria of the respective infinite regular n -trees:*

- (a) *Directed networks:* $\bar{\tau}^I \rightarrow \bar{\tau}$ and $\bar{b}_t^I \rightarrow n\bar{a}_t$ for all $t \neq \bar{\tau}$.
- (b) *Undirected networks:* $\bar{\tau}^I \rightarrow \bar{\tau}$ and $\bar{b}_t^I \rightarrow n\bar{a}_t$ for all $t \neq \bar{\tau}$.
- (c) *Triangle networks:* $\hat{\tau}^I \rightarrow \hat{\tau}$ and $\hat{b}_t^I \rightarrow n\hat{a}_t$ for all $t \neq \hat{\tau}$.

Proof. See Appendix C.2. □

6 Conclusion

This paper studies a simple model of experimentation in networks. We characterize individual experimentation, social learning curves, asymptotic information, and welfare in large random networks and core-periphery networks. We show that asymptotic information falls in network density, but welfare is single-peaked. We also compare directed, undirected and clustered links in trees. Relative to the literature, we go beyond long-term outcomes by describing learning dynamics and welfare, and perform comparative statics across networks.

Our main result considers two canonical classes of networks, which admit natural density measures and give rise to unique limit behavior. But the economic forces underlying our analysis transcend these two classes. For example, the general conflict between learning and welfare for pessimistic priors $p_0 < \bar{p}$, is apparent from the welfare benchmark, $V^* = \mathcal{V}(0, 0)$, which requires individual experimentation to vanish, undermining asymptotic learning. The details of how these forces play out does depend on the structure of the network. For example, in core-periphery networks, asymptotic learning and second-best welfare are generically incompatible, but in random networks the larger diameter allows them to coexist for a range of intermediate network densities.

This paper focuses on the role of networks in facilitating social learning. One can also use the model to study the impact of communication more directly, by assuming that agents observe each other imperfectly. The simplest such model has two agents, Iris and John, linked (undirectedly) with probability γ . Thus Iris and John do not know if they are experimenting by themselves, or if the other is working on the same problem. Over time, failure to observe a success makes Iris pessimistic about her chance of being linked to John, so social learning $b_{i,t} = E^H[j \in N_i(G) | t < T_i, S_i] = \frac{\gamma e^{-2t}}{\gamma e^{-2t} + (1-\gamma)e^{-t}}$ falls over $[0, \tau]$ before dropping to 0 forever. An increase in γ raises social learning $\{B_t\}$ for fixed τ , so lowers the equilibrium cutoff τ , and hence raises welfare $\mathcal{V}(\tau, \bar{\tau} - \tau)$.

Another possibility is that $I \rightarrow \infty$ agents in the clique network observe others' successes with a fixed delay $\sigma > 0$. Fix $p_0 < \bar{p}$ and define σ^* as in equation (9). When $\sigma > \sigma^*$, initial experimentation incentives are positive, so $\tau := \lim \tau^I > 0$ solves $P^\emptyset(\tau)(x + (1 - e^{-r(\sigma-\tau)})y) = c$. Agents learn perfectly at σ , $B = \infty$, but welfare is below second-best $\mathcal{V}(\tau, 0) < V^*$. Conversely, when $\sigma < \sigma^*$, perfect learning at σ would eliminate experimentation incentives for finite I , so total information $B < \infty$ solves (11). Since $\tau^I \approx B/I \rightarrow 0$, welfare is second-best $\mathcal{V}(0, 0) = V^*$. As in core-periphery networks, perfect learning and the welfare benchmark are generically incompatible. In contrast, in large random networks the learning time σ is endogenous, and equals σ^* for a wide range of intermediate network densities.

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A Appendix: Proofs from Section 3

A.1 Proof of Proposition 1 (Cutoff strategies)

To formalize the discussion surrounding the statement of Proposition 1, we write Iris's payoff from an arbitrary experimentation strategy as $\Pi = \Pi(\{a_s\}, \{b_s\})$. We will show that front-loading incentives are positive, equal to

$$-\frac{d}{dt} \frac{\partial \Pi}{\partial a_t} = (r(p_t(x+y) - c) + p_t b_t(x-c)) e^{-\int_0^t r+p_u(a_u+b_u)du} \quad (23)$$

The term $r(p_t(x+y) - c)$ is the time-value of front-loading own experimentation from $t+\delta$ to t , while $p_t b_t(x-c)$ captures the value of additional experimentation that arises when a neighbor succeeds in $[t, t+\delta]$; finally, the discount-term $e^{-\int_0^t r+p_u(a_u+b_u)du}$ reflects that (23) evaluates the effect of front-loading time- t effort from the time-0 perspective. Equation (23) implies that agents maximally front-load their effort, and so cutoff strategies are optimal.

To establish the second derivative (23), we first derive convenient expression for Π and its various first derivatives. Truncating (3) at time- t , we get

$$\Pi = \int_0^t (p_s(a_s(x+y) + b_s y) - a_s c) e^{-\int_0^s r+p_u(a_u+b_u)du} ds + \Pi_t e^{-\int_0^t r+p_u(a_u+b_u)du}. \quad (24)$$

We next establish two convenient expressions for the continuation payoff

$$\Pi_t = \int_t^\infty (p_s(a_s(x+y) + b_s y) - a_s c) e^{-\int_t^s r+p_u(a_u+b_u)du} ds \quad (25)$$

$$= \int_t^\infty e^{-r(s-t)} \left(p_t e^{-\int_t^s (a_u+b_u)du} (a_s(x+y-c) + b_s y) - (1-p_t) a_s c \right) ds \quad (26)$$

Equation (25) follows the same logic as (3): the discounted chance of no success on $[t, s]$ is $e^{-\int_t^s r+p_u(a_u+b_u)du}$; at time- s we condition on θ and realize expected flow benefits $a_s(x+y)+b_s y$ if $\theta = H$ and flow costs $a_s c$ for either θ . Equation (26) conditions on θ already at time- t . For $\theta = H$ no success arrives by s with probability $e^{-\int_t^s (a_u+b_u)du}$, and net flow benefits are $a_s(x+y-c) + b_s y$; for $\theta = L$, no success arrives ever, and Iris incurs flow costs $a_s c$.

Write $\alpha_t := \int_t^\infty e^{-r(s-t)} (a_s c) ds$ with time-derivative $\dot{\alpha}_t = r\alpha_t - a_t c$. By (26), $\Pi_t + \alpha_t$ is a linear function of the posterior belief p_t , and so

$$\frac{\partial \Pi_t}{\partial p_t} = \frac{1}{p_t} (\Pi_t + \alpha_t) \quad (27)$$

To compute $\partial \Pi / \partial a_t$, define the derivative of the posterior belief $p_t = P^\theta \left(\int_0^t (a_s + b_s) ds \right)$

with respect to “experimentation just before t ”,

$$\frac{\partial p_t(\{a_s\}_{s \geq 0})}{\partial a_t} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (p_t(\{a_s^{t,\epsilon}\}_{s \geq 0}) - p_t(\{a_s\}_{s \geq 0})) = -p_t(1 - p_t)$$

where $a_s^{t,\epsilon} := a_s + \mathbb{I}_{\{s \in [t-\epsilon, t]\}}$. Similarly differentiating payoff (24) wrt a_t and using (27),

$$\begin{aligned} \frac{\partial \Pi}{\partial a_t} &= \left(p_t(x + y) - c + \frac{\partial \Pi_t}{\partial p_t} \frac{\partial p_t}{\partial a_t} - p_t \Pi_t \right) e^{-\int_0^t r + p_u(a_u + b_u) du} \\ &= (p_t(x + y) - c - (1 - p_t)\alpha_t - \Pi_t) e^{-\int_0^t r + p_u(a_u + b_u) du} \end{aligned} \quad (28)$$

Turning to the time-derivatives, we first note $\dot{p}_t = -(a_t + b_t)p_t(1 - p_t)$, differentiate (25)

$$\dot{\Pi}_t = -(p_t(a_t(x + y) + b_t y - a_t c) + (r + p_t(a_t + b_t))\Pi_t).$$

and then (28) to get (23)

$$\begin{aligned} e^{\int_0^t r + p_u(a_u + b_u) du} \frac{d}{dt} \frac{\partial \Pi}{\partial a_t} &= -p_t(1 - p_t)(a_t + b_t)(x + y + \alpha_t) - (1 - p_t)(r\alpha_t - a_t c) \\ &\quad + (p_t(a_t(x + y) + b_t y) - a_t c) - (r + p_t(a_t + b_t))\Pi_t \\ &\quad - (r + p_t(a_t + b_t))(p_t(x + y) - c - (1 - p_t)\alpha_t - \Pi_t) \\ &= -r(p_t(x + y) - c) - p_t b_t(x - c). \end{aligned}$$

Having established that cutoff strategies are optimal, we now show that the optimal cutoff is the unique solution of $\psi_\tau = 0$. For cutoff strategies $a_s = \mathbb{I}_{\{s \leq t\}}$ we have $\alpha_t = \int_t^\infty e^{-r(s-t)} (a_s c) ds = 0$ and (26) simplifies to

$$\Pi_t = p_t y \int_t^\infty e^{-r(s-t)} b_s e^{-\int_t^s b_u du} ds = p_t y \left(1 - r \int_t^\infty e^{-\int_t^s (r + b_u) du} ds \right),$$

where the last equality uses integration by parts. Then (28) simplifies to

$$e^{\int_0^t r + p_u(a_u + b_u) du} \frac{\partial \Pi(\mathbb{I}_{\{s \leq t\}})}{\partial a_t} = p_t(x + y) - c - \Pi_t = p_t \left(x + r y \int_t^\infty e^{-\int_t^s (r + b_u) du} ds \right) - c = \psi_t. \quad (29)$$

Differentiating the LHS wrt t , we see that ψ_t strictly single-crosses from above since

$$\dot{\psi}_t = (r + p_t(a_t + b_t))\psi_t + e^{\int_0^t r + p_u(a_u + b_u) du} \frac{d}{dt} \frac{\partial \Pi(\mathbb{I}_{\{s \leq t\}})}{\partial a_t}$$

is negative whenever $\psi_t = 0$ by (23). □

For future reference we summarize some properties of

$$\psi_\tau(\{B_t\}) = P^\emptyset(\tau + B_\tau) \left(x + ry \int_t^\infty e^{-r(s-\tau)-(B_s-B_\tau)} ds \right) - c. \quad (30)$$

First, note that while (4) and (5) express ψ instead as a function of the social learning rate $\{b_t\}$, the definition and most properties of ψ extend to any increasing (not necessarily continuous or positive) cumulative social learning curve B_t .

Lemma 6. *Properties of $\psi_\tau(\{B_t\})$.*

- (a) Equation (30) falls in $\{B_t\}$, and thus also in $\{b_t\}$ with partial derivative given in (5).
- (b) Equation (30) strictly single-crosses from above in τ , and is equi-Lipschitz continuous in τ for all uniformly bounded $\{b_t\}$.
- (c) The root τ of $\psi_\tau = 0$ falls in $\{B_t\}$, and strictly falls in $\{b_t\}$.

A.2 Proof of Lemma 2 (Characterization of \mathcal{V})

We first derive (6)

$$\begin{aligned} V &= \left(p_0 \int_0^\tau e^{-\int_0^t (r+b_s+1)ds} (x + (b_t + 1)y - c) dt \right) - \left((1 - p_0) \int_0^\tau e^{-rt} c dt \right) + e^{-r\tau} (p_0 e^{-B_\tau - \tau} + (1 - p_0)) V_\tau \\ &= p_0 y (1 - e^{-B_\tau - (r+1)\tau}) - (1 - p_0) c \frac{1 - e^{-r\tau}}{r} + e^{-r\tau} (p_0 e^{-B_\tau - \tau} (x + y - c) - (1 - p_0) c) \\ &= \frac{p_0 x - c}{r} + e^{-r\tau} \left(p_0 e^{-B_\tau - \tau} (x - c) - (1 - p_0) c \frac{r - 1}{r} \right). \end{aligned}$$

The first line conditions on θ at time-0 and truncates flow payoffs (3) at $t = \tau$. The second line evaluates the first integral using $x - c = ry$, and the last term using $p_0 e^{-B_\tau - \tau} + (1 - p_0) = p_0 e^{-B_\tau - \tau} / p_\tau$ by Bayes' rule, and $V_\tau = p_\tau y \int_\tau^\infty b_t \exp^{-\int_t^\infty (r+b_s)ds} dt = p_\tau (x + y) - c$ (using $\psi_\tau = 0$). The last line uses $y = (x - c)/r$ and reorders terms.

The monotonicity in B_τ is immediate from (6). To see the monotonicity in τ , note that the first term in (6) is the payoff from experimenting forever. Thus, the second term is the option value of stopping earlier, which must be positive. Then

$$\begin{aligned} \partial_\tau \mathcal{V} &= -r e^{-r\tau} \left(p_0 e^{-B_\tau - \tau} (x - c) - (1 - p_0) c \frac{r - 1}{r} \right) - e^{-r\tau} p_0 e^{-B_\tau - \tau} (x - c) \\ &< -e^{-r\tau} p_0 e^{-B_\tau - \tau} (x - c) = \partial_B \mathcal{V} < 0. \end{aligned} \quad (31)$$

A.3 Proof of Proposition 3 (Equal Cutoffs of Equals)

We first establish two Lemmas that are of some independent interest and clarify the proof logic. For social learning $\{B_t\}$ and the associated optimal cutoff τ , define total learning

$B_t + \min\{t, \tau\}$. So defined, $\Pr^H(\min\{S, T\} \leq t) = 1 - \exp(-(B_t + \min\{t, \tau\}))$.

Lemma 7. *Higher total learning, $B_t + \min\{\tau, t\} \geq \hat{B}_t + \min\{\hat{\tau}, t\}$ for all t , is associated with lower cutoffs, $\tau \leq \hat{\tau}$.*

This is closely related to Lemma 1, that lower social learning $\{B_t\} \leq \{\hat{B}_t\}$ implies higher cutoffs $\tau \geq \hat{\tau}$. Lemma 7 shows additionally that the higher cutoff cannot lead to higher total learning. Intuitively, all learning (both social and own) crowds out incentives.

Lemma 8. *Fix a network G , cutoffs $\{\tau_k\}_{k \neq i, j}$ and $\tau_* < \tau^*$, and write k 's first success time as $\{T_k\}$ if $\tau_i = \tau^*, \tau_j = \tau_*$, and $\{T'_k\}$ if $\tau_i = \tau_*, \tau_j = \tau^*$. $\min\{T_i, S_i\} \stackrel{D}{\preceq} \min\{T'_i, S'_i\}$.³⁵*

Lemma 8 is intuitive: Additional experimentation during $[\tau_*, \tau^*]$ is more immediate and useful to i when done by i herself instead of j .

Proof of Proposition 3. By contradiction, assume $\tau_i > \tau_j$. Symmetry, $G_{i \leftrightarrow j} = G$, implies $\min\{T_j, S_j\} \stackrel{D}{=} \min\{T'_i, S'_i\}$. Lemma 8 then implies $\min\{T_i, S_i\} \stackrel{D}{\preceq} \min\{T_j, S_j\}$. Noting the connection between total learning and the time of the first observed success, $\Pr^H(\min\{S, T\} \leq t) = 1 - \exp(-(B_t + \min\{\tau, t\}))$, this implies $\{B_{i,t} + \min\{\tau_i, t\}\} \geq \{B_{j,t} + \min\{\tau_j, t\}\}$ and so, by Lemma 7, $\tau_i \leq \tau_j$. \square

Proof of Lemma 7. Lemmas 1 and 6 study incentives ψ_τ as a function of social learning $\{B_t\}$; we now study ψ_τ as a function of total learning $\{B_t + \min\{t, \tau\}\}$.

By contradiction assume that $B_t + \min\{\tau, t\} \geq \hat{B}_t + \min\{\hat{\tau}, t\}$ for all t , yet $\tau > \hat{\tau}$. Define $\tilde{B}_t := \hat{B}_t - (\tau - \hat{\tau})$; clearly $\tilde{B}_t \leq B_t$, and so

$$\psi_\tau(\{\tilde{B}_t\}) \geq \psi_\tau(\{B_t\}) = 0.$$

Since $\tilde{B}_\tau + \tau = \hat{B}_\tau + \hat{\tau}$ and $\tilde{b}_u = \hat{b}_u$ for $u \geq \tau$, time- τ experimentation incentives for the social learning curve $\{\hat{B}_t\}$ are also positive

$$e^{\int_0^\tau r + p_u(\hat{a}_u + \hat{b}_u) du} \frac{\partial \hat{\Pi}(\mathbb{I}_{\{t \leq \hat{\tau}\}})}{\partial a_\tau} = P^\emptyset(\hat{B}_\tau + \hat{\tau}) \left(x + ry \int_\tau^\infty e^{-\int_\tau^s (r + \hat{b}_u) du} ds \right) - c = \psi_\tau(\{\tilde{B}_t\}) \geq 0$$

where the first equality follows as in (29), using $\hat{a}_u = 0$ at $u \geq \tau$ since $\tau > \hat{\tau}$. Front-loading, (23), then implies

$$\frac{\partial \hat{\Pi}(\mathbb{I}_{\{t \leq \hat{\tau}\}})}{\partial a_{\hat{\tau}}} > \frac{\partial \hat{\Pi}(\mathbb{I}_{\{t \leq \hat{\tau}\}})}{\partial a_\tau} \geq 0$$

³⁵As always, $S_i = \min_{j \in N_i(G)} \{T_j\}$ and $S'_i = \min_{j \in N_i(G)} \{T'_j\}$.

contradicting the optimality of cutoff $\hat{\tau}$. \square

Proof of Lemma 8. As a baseline, write \bar{T}_k for k 's first success time in network G when i and j both use cutoff τ_* . For each realization of (\bar{T}_i, \bar{S}_i) , we dynamically realize $\{T_k, T'_k\}_k$ as follows. In a first step, raising τ_i (or τ_j) from τ_* to τ^* begets new success opportunities on $[\tau_*, \min\{\tau^*, \bar{T}_i, \bar{S}_i\}]$ (successes after $\min\{\bar{T}_i, \bar{S}_i\}$ have already been realized in the baseline). Thus, we draw an exponential random variable $Z \sim \text{Exp}(1)$, and set

$$T_i, T'_j = \begin{cases} \tau_* + Z & \text{if } \tau_* + Z \leq \min\{\tau^*, \bar{T}_i, \bar{S}_i\}, \\ \bar{T} & \text{otherwise.} \end{cases}$$

In subsequent steps, we trace the effects of additional successes in the first step through the network. Since this cascade starts at $\tau_* + Z$ (if at all) and successes are not instantaneous, we have $T_k \in (\tau_* + Z, \bar{T}_k]$ for all $k \neq i$ and $T'_k \in (\tau_* + Z, \bar{T}_k]$ for all $k \neq j$.

So defined, if $\tau_* + Z > \min\{\tau^*, \bar{T}_i, \bar{S}_i\}$, no additional successes realize, so $T_k = T'_k = \bar{T}_k$ for all k ; a fortiori $\min\{T_i, S_i\} = \min\{T'_i, S'_i\} = \min\{\bar{T}_i, \bar{S}_i\}$. If $\tau_* + Z \leq \min\{\tau^*, \bar{T}_i, \bar{S}_i\}$ we have $\min\{T_i, S_i\} = \tau_* + Z \leq \min\{T'_i, S'_i\}$ with equality iff j is a neighbor of i . All told, $\min\{T_i, S_i\} \stackrel{D}{\preceq} \min\{T'_i, S'_i\}$ with equality iff j is a neighbor of i . \square

B Appendix: Proofs from Section 4

B.1 Proof of Lemma 3 (Links in Large Random Networks)

Part (a): We will show separately that for every $\epsilon > 0$

$$\Pr \left[N^I \geq (1 + \epsilon)I(1 - e^{-\hat{n}^I/I}) \right] \rightarrow 0, \quad (32)$$

$$\Pr \left[N^I \leq (1 - \epsilon)I(1 - e^{-\hat{n}^I/I}) \right] \rightarrow 0. \quad (33)$$

This implies that the number of links converges to $1 - e^{-\hat{n}^I/I}$ in distribution, $N^I/(I(1 - e^{-\hat{n}^I/I})) \xrightarrow{D} 1$, and a fortiori in expectation, $n^I/(I(1 - e^{-\hat{n}^I/I})) \rightarrow 1$.

Start with the upper bound, (32). We can restrict attention to $\hat{\rho} = \lim \hat{n}^I/I < \infty$; for $\hat{\rho} = \infty$, we have $(1 + \epsilon)I(1 - e^{-\hat{n}^I/I}) > I$ for any $\epsilon > 0$ and large enough I , so trivially $\Pr[N^I \geq (1 + \epsilon)I(1 - e^{-\hat{n}^I/I})] = 0$.

Realize Iris's \hat{n}^I stubs k one after another, and keep track of the number of stubs $K^I(m)$ used to reach degree m ; if i has less than m neighbors set $K^I(m) := \hat{n}^I + 1$. When connecting Iris's k^{th} stub to her m^{th} neighbor, $I - m$ potential new neighbors with $\hat{n}^I(I - m)$ stubs

compete with $\hat{n}^I m - (2k - 1)$ remaining stubs of Iris and her $m - 1$ neighbors, sandwiching the success rate between $\frac{I-m}{I}$ and $\frac{I-m}{I-2}$. Writing X_ℓ^I for independent (shifted) geometric random variables with success rate $\frac{I-\ell}{I}$ we can thus upper-bound $K^I(m) \stackrel{D}{\preceq} \sum_{\ell=1}^m X_\ell^I$.

The chance of m or more neighbors is then upper-bounded by

$$\begin{aligned} \Pr [N^I \geq m] &= \Pr [K^I(m) \leq \hat{n}^I] \leq \Pr \left[\sum_{\ell=1}^m X_\ell^I \leq \hat{n}^I \right] \leq \inf_{\xi \geq 0} \exp \left(\xi \hat{n}^I + \sum_{\ell=1}^m \log E[e^{-\xi X_\ell^I}] \right) \\ &= \inf_{\xi \geq 0} \exp \left(\xi (\hat{n}^I - m) - \sum_{\ell=1}^m \log \frac{1 - e^{-\xi \ell/I}}{1 - \ell/I} \right) \end{aligned} \quad (34)$$

where the second inequality is a Chernoff-bound, and the final equality evaluates the moment generating function of the shifted geometric distribution, $E[e^{-\xi X_\ell^I}] = \frac{e^{-\xi(1-\ell/I)}}{1 - e^{-\xi \ell/I}}$.

Since $\log \frac{I - e^{-\xi \ell}}{I - \ell}$ rises in ℓ , the last term in (34) is lower-bounded by

$$\begin{aligned} \sum_{\ell=1}^m \log \frac{1 - e^{-\xi \ell/I}}{1 - \ell/I} &\geq \int_0^m \left(\int_{1-\ell/I}^{1-e^{-\xi \ell/I}} \frac{1}{x} dx \right) d\ell = \int_{1-m/I}^1 \left(\int_{I(1-x)}^{\min\{e^\xi I(1-x), m\}} \frac{1}{x} d\ell \right) dx \\ &= \int_{1-m/I}^{1-e^{-\xi m/I}} \frac{m - I(1-x)}{x} dx + \int_{1-e^{-\xi m/I}}^1 \frac{I(1-x)(e^\xi - 1)}{x} dx \\ &= I \left[(1 - m/I) \log(1 - m/I) - e^\xi (1 - e^{-\xi m/I}) \log(1 - e^{-\xi m/I}) \right]. \end{aligned}$$

For any $\epsilon > 0$, we now set $m = m^I := \left\lceil (1 + \epsilon)I(1 - e^{-\hat{n}^I/I}) \right\rceil$, substitute back into the term in parentheses in (34), and divide by I

$$\xi \frac{\hat{n}^I - m^I}{I} - (1 - m^I/I) \log(1 - m^I/I) + e^\xi (1 - e^{-\xi m^I/I}) \log(1 - e^{-\xi m^I/I}) =: \Gamma^I(\xi, \epsilon)$$

with limit $\Gamma(\xi, \epsilon)$ as $I \rightarrow \infty$. So defined, (34) becomes

$$\Pr [N^I \geq (1 + \epsilon)I(1 - e^{-\hat{n}^I/I})] \leq \inf_{\xi \geq 0} \exp (I\Gamma^I(\xi, \epsilon)) \quad (35)$$

The derivative $\Gamma_\xi(0, \epsilon) = \hat{\rho} + \log(1 - (1 + \epsilon)(1 - e^{-\hat{\rho}}))$ vanishes for $\epsilon = 0$ and falls in ϵ . Thus, for any $\epsilon > 0$ we have $\Gamma_\xi(0, \epsilon) < 0$. Also, $\Gamma(0, \epsilon) = 0$, and so $\Gamma(\xi, \epsilon) < 0$ for small ξ , and $\Gamma^I(\xi, \epsilon) < 0$ uniformly for large I . Thus, (35) vanishes for $I \rightarrow \infty$, implying (32).

The lower bound (33) follows analogously.

Part (b): By the proof of part (a) $n^I = E[N^I]$ approximates $I(1 - e^{-\hat{n}^I/I})$, and so $\lim n^I/I =$

$1 - e^{-\hat{\rho}}$. Further, since

$$I(1 - \exp^{-\hat{n}^I/I}) = I \left(\frac{\hat{n}^I}{I} - \frac{1}{2} \left(\frac{\hat{n}^I}{I} \right)^2 + \frac{1}{6} \left(\frac{\hat{n}^I}{I} \right)^3 - \dots \right).$$

$n^I/\hat{n}^I - 1$ is of order \hat{n}^I/I , which vanishes for $\hat{\rho} = 0$.

Part (c): Since $A_t = 1$ for $t < \tau$, we have $B_{\tau^I}^I = \int_0^{\tau^I} b_t^I dt = \int_0^{\tau^I} E^H[N^I|t < T^I, S^I] dt$. For I finite, $E^H[N^I|t < T^I, S^I] < n^I$ (and so $B_{\tau^I}^I < n^I \tau^I$) because lack of success, $t < T^I, S^I$, indicates fewer neighbors N^I . To bound the effect of such updating, we note that conditional on $|N^I - n^I| \leq \epsilon n^I$, and so $N^I \leq (1 + \epsilon)n^I$, we have $\Pr^H(t < T^I, S^I) \geq e^{-((1+\epsilon)n^I+1)t}$. Thus

$$\frac{\Pr^H(|N^I - n^I| \leq \epsilon n^I | t < T^I, S^I)}{\Pr^H(|N^I - n^I| \geq \epsilon n^I | t < T^I, S^I)} \geq \frac{\Pr^H(|N^I - n^I| \leq \epsilon n^I)}{\Pr^H(|N^I - n^I| \geq \epsilon n^I)} e^{-((1+\epsilon)n^I+1)t}. \quad (36)$$

We show below that $n^I \tau^I$ is bounded. This bounds $e^{-((1+\epsilon)n^I+1)t}$ away from 0 for all $t \leq \tau^I$. Thus, as the prior likelihood-ratio of $|N^I - n^I| \leq \epsilon n^I$ on the RHS of (36) diverges as $I \rightarrow \infty$ (by part (a)), so does the posterior likelihood-ratio on the LHS of (36), implying $E^H[N^I|t < T^I, S^I]/n^I \rightarrow 1$ and so $B_{\tau^I}^I/(n^I \tau^I) \rightarrow 1$, finishing the proof of part (c).

To show that $n^I \tau^I$ is bounded, assume it was not. Then we could choose $\hat{\tau}^I < \tau^I$ such that $n^I \hat{\tau}^I$ is bounded, but with limit $\lim n^I \hat{\tau}^I > \bar{\tau}$. Applying the above argument to $n^I \hat{\tau}^I$ instead of $n^I \tau^I$, we get $\lim B_{\hat{\tau}^I}^I = \lim n^I \hat{\tau}^I > \bar{\tau}$, leading to the contradiction that $p_{\tau^I} < p_{\hat{\tau}^I} = P^\emptyset(B_{\hat{\tau}^I}^I + \hat{\tau}^I) < \underline{p}$ for large I .

B.2 Proof of Theorem 1 (Large Random Networks)

For finite degrees $\nu < \infty$, the proof of Theorem 1 relies on results in Section 5. Proposition 4 shows that the component size of a typical node explodes with I ; Theorem 3 shows that equilibrium cutoffs $\tau = \lim \tau^I > 0$ (and fall in ν) so $B = \infty$, and that V rises in ν .

For $\nu = \infty$, equilibrium cutoffs must vanish, $\tau^I \rightarrow 0$; otherwise the posterior-belief at the cutoff $P^\emptyset((n^I + 1)\tau^I) \rightarrow 0$, choking off experimentation incentives. Lemma 4 characterizes the limit of social learning curves as step functions $B_t = B \mathbb{I}_{\{t \geq \sigma\}}$; if $\sigma = 0$, we can distinguish two bursts, with pre-cutoff information $B_\tau := \lim n^I \tau^I$, and post-cutoff information $B - B_\tau$. The equilibrium indifference condition becomes

$$\begin{aligned} \psi_\tau &= \lim \psi_{\tau^I}^I = p(B_\tau) \left(x + ry \int_0^\infty e^{-rt - (B_t - B_\tau)} dt \right) - c \\ &= p(B_\tau) \left(x + (1 - e^{-r\sigma}(1 - e^{-(B - B_\tau)})y) \right) - c = 0. \end{aligned} \quad (37)$$

To solve for B_τ, B, σ we complement (37) with the simple observation that

$$\frac{B_\tau}{B} = \frac{\lim n^I \tau^I}{\lim I \tau^I} = \lim \frac{n^I}{I} = \rho, \quad (38)$$

and two conditions linking the learning time $\sigma = \lim \frac{-\log \tau^I}{n^I}$ to pre-cutoff learning B_τ and total learning B : First, bounded total learning implies the learning time equals the network's time-diameter

$$\text{If } B = \lim I \tau^I < \infty, \text{ then } \sigma = \lim \frac{\log(I \tau^I) - \log \tau^I}{n^I} = \frac{1}{\lambda}. \quad (39)$$

Second, non-vanishing pre-cutoff learning implies immediate learning

$$\text{If } B_\tau = \lim n^I \tau^I > 0, \text{ then } \sigma = \lim \frac{\log(n^I \tau^I) - \log \tau^I}{n^I} = 0. \quad (40)$$

Case 1: $\rho = 0$ and $p_0 > \bar{p}$. The optimistic prior together with the equilibrium condition (37) require non-vanishing pre-cutoff learning, $B_\tau > 0$, and so by (40) immediate learning, $\sigma = 0$. The sparsity of the network together with (38) implies perfect learning $B = \infty$. Perfect immediate learning, $B = \infty, \sigma = 0$, in turn implies the welfare benchmark $V = p_0 y = V^*$.

Case 2: $\rho = 0$ and $p_0 \leq \bar{p}$. We first observe $B_\tau = 0$. Otherwise, if $B_\tau > 0$, the proof for Case 1 implies $B = \infty$ and $\sigma = 0$, and so experimentation incentives $\psi_\tau < p_0 x - c \leq 0$, contradicting equilibrium. This implies the welfare benchmark, $\lim \mathcal{V}(\tau^I, n^I \tau^I) = \mathcal{V}(0, 0) = V^*$.

Turning to asymptotic information, we now show that B attains the benchmark ∞ iff $\lambda \leq 1/\sigma^*$, and strictly decreases above. For sparse networks $\lambda \leq 1/\sigma^*$,³⁶ if by contradiction learning was imperfect $B < \infty$, social learning happens too late, at $\sigma = 1/\lambda \geq \sigma^*$ by (39), so experimentation incentives are strictly positive

$$\psi_\tau = p_0 (x + (1 - e^{-r\sigma}(1 - e^{-B}))y) - c > p_0 (x + (1 - e^{-r\sigma^*})y) - c = 0,$$

contradicting equilibrium.

Conversely, dense networks $\lambda > 1/\sigma^*$ induce early social learning $\sigma \leq 1/\lambda < \sigma^*$, and equilibrium indifference

$$p_0 (x + (1 - e^{-r\sigma}(1 - e^{-B}))y) - c = \psi_\tau = 0 = p_0 (x + (1 - e^{-r\sigma^*})y) - c$$

requires $B < \infty$. Moreover, $B = B(\sigma)$ rises in σ , and so falls in $\lambda = 1/\sigma$.

³⁶For $p_0 = \bar{p}$, we have $\sigma^* = 0$, so this condition is always satisfied.

Case 3: $\rho > 0$. Then $B_\tau = \hat{\rho}B > 0$; (40) then implies $\sigma = 0$, and so (37) becomes

$$p(\hat{\rho}B) (x + e^{-(1-\hat{\rho})B}y) = c.$$

In the notation of (52) in the proof of Theorem 2, this means $\Psi(\hat{\rho}B, (1 - \hat{\rho})B) = 0$, so by (53), information $B = B(\hat{\rho})$ falls in $\hat{\rho}$. Then also welfare $V = p_0(1 - e^{-B})$ falls in $\hat{\rho}$.

B.3 Proof of Lemma 4 (Degenerate Exposure Time)

With probability $e^{-I\tau^I}$, no agent succeeds by τ^I , and so $S^I = \infty$; from here on we condition on the complementary event that at least one agent succeeds during experimentation, triggering a contagion process. For now, we also restrict attention to $\lim \hat{n}^I/I = 0$, so that $\hat{n}^I/n^I \rightarrow 1$ by Lemma 3(a). This allows us to work with \hat{n}^I for finite I , but switch to n^I in the limit where $\sigma := \lim \frac{-\log \tau^I}{n^I}$. We discuss the case $\lim \hat{n}^I/I > 0$ later.

The overarching proof strategy is to separate the “geographical”/network aspects of the contagion process from its timing. Specifically, we realize the randomness of the network G^I as agents succeed. To emphasize the analogy to epidemiological SI contagion processes, we refer to agents who have succeeded as *infected*. When k agents are infected, let E_k^I be the random number of *exposed agents*, i.e. that have observed a success but have yet to succeed themselves. Clearly $E_k^I \leq \hat{n}^I k$; a (relative) *exposure gap*, $\Gamma_k^I := \frac{\hat{n}^I k - E_k^I}{\hat{n}^I k} > 0$, opens up after an exposed j agent succeeds because the exposing agent i already succeeded and cannot be re-exposed, or a stub of a succeeding agent connects to an already exposed agent. For $\epsilon > 0$, write $\mathcal{E}^I(\epsilon) := \{\Gamma_k^I < 3\epsilon \text{ for all } k \leq \epsilon I / \hat{n}^I\}$ for the event that the gap process remains bounded in early stages of the contagion

Lemma 9. *For any $\epsilon > 0$, $\lim_{I \rightarrow \infty} \Pr(\mathcal{E}^I(\epsilon)) = 1$.*

We postpone the proof of Lemma 9; the idea is that with $E_k^I \leq \epsilon I$ exposed agents, ϵ small, and \hat{n}^I large, most stubs expose new agents.

For small ϵ , Lemma 9 means that after the approximately $\tau^I I$ initial infections in the experimentation phase, the contagion process resembles a collection of tree networks emanating from these “seeds” at exponential rate \hat{n}^I . We now argue that as $\hat{n}^I \rightarrow \infty$, this contagion process reaches a negligible fraction of all agents at any $\underline{t} < \sigma = \lim \frac{-\log \tau^I}{\hat{n}^I}$, but approximately all agents at any $\bar{t} > \sigma$.

Specifically, write T_k^I for the k^{th} infection time, and K^I for the (random) number of infected agents at τ^I . Also define inter-arrival times in the contagion phase $\Delta_k^I := T_{k+1}^I - T_k^I$ for $k > K^I$ and $\Delta_k^I := T_{k+1}^I - \tau^I$ for $k = K^I$. The proof idea is to apply Chernoff bounds to $T_k^I - \tau^I = \sum_{\ell=K^I}^{k-1} \Delta_\ell^I$. Towards this goal, note that conditional on the realization of the

“geographical exposure process” $\{E_k^I\}_{k \in [K^I, \epsilon I / \hat{n}^I]}$, inter-arrival times Δ_k^I are independent with arrival rate E_k^I . Conditional on $\mathcal{E}^I(\epsilon)$ we have $E_k^I \in [(1 - 3\epsilon)\hat{n}^I k, \hat{n}^I k]$, and so

$$\mathbb{E}[e^{-\xi \Delta_k^I} | \mathcal{E}^I(\epsilon)] \leq \frac{\hat{n}^I k}{\hat{n}^I k + \xi} \quad \text{for all } \xi \geq 0, \quad (41)$$

$$\mathbb{E}[e^{\xi \Delta_k^I} | \mathcal{E}^I(\epsilon)] \leq \frac{(1 - 3\epsilon)\hat{n}^I k}{(1 - 3\epsilon)\hat{n}^I k - \xi} \quad \text{for all } \xi \in [0, (1 - 3\epsilon)\hat{n}^I k]. \quad (42)$$

We now derive upper and lower bounds for the k^{th} success time T_k^I in the contagion phase $k \in [K^I, \epsilon I / \hat{n}^I]$; in the limit $I \rightarrow \infty$ these bounds are then shown to imply vanishing chances of getting exposed before σ and after σ , respectively. The upper bound is as follows

$$\begin{aligned} \Pr(T_k^I \leq \tau^I + \delta | \mathcal{E}^I(\epsilon), K^I) &= \Pr\left(\sum_{\ell=K^I}^{k-1} \Delta_\ell^I \leq \delta | \mathcal{E}^I(\epsilon)\right) \leq \inf_{\xi \geq 0} e^{\xi \delta} \prod_{\ell=K^I}^{k-1} \mathbb{E}[e^{-\xi \Delta_\ell^I} | \mathcal{E}^I(\epsilon)] \\ &\leq \inf_{\xi \geq 0} \exp\left(\xi \delta - \sum_{\ell=K^I}^{k-1} \left(\log(\hat{n}^I \ell + \xi) - \log(\hat{n}^I \ell)\right)\right) \\ &\leq \inf_{\xi \in [0, \hat{n}^I]} \exp\left(\xi \delta - \sum_{\ell=K^I}^{k-1} \frac{\xi}{\hat{n}^I} \left(\log(\hat{n}^I(\ell + 1)) - \log(\hat{n}^I \ell)\right)\right) \\ &= \inf_{\xi \in [0, \hat{n}^I]} \exp\left(\xi \left(\delta - \frac{\log k - \log K^I}{\hat{n}^I}\right)\right) \end{aligned} \quad (43)$$

The first equality drops the τ^I to focus on time since the cutoff, the first inequality is a Chernoff-bound, the second uses (41), the third uses the concavity of the logarithm, and the final equality collapses the telescopic sum.

Next, we argue that for fixed $\epsilon > 0$ and the integer floor $k = \lfloor \epsilon I / \hat{n}^I \rfloor$, the fraction on the RHS of (43) (which approximates the time for the contagion process to reach k agents) converges to $\sigma = \lim_{\hat{n}^I \rightarrow \infty} \frac{-\log \tau^I}{\hat{n}^I}$:

$$\frac{\log \lfloor \epsilon I / \hat{n}^I \rfloor - \log K^I}{\hat{n}^I} \xrightarrow{D} \sigma \quad (44)$$

For $\bar{B} = \lim I \tau^I < \infty$, this follows because K^I is almost surely bounded above, so as $\hat{n}^I \rightarrow \infty$, all terms other than $\frac{\log I}{\hat{n}^I}$ vanish, and $\lim_{\hat{n}^I \rightarrow \infty} \frac{\log I}{\hat{n}^I} = \lim_{\hat{n}^I \rightarrow \infty} \frac{\log I - \log \bar{B}}{\hat{n}^I} = \lim_{\hat{n}^I \rightarrow \infty} \frac{-\log \tau^I}{\hat{n}^I} = \sigma$. For $\bar{B} = \infty$, it follows because, by the law of large numbers, $\frac{K^I}{I \tau^I} \xrightarrow{D} 1$; equivalently, $\log K^I - \log I - \log \tau^I \xrightarrow{D} 0$ so the LHS of (44) becomes $\frac{-\log \tau^I}{\hat{n}^I}$, whose limit is σ .

Exposing any positive fraction $\epsilon > 0$ of nodes requires infecting at least $\epsilon I / \hat{n}^I$ agents, and the chance of this at any time $\underline{t} < \sigma$ vanishes

$$\lim_{I \rightarrow \infty} \Pr(T_{\lfloor \epsilon I / \hat{n}^I \rfloor}^I \leq \tau^I + \underline{t}) = \lim_{I \rightarrow \infty} \Pr(T_{\lfloor \epsilon I / \hat{n}^I \rfloor}^I \leq \tau^I + \underline{t} | \mathcal{E}^I(\epsilon)) \leq \inf_{\xi \geq 0} \exp(\xi(\underline{t} - \sigma)) = 0.$$

The equality uses Lemma 9, and the inequality (43) and (44). A fortiori $\Pr(T_{\lfloor \epsilon I / \hat{n}^I \rfloor}^I \leq \underline{t}) \rightarrow 0$.

Finally, for any population share $\epsilon > 0$, the probability that a given agent i has been exposed by time \underline{t} is bounded above by the sum of that share ϵ and the probability that more than share ϵ has been exposed by time \underline{t} , $\Pr(S^I \leq \underline{t}) \leq \epsilon + \Pr(T_{\lfloor \epsilon I / \hat{n}^I \rfloor}^I \leq \underline{t})$. Since this inequality holds for any $\epsilon > 0$, we have

$$\lim_{I \rightarrow \infty} \Pr(S^I \leq \underline{t}) \leq \lim_{\epsilon \rightarrow 0} \lim_{I \rightarrow \infty} (\epsilon + \Pr(T_{\lfloor \epsilon I / \hat{n}^I \rfloor}^I \leq \underline{t})) = 0. \quad (45)$$

Turning to the lower bound for T_k^I , using the same steps as for (43), but with (42) substituting for (41) for the second inequality

$$\begin{aligned} \Pr(T_k^I \geq \tau^I + \delta | \mathcal{E}^I(\epsilon), K^I) &\leq \inf_{\xi \geq 0} e^{-\xi \delta} \prod_{\ell=K^I}^{k-1} \mathbb{E}[e^{\xi \Delta_\ell} | \mathcal{E}^I(\epsilon)] \\ &\leq \inf_{\xi \geq 0} \exp \left(-\xi \delta + \sum_{\ell=K^I}^{k-1} (\log((1-3\epsilon)\hat{n}^I \ell) - \log((1-3\epsilon)\hat{n}^I \ell - \xi)) \right) \\ &\leq \inf_{\xi \in [0, (1-3\epsilon)\hat{n}^I]} \exp \left(-\xi \delta + \sum_{\ell=K^I}^{k-1} \frac{\xi}{(1-3\epsilon)\hat{n}^I} (\log((1-3\epsilon)\hat{n}^I \ell) - \log((1-3\epsilon)\hat{n}^I (\ell-1))) \right) \\ &= \inf_{\xi \in [0, (1-3\epsilon)\hat{n}^I]} \exp \left(-\xi \left(\delta - \frac{\log(k-1) - \log(K^I-1)}{(1-3\epsilon)\hat{n}^I} \right) \right) \end{aligned}$$

As for the upper bound, for $k = \lfloor \epsilon I / \hat{n}^I \rfloor$ the fraction on the RHS converges, $\frac{\log(\epsilon I / \hat{n}^I - 1) - \log(K^I - 1)}{(1-3\epsilon)\hat{n}^I} \xrightarrow{D} \sigma / (1-3\epsilon)$, so for any $\bar{\delta} > \sigma / (1-3\epsilon)$ in the limit

$$\lim_{I \rightarrow \infty} \Pr(T_{\lfloor \epsilon I / \hat{n}^I \rfloor}^I \geq \tau^I + \bar{\delta} | \mathcal{E}^I(\epsilon)) \leq \inf_{\xi \geq 0} \exp \left(-\xi \left(\bar{\delta} - \frac{\sigma}{1-3\epsilon} \right) \right) = 0.$$

Conditional on $\mathcal{E}^I(\epsilon)$, $\lfloor \epsilon I / \hat{n}^I \rfloor$ infections guarantee $\epsilon(1-3\epsilon)I$ exposures by $\tau^I + \bar{\delta}$. For $\epsilon' > 0$ small, approximately $\epsilon'\epsilon(1-3\epsilon)I$ of these get infected by $\tau^I + \bar{\delta} + \epsilon'$, generating approximately $\hat{n}^I \epsilon' \epsilon (1-3\epsilon)I$ new exposure possibilities; that is, an exploding number $\hat{n}^I \epsilon' \epsilon (1-3\epsilon) \rightarrow \infty$ for every agent. Now, for any $\bar{t} > \sigma$, we choose $\epsilon, \epsilon' > 0$ small enough, and I large enough that $\tau^I + \bar{\delta} + \epsilon' < \bar{t}$ for $\bar{\delta} := \sigma / (1-3\epsilon) + \epsilon' > \sigma$. As $I \rightarrow \infty$, all remaining nodes get exposed before $\tau^I + \bar{\delta} + \epsilon'$ and thus before \bar{t} with probability

$$\lim_{I \rightarrow \infty} \Pr(S^I \leq \bar{t}) = 1. \quad (46)$$

Jointly, (45) and (46) for any $\underline{t} < \sigma < \bar{t}$ establish Lemma 4.

The case $\lim \hat{n}^I / I > 0$. So far we assumed $\lim \hat{n}^I / I > 0$ so that $\lim n^I / \hat{n}^I = 1$. Otherwise,

we have $\rho = \lim n^I/I = 1 - \exp(-\lim \hat{n}^I/I) > 0$, implying $B_\tau = \rho B > 0$ and so the desired learning time equals $\sigma = 0$ by (40). To see that learning is indeed immediate, note that the first infection exposes fraction $\rho > 0$ of nodes. The paragraph preceding (46) then implies that everyone is exposed immediately thereafter. Similarly, if $\lim \tau^I > 0$, a non-vanishing share of agents gets infected, and the entire population gets exposed at any $t > 0$, so $\sigma = 0$.

Proof of Lemma 9. We will construct $p(\epsilon) < 1$ such that for large I and any $k \leq \epsilon I/\hat{n}^I$ the chance of a large exposure gap is bounded above via

$$\Pr(\Gamma_k^I > 3\epsilon) < p(\epsilon)^{\hat{n}^I k}. \quad (47)$$

Since $\mathcal{E}^I(\epsilon)$ is the complement of the union of these events over $k \geq 1$, Boole's inequality implies $1 - \Pr(\mathcal{E}^I(\epsilon)) \leq \sum_{k=1}^{\infty} p(\epsilon)^{\hat{n}^I k} = p(\epsilon)^{\hat{n}^I} / (1 - p(\epsilon)^{\hat{n}^I}) \rightarrow 0$, which implies (47).

We construct $p(\epsilon)$ and show (47) with the help of Chernoff bounds. The increment $E_k^I - E_{k-1}^I$ counts the newly exposed agents at the k^{th} infection. If j was exposed himself, he exposes $\hat{n}^I - 1$ others and is himself deducted from E_k^I ; if j was not exposed, he exposes \hat{n}^I others. Each agent exposed by j was already exposed with probability at most $k\hat{n}^I/I$. Thus, writing X_ν for iid binary random variables with $\Pr(X_\nu = 1) = k\hat{n}^I/I$, and $X_\nu = 0$ else, we can upper bound the absolute exposure gap

$$\hat{n}^I k \Gamma_k^I = \hat{n}^I k - E_k^I = \sum_{\ell=1}^k (\hat{n}^I - (E_\ell^I - E_{\ell-1}^I)) \stackrel{D}{\preceq} 2k + \sum_{\nu=1}^{k\hat{n}^I} X_\nu \quad (48)$$

Now define $p(\epsilon) := \inf_{\xi \geq 0} \left(\frac{\mathbb{E}[e^{X_\nu \xi}]}{e^{2\epsilon \xi}} \right)$. We have $p(\epsilon) < 1$ since $\mathbb{E}[X_\nu] = k\hat{n}^I/I < \epsilon$, and so $\frac{\mathbb{E}[e^{X_\nu \xi}]}{e^{2\epsilon \xi}} \approx \frac{1 + \mathbb{E}[X_\nu] \xi}{1 + 2\epsilon \xi} \leq \frac{1 + \epsilon \xi}{1 + 2\epsilon \xi} < 1$ for small $\xi > 0$. For I large, such that $\epsilon \hat{n}^I > 2$, we then get the following Chernoff-upper bound for the RHS of (48)

$$\Pr \left(2k + \sum_{\nu=1}^{k\hat{n}^I} X_\nu > 3\epsilon \hat{n}^I k \right) \leq \Pr \left(\sum_{\nu=1}^{k\hat{n}^I} X_\nu > 2\epsilon \hat{n}^I k \right) \leq \inf_{\xi \geq 0} \left(\frac{\mathbb{E}[e^{X_\nu \xi}]}{e^{2\epsilon \xi}} \right)^{\hat{n}^I k} = p(\epsilon)^{\hat{n}^I k}$$

which together with (48) implies (47), and hence Lemma 9. \square

B.4 Proof of Equation (14)

We apply Bayes' rule

$$1 - a_t = \frac{\Pr^H(\forall k, \ell' : t < T_k, T_{\ell'})}{\Pr^H(t < T_\ell, \forall k : t < T_k)} = \begin{cases} \frac{\exp(-(K+L)t)}{\exp(-(K+1)t)} = \exp(-(L-1)t) & t < \tau_k \\ \frac{\exp(-K\tau_k - Lt)}{\exp(-K(\tau_k + \int_{\tau_k}^t a_s ds) - t)} = \exp\left(-(L-1)t + K \int_{\tau_k}^t a_s ds\right) & t \in (\tau_k, \tau_\ell) \\ \frac{\exp(-K\tau_k - L\tau_\ell)}{\exp(-K(\tau_k + \int_{\tau_k}^{\tau_\ell} a_s ds) - \tau_\ell)} = \exp\left(-(L-1)\tau_\ell + K \int_{\tau_k}^{\tau_\ell} a_s ds\right) & t > \tau_\ell \end{cases}$$

and then differentiate wrt t .

Alternatively, we can derive (14) from peripherals' cumulative social learning $B_{\ell,t} = \int_0^t K a_s ds$. The latter is given by

$$e^{-B_{\ell,t}} = e^{-K \min\{\tau_k, t\}} \left(1 - \int_0^{\min\{t, \tau_\ell\}} (L-1) e^{-(L-1)s} (1 - e^{-K(t - \max\{s, \tau_k\})}) ds \right).$$

The LHS is the probability a peripheral does not socially learn by t . The RHS is the probability of the joint event that (i) no core agent succeeds during experimentation before t , and (ii) no peripheral succeeds at some $s < t$ during experimentation and triggers a core agent to succeed during contagion. The latter expression is useful to solve for equilibrium numerically.

B.5 Proof of Theorem 2 (Core-Periphery Networks)

The challenge with this proof is the complexity of characterizing two outcome variables, asymptotic information and welfare, for a myriad of cases. Specifically we must consider six different network densities $\kappa \lesseqgtr \kappa^*$, $\rho = 0$, $\in (0, 1)$, or $= 1$, and pessimistic priors $p_0 < \bar{p}$ as well as optimistic ones. While some arguments apply to all of these cases, each case also has its idiosyncrasies.

We structure the exposition in order of increasing network density, characterizing asymptotic information and welfare in parallel and emphasizing the case of pessimistic priors $p_0 < \bar{p}$. But to avoid repetitions, we sometimes break this linear narrative by bracketing out arguments that apply more broadly.

As in the paper body, we superscript variables in finite networks with the network size I , e.g. τ_ℓ^I , and drop the superscript in the limit, e.g. $\tau_\ell := \lim_{I \rightarrow \infty} \tau_\ell^I$. A priori the limit is well-defined only for some subsequence, but the analysis characterizes all limits under consideration uniquely.

Asymptotic information equals $B = \lim B^I = \lim(K^I \tau_k^I + L^I \tau_\ell^I)$ since the network is connected and each agent's own experimentation $\tau_{k,\ell}^I$ (which in principle is excluded from

the social information B) is negligible as $I \rightarrow \infty$. It will be useful to decompose B into core agents' pre-cutoff learning $\Upsilon_k^I := I\tau_k^I$ and post cutoff learning $\Upsilon_\ell^I := L^I(\tau_\ell^I - \tau_k^I)$.

We can already note two bounds on $\Upsilon_k, \Upsilon_\ell$: Total information $B = \Upsilon_k + \Upsilon_\ell$ is strictly positive: By contradiction, $B = 0$ means agents face the single-agent problem, choose $\tau_k = \tau_\ell = \bar{\tau} > 0$ and so $B = \infty$. Any agent's pre-cutoff learning B_τ is no larger than $\bar{\tau}$, recalling from (4) that $P^\emptyset(B_\tau)(x + y) - c \geq \psi_\tau = 0$. For core agents, this means $\Upsilon_k \leq \bar{\tau}$. Thus, there is asymptotic learning iff $\Upsilon_\ell = \infty$; a sufficient (but not necessary) condition is $\tau_\ell > 0$.

B.5.1 Case 1: Bounded core size $\kappa < \infty$

Preliminaries. We first establish a necessary and sufficient condition for maximal social learning by peripherals

$$B_{\ell,t} \equiv \kappa t \quad \text{iff} \quad \Upsilon_\ell = \infty. \quad (49)$$

If $\Upsilon_\ell = \infty$, core agents immediately observe a peripheral succeed, and then work forever after. If $\Upsilon_\ell < \infty$, the probability of a success $1 - e^{-(\Upsilon_k + \Upsilon_\ell)}$ is less than one, bounding above $b_{\ell,t} \leq \kappa(1 - e^{-(\Upsilon_k + \Upsilon_\ell)}) < \kappa$ for $t > \tau_k$.

By Lemma 1, the social learning upper-bound (49) implies an incentive lower-bound

$$\psi_{\ell,0} \geq \underline{\psi}_{\ell,\kappa,0} := p_0 \left(x + \frac{r}{r + \kappa} y \right) - c \quad (50)$$

with equality iff $\Upsilon_\ell = \infty$.

We distinguish three cases, $\kappa \lesseqgtr \kappa^*$; for optimistic priors $p_0 \geq \bar{p}$, we have $\kappa^* = \infty$, and so only case 1a) $\kappa < \infty$ is relevant.

Case 1a: $\kappa < \kappa^*$. Since $\underline{\psi}_{\ell,\kappa,0}$ falls in κ , we have $\underline{\psi}_{\ell,\kappa,0} > \underline{\psi}_{\ell,\kappa^*,0} = 0$, so $\psi_{\ell,0} > 0$, and continuity of $\psi_{\ell,0}$ implies $\tau_\ell > 0$, and asymptotic learning $\Upsilon_\ell = \infty$. By Lemma 2, welfare is bounded away from second-best $\mathcal{V}(\tau_\ell, \kappa\tau_\ell) < \mathcal{V}(0, 0) = V^*$. Quantitatively, $\Upsilon_\ell = \infty$ and (49) imply $B_{\ell,t} = \kappa t$, so welfare increases in κ by Lemma 1.

For $p_0 \geq \bar{p}$, only one argument needs adapting: the welfare benchmark now equals $V^* = p_0 y$ which requires immediate and perfect social learning, $B_t = \infty$ for $t > 0$. Clearly, $B_{\ell,t} = \kappa t$ falls short of this benchmark.

Case 1b: $\kappa = \kappa^*$. Now $\underline{\psi}_{\ell,\kappa,0} = 0$. We show asymptotic learning, $\Upsilon_\ell = \infty$, by contradiction: By (50), $\Upsilon_\ell < \infty$ would imply $\psi_{\ell,0} > 0$ and so $\tau_\ell > 0$, leading to the contradiction that $\Upsilon_\ell = \infty$. In turn, $\Upsilon_\ell = \infty$ implies by (49) and (50) that $\psi_{\ell,0} = \underline{\psi}_{\ell,\kappa,0} = 0$ and so $\tau_\ell = 0$ and $\kappa\tau_\ell = 0$, attaining the welfare upper bound $\mathcal{V}(0, 0) = V^*$.

Case 1c: $\kappa \in (\kappa^*, \infty)$. Now $\underline{\psi}_{\ell, \kappa, 0} < 0$. Asymptotic learning fails because $\Upsilon_\ell = \infty$ would imply by (49) and (50) that $\psi_{\ell, 0} = \underline{\psi}_{\ell, \kappa, 0} < 0$ and so $\tau_\ell^I = 0$ for large I and $\Upsilon_\ell = 0$. In turn, $\Upsilon_\ell < \infty$ implies $\tau_\ell = 0$ and $\psi_{\ell, 0} = 0$. To quantify information, we first claim that $\Upsilon_k = \lim I\tau_k^I = 0$: Indeed, core agents receive all social information immediately, $B_{k,t} = \Upsilon_k + \Upsilon_\ell$ for all $t > 0$, while peripherals' learning is bounded by $B_{\ell,t} \leq \kappa t$. This bounds incentives of core agents above $\psi_{k,0} < \psi_{\ell,0} = 0$, and so $\tau_k^I = 0$ for large I .³⁷

Social information thus equals Υ_ℓ . We now show this falls in κ : Peripherals observe a success by time t iff at least one peripheral succeeds during experimentation, and then a core agent succeeds during $(0, t]$; thus $1 - e^{-B_{\ell,t}} = (1 - e^{-\Upsilon_\ell})(1 - e^{-\kappa t})$.³⁸ Since the RHS rises with both κ and Υ_ℓ and experimentation incentives $\psi_{\ell,0}$ fall in $\{B_{\ell,t}\}$, the equilibrium condition $\psi_{\ell,0} = 0$ implies that a rise in information transmission κ must be compensated by a fall in aggregate information Υ_ℓ . For future reference, we note that as $\kappa \rightarrow \infty$, the learning curve $B_{\ell,t}$ converges to Υ_ℓ for each $t > 0$, and so peripherals' indifference condition converges to $p_0(x + e^{-\Upsilon_\ell}y) = c$, pinning down aggregate information Υ_ℓ .

Finally, since $\tau_\ell = \kappa\tau_\ell = 0$, welfare attains the upper bound $\mathcal{V}(0, 0) = V^*$.

B.5.2 Case 2: Exploding core $\kappa = \infty$

Preliminaries. For simplicity we first assume $\rho < 1$, and cover the case $\rho = 1$ separately. We prepare the ground with two preliminary lemmas.

Lemma 10. *Assume $\kappa = \infty$, $\rho < 1$, and any prior $p_0 > \underline{p}$.*

- (a) *Individual learning vanishes:* $\tau_k^I, \tau_\ell^I \rightarrow 0$.
- (b) *Social learning is immediate:* For all $t > 0$, $B_{k,t}^I, B_{\ell,t}^I \rightarrow \Upsilon_k + \Upsilon_\ell$.

Proof. Part (a) follows by the upper bound on pre-cutoff learning $B_\tau \leq \bar{\tau}$. For core agents, $B_{k,t}^I = (I - 1)\tau_k^I \leq \bar{\tau}$. For peripherals,

$$B_{\ell, \tau_\ell^I}^I := K^I \tau_k^I + \int_{\tau_k^I}^{\tau_\ell^I} K^I a_t^I dt \quad (51)$$

where core agents' expected effort a_t^I from (14) drifts towards $\min\{(L^I - 1)/K^I, 1\}$ and is hence bounded away from 0 by our assumption that $\rho < 1$. The upper bound, $B_{\ell, \tau_\ell^I}^I < \bar{\tau}$ thus requires the domain to vanish, $\tau_\ell^I \rightarrow 0$, as the integrand explodes, $K^I \rightarrow \infty$.

³⁷We also get $\tau_k^I = 0$ for large I and $\Upsilon_k = 0$ in cases 1a,b with $p_0 < \bar{p}$, where $\psi_{k,0} < 0$ is ensured by $B_{k,t} = \infty$ for all $t > 0$.

³⁸Solving for $B_{\ell,t}$ and differentiating yields $b_{\ell,t} = \kappa \frac{e^{-\kappa t}(1 - e^{-\Upsilon_\ell})}{e^{-\kappa t}(1 - e^{-\Upsilon_\ell}) + e^{-\Upsilon_\ell}}$, generalizing (49).

Turning to part (b), the conditional probability that some agent i has observed a neighbor succeed by $t < \tau_\ell^I$ is bounded via

$$(1 - \exp(-(I\tau_k^I + (L^I - 1)\tau_\ell^I))) (1 - \exp(-K^I(t - \tau_\ell^I))) < 1 - \exp(-B_t^I) < 1 - \exp(-(\Upsilon_k^I + \Upsilon_\ell^I))$$

The upper bound is the probability that any agent succeeds. The lower bound is the probability that some agent $j \neq i$ succeeds during experimentation, times the probability that a core agent succeeds in $[\tau_\ell^I, t]$. Both bounds converge to $1 - \exp(-(\Upsilon_k + \Upsilon_\ell))$ as $I \rightarrow \infty$. \square

Lemma 10(b) implies that welfare of both core agents and peripherals equals

$$V_k = V_\ell = (1 - \exp(-(\Upsilon_k + \Upsilon_\ell))) p_0 y$$

and thus rises in social information $\Upsilon_k + \Upsilon_\ell$. All of our monotonicity results for social information thus apply equally to welfare.

Lemma 10b implies that social learning of both core agents and peripherals occurs in two bursts: one before the cutoff and one immediately after, and both approaching $t = 0$. For such learning with burst sizes B^- and B^+ , the indifference condition (4) becomes

$$\Psi(B^-, B^+) := P^\emptyset(B^-)(x + e^{-B^+} y) - c = 0. \quad (52)$$

Recalling the effects of social learning on experimentation incentives (5) and $ry = x - c$, the solution of (52) has slope

$$-\frac{dB^+}{dB^-} = \frac{\partial_{B^-} \Psi}{\partial_{B^+} \Psi} = \frac{e^{-B^+} y + x - c}{e^{-B^+} y} = 1 + re^{B^+}. \quad (53)$$

To apply (52) to core agents and peripherals, write asymptotic pre-cutoff learning $B_{\ell, \tau_\ell} = \lim B_{\ell, \tau_\ell}^I$, experimentation incentives as $\psi_{\ell, \tau_\ell} = \lim \psi_{\ell, \tau_\ell}^I$, and similarly for core agents, substituting “ k ” for “ ℓ ”.³⁹ For core agents, $B^- = B_{k, \tau_k} = \Upsilon_k$, $B^+ = \Upsilon_\ell$, and (52) coincides with the limit of (13) as $L \rightarrow \infty$.⁴⁰

Lemma 11. *Assume $\kappa = \infty$, $\rho < 1$, and any prior $p_0 > \underline{p}$.*

(a) *Core agents are initially indifferent*

$$\Psi(\Upsilon_k, \Upsilon_\ell) = P^\emptyset(\Upsilon_k) (x + e^{-\Upsilon_\ell} y) - c = 0. \quad (54)$$

(b) *Pre-cutoff learning of core agents and peripherals coincides: $B_{\ell, \tau_\ell} = \Upsilon_k$.*

³⁹Note that even though $\tau_\ell^I \rightarrow \tau_\ell = 0$, this is distinct from, and generally greater than the other limit $B_{\ell, 0} = \lim B_{\ell, 0}^I = 0$.

⁴⁰For peripherals, we get an explicit expression of B_{ℓ, τ_ℓ} in $\Upsilon_k, \Upsilon_\ell$ only for $\rho > 0$, (58).

Proof. Part (a): For internal cutoffs $\tau_k^I > 0$, the indifference conditions $\psi_{k,\tau_k^I}^I = 0$ converge to (54). By contradiction, assume that $\tau_k^I = 0$ for large I , so that $\Upsilon_k = 0$ and $\psi_{k,\tau_k} = \psi_{k,0} = p_0(x + e^{-\Upsilon_\ell}y) - c < 0$. Using Lemma 10(b) (immediate learning by both core agents and peripherals) and the greater importance of pre-cutoff learning (53), strict shirking incentives by core agents carry over to peripherals⁴¹

$$\psi_{\ell,\tau_\ell} = \Psi(B_{\ell,\tau_\ell}, \Upsilon_\ell - B_{\ell,\tau_\ell}) \leq \Psi(0, \Upsilon_\ell) = \psi_{k,\tau_k} < 0.$$

Thus $\tau_\ell^I = 0$ for large I , leading to the contradiction that $\Upsilon_k + \Upsilon_\ell = 0$ and $\psi_{k,0} = \psi_{\ell,0} = p_0(x + y) - c = \Psi(0, 0) > 0$.

Part (b): The indifference condition of core and peripheral agents imply

$$\Psi(\Upsilon_k, \Upsilon_\ell) = \psi_{k,\tau_k} = 0 = \psi_{\ell,\tau_\ell} = \Psi(B_{\ell,\tau_\ell}, \Upsilon_k + \Upsilon_\ell - B_{\ell,\tau_\ell}) = \Psi(\Upsilon_k - (\Upsilon_k - B_{\ell,\tau_\ell}), \Upsilon_\ell + (\Upsilon_k - B_{\ell,\tau_\ell})).$$

The greater effect of pre-cutoff learning on incentives (53) thus implies $\Upsilon_k - B_{\ell,\tau_\ell} = 0$. \square

Lemma 11 establishes two equations for $\Upsilon_k, \Upsilon_\ell$. Below we show they admit a unique solution; a corner solution for $\rho = 0$, and an internal one for $\rho \in (0, 1)$.

Case 2a: $\rho = 0$. In this case we get a corner solution for $\Upsilon_k, \Upsilon_\ell$ with $\Upsilon_k/\Upsilon_\ell = 0$. Indeed, using Lemma 11(b), pre-cutoff learning is a vanishing proportion of post-cutoff learning

$$\Upsilon_k = B_{\ell,\tau_\ell} = \lim B_{\ell,\tau_\ell^I}^I < \lim K^I \tau_\ell^I = \lim \frac{K^I}{L^I} L^I \tau_\ell^I \leq \left(\frac{\rho}{1 - \rho} (\Upsilon_k + \Upsilon_\ell) \right). \quad (55)$$

Since $\rho = 0$, we must have either $\Upsilon_k = 0$ or $\Upsilon_\ell = \infty$ (then the last term “ $0 \cdot \infty$ ” is not well defined), or both.

For pessimistic priors $p_0 < \bar{p}$, core agents’ indifference (54) rules out asymptotic learning, so $\Upsilon_\ell < \infty$ and (55) implies $\Upsilon_k = 0$. In turn, aggregate information Υ_ℓ solves $\Psi(0, \Upsilon_\ell) = p_0(x + e^{-\Upsilon_\ell}y) - c = 0$.⁴²

For $p_0 \geq \bar{p}$, Υ_k solves $P^\emptyset(\Upsilon_k) = \bar{p}$ and $\Upsilon_\ell = \infty$.⁴³ Core agents’ indifference (54) clearly requires experimentation until the myopic threshold, $P^\emptyset(\Upsilon_k) \leq \bar{p}$. If, by contradiction, core agents experiment past the myopic threshold, $P^\emptyset(\Upsilon_k) < \bar{p}$, then (54) implies $\Upsilon_\ell < \infty$, and (55) leads to the contradiction that $\Upsilon_k = 0$.

⁴¹Note the contrast to the case with bounded core size $\kappa < \infty$ (and $p_0 < \bar{p}$), where peripherals learn slower than core agents, so that $\psi_{k,\tau_k} < \psi_{\ell,\tau_\ell} = 0$.

⁴²This is the same indifference condition we found in case 1c as $\kappa \rightarrow \infty$, so aggregate information is continuous in this limit.

⁴³In the borderline case with $p_0 = \bar{p}$, we get both $\Upsilon_k = 0$ and $\Upsilon_\ell = \infty$.

Case 2b: $\rho \in (0, 1)$. In this case we get an internal solution for $\Upsilon_k, \Upsilon_\ell$. We first further operationalize Lemma 11(b) by replacing the upper bound in (55) with an explicit expression of peripherals' pre-cutoff learning B_{ℓ, τ_ℓ} in terms of $\Upsilon_k, \Upsilon_\ell$, (58). To analyze (51) as the integrand $K^I a_t^I$ explodes and the integration domain $[\tau_k^I, \tau_\ell^I]$ vanishes, we rescale time $\alpha_t^I := a_{t/I}^I$. The ODE (14) for core agents' experimentation intensity thus becomes

$$I \frac{\dot{\alpha}_t^I}{1 - \alpha_t^I} = \begin{cases} L^I - 1 & t < I\tau_k^I \\ L^I - 1 - K^I \alpha_t^I & t \in (I\tau_k^I, I\tau_\ell^I) \\ -K^I \alpha_t^I & t > I\tau_\ell^I \end{cases} \quad (56)$$

Recalling the definition of $\rho, \Upsilon_k, \Upsilon_\ell$, as $I \rightarrow \infty$, the solution α_t^I converges to the solution α_t of

$$\frac{\dot{\alpha}}{1 - \alpha} = \begin{cases} 1 - \rho & t < \Upsilon_k \\ 1 - \rho - \rho\alpha & t \in (\Upsilon_k, \Upsilon_k + \Upsilon_\ell/(1 - \rho)) \\ -\rho\alpha & t > \Upsilon_k + \Upsilon_\ell/(1 - \rho) \end{cases} \quad (57)$$

Peripherals' pre-cutoff learning (51) then converges to

$$B_{\ell, \tau_\ell} = \rho \left(\Upsilon_k + \int_{\Upsilon_k}^{\Upsilon_k + \Upsilon_\ell/(1 - \rho)} \alpha_t dt \right), \quad (58)$$

so we can rewrite Lemma 11(b) as

$$\Phi(\rho, \Upsilon_k, \Upsilon_\ell) := \rho \left(\Upsilon_k + \int_{\Upsilon_k}^{\Upsilon_k + \Upsilon_\ell/(1 - \rho)} \alpha_t dt \right) - \Upsilon_k = 0. \quad (59)$$

We can now characterize equilibrium learning.

Lemma 12. *For all $\rho \in [0, 1]$, equations (54), (59) admit a unique solution $(\Upsilon_k, \Upsilon_\ell)$. This solution satisfies $0 < \Upsilon_k, \Upsilon_\ell < \infty$, and aggregate information $\Upsilon_k + \Upsilon_\ell$ falls in ρ .*

The proof of Lemma 12 relies on the following generalization of Leibniz's integral rule: For Lipschitz-continuous functions f, g and some cutoff $s > 0$, let x_t be the continuous solution of an ODE

$$\dot{x} = \begin{cases} f(x) & \text{for } t < s \\ g(x) & \text{for } t > s \end{cases}$$

with initial condition x_0 . We write $x_t(s)$ to emphasize the importance of the cutoff, and assume $g(x_s(s)) \neq 0$.

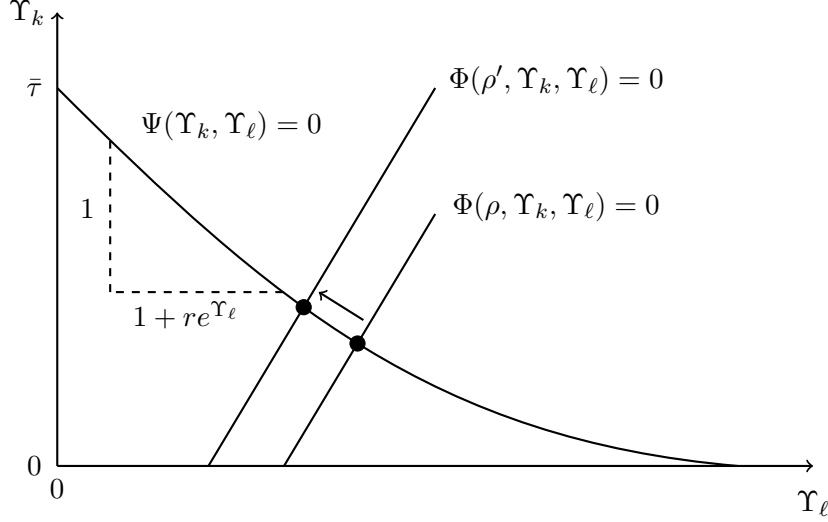


Figure 8: **Solutions of $\Phi(\rho, \Upsilon_k, \Upsilon_\ell) = 0$ and $\Psi(\Upsilon_k, \Upsilon_\ell) = 0$.**

Lemma 13. *For any $\Delta > 0$*

$$\frac{\partial}{\partial s} \int_s^{s+\Delta} x_t(s) dt = \frac{f(x_s(s))}{g(x_s(s))} (x_s(s + \Delta) - x_s(s)) \quad (60)$$

Proof of Lemma 12. Equation (59) together with $\Upsilon_k + \Upsilon_\ell > 0$ and the fact that the solution α of (57) is bounded away from zero imply $\Upsilon_k > 0$, and in turn that $0 < \Upsilon_\ell < \infty$. Thus, asymptotic learning fails.

To solve (54), (59), we note that Φ clearly rises in ρ and Υ_ℓ . We show below that it falls in Υ_k . Hence zero-sets of Φ in $(\Upsilon_\ell, \Upsilon_k)$ -space are increasing and shift left when ρ rises to ρ' , as illustrated in Figure 8. Recalling from (53) that zero-sets of Ψ are decreasing with slope $-1/(1 + re^{-\Upsilon_\ell}) > -1$, equations (54), (59) admit a unique solution $(\Upsilon_k, \Upsilon_\ell)$. A rise in ρ shifts this solution left on the zero-set of Ψ , so $\Upsilon_k + \Upsilon_\ell$ falls.

In fact, the monotonicity of $\Upsilon_k + \Upsilon_\ell$ extends to the boundary points $\rho = 0, 1$: We recall that for $\rho = 0$ all learning is post-cutoff, $\Upsilon_k = 0, \Psi(0, \Upsilon_\ell) = 0$,⁴⁴ and anticipate that for $\rho = 1$ all learning is pre-cutoff, $\Upsilon_\ell = 0, \Psi(\Upsilon_k, 0) = 0$, thus attaining the extreme points on the zero set of $\Psi(\Upsilon_k, \Upsilon_\ell) = 0$ as illustrated in Figure 8.

To show that Φ falls in Υ_k , we write $\alpha_* = \alpha_{\Upsilon_k}$ and $\alpha^* = \alpha_{\Upsilon_k + \Upsilon_\ell/(1-\rho)}$, assume that

⁴⁴This assumes $p_0 < \bar{p}$. For $p_0 \geq \bar{p}$, asymptotic information is infinite for $\rho = 0$, and hence trivially greater than the finite learning for $\rho > 0$.

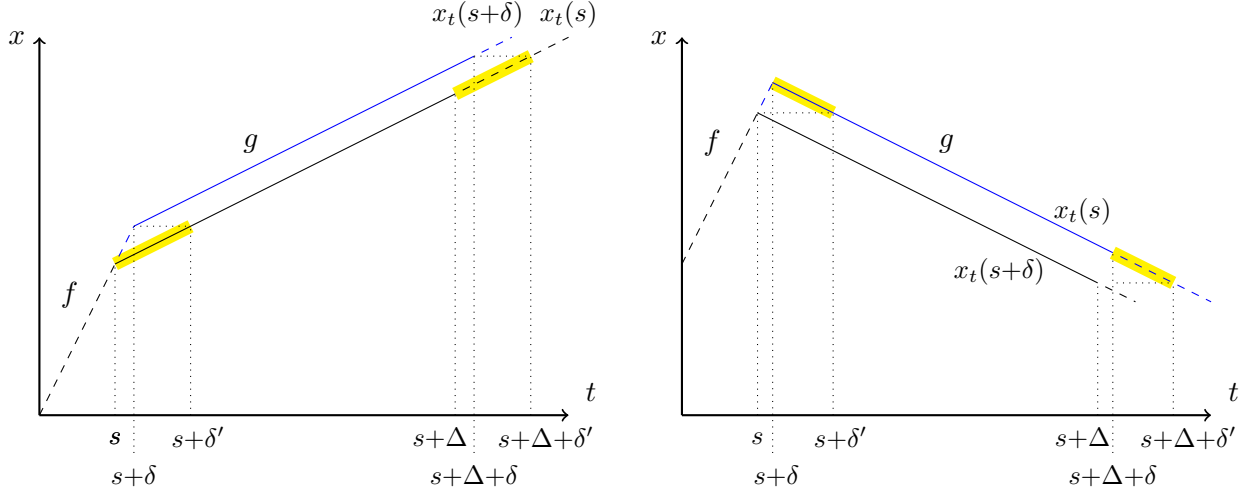


Figure 9: **Proof of Leibniz Rule.** In both figures the difference of between the integral of the upper solid line, $x_t(s+\delta)$ over $t \in [s+\delta, s+\delta+\Delta]$, and the lower solid line, $x_t(s)$ over $t \in [s, s+\Delta]$, equals the difference in the integrals of the shaded lines. E.g. In the left picture this is difference between $x_t(s)$ over $t \in [s+\Delta, s+\delta'+\Delta]$ and $x_t(s)$ over $t \in [s, s+\delta']$, which is the RHS of (62) after substituting $t = s + \tilde{\delta}$.

$1 - \rho - \rho\alpha_* \neq 0$, and then argue⁴⁵

$$\frac{\partial \Phi}{\partial \Upsilon_k} = -(1 - \rho) + \rho \frac{1 - \rho}{1 - \rho - \rho\alpha_*} (\alpha^* - \alpha_*) = -(1 - \rho) \frac{1 - \rho - \rho\alpha^*}{1 - \rho - \rho\alpha_*} < 0.$$

The first equality follows from Lemma 13 by substituting $s = \Upsilon_k$ and $\Delta = \Upsilon_\ell / (1 - \rho)$ for the integral boundaries, $x_t = \alpha_t$ for the trajectory, $f(\alpha) = (1 - \rho)(1 - \alpha)$ for the law-of-motion before $s = \Upsilon_k$, and $g(\alpha) = (1 - \rho - \rho\alpha)(1 - \alpha)$ after Υ_k .

The middle equality is an elementary algebraic transformation, and the final inequality owes to the fact that $\dot{\alpha}/(1 - \alpha) = 1 - \rho - \rho\alpha$ cannot switch signs on $[\Upsilon_k, \Upsilon_k + \Upsilon_\ell / (1 - \rho)]$, cf (57), so that $\frac{1 - \rho - \rho\alpha^*}{1 - \rho - \rho\alpha_*} > 0$. \square

Proof of Lemma 13. The Leibniz rule evaluates the LHS of (60) “vertically”, computing $\frac{\partial}{\partial s} x_t(s) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} (x_t(s + \delta) - x_t(s))$ for fixed $t \in [s, s + \Delta]$. Since the ODE $\dot{x} = g(x)$ is autonomous, it is more economical to compare the trajectories $\{x_t(s + \delta)\}_t$ and $\{x_t(s)\}_t$ “horizontally”, as illustrated in Figure 9.

Formally, assume first that $f(s)$ and $g(s)$ have the same sign, and for $\delta > 0$ small, let $\delta' > 0$ solve $x_{s+\delta'}(s) = x_{s+\delta}(s + \delta)$. At $s + \delta'$ the original trajectory “merges” with the shifted trajectory and since $\dot{x} = g(x)$ is autonomous we get $x_{s+\delta'+\hat{\delta}}(s) = x_{s+\delta+\hat{\delta}}(s + \delta)$, as illustrated

⁴⁵Since $\alpha_t = 1 - \exp(-(1 - \rho)t)$ for $t < \Upsilon_k$, there exists at most one value of Υ_k with $1 - \rho - \rho\alpha_{\Upsilon_k} = 0$. Since Φ is continuous in Υ_k and decreasing in Υ_k everywhere else, it decreases everywhere.

in Figure 9(left). Thus

$$\int_{\delta}^{\delta+\Delta} x_{s+\tilde{\delta}}(s+\delta) d\tilde{\delta} = \int_0^{\Delta} x_{s+\delta+\hat{\delta}}(s+\delta) d\hat{\delta} = \int_0^{\Delta} x_{s+\delta'+\hat{\delta}}(s) d\hat{\delta} = \int_{\delta'}^{\delta'+\Delta} x_{s+\tilde{\delta}}(s) d\tilde{\delta} \quad (61)$$

using the change of variable $\tilde{\delta} = \delta + \hat{\delta}$ in the first equality, and $\tilde{\delta} = \delta' + \hat{\delta}$ in the last. Thus

$$\begin{aligned} \int_{s+\delta}^{s+\Delta+\delta} x_t(s+\delta) dt - \int_s^{s+\Delta} x_t(s) dt &= \int_{\delta}^{\delta+\Delta} x_{s+\tilde{\delta}}(s+\delta) d\tilde{\delta} - \int_0^{\Delta} x_{s+\tilde{\delta}}(s) d\tilde{\delta} \\ &= \int_{\delta'}^{\delta'+\Delta} x_{s+\tilde{\delta}}(s) d\tilde{\delta} - \int_0^{\Delta} x_{s+\tilde{\delta}}(s) d\tilde{\delta} = \int_{\Delta}^{\Delta+\delta'} x_{s+\tilde{\delta}}(s) d\tilde{\delta} - \int_0^{\delta'} x_{s+\tilde{\delta}}(s) d\tilde{\delta} \end{aligned} \quad (62)$$

where the first equality uses the change of variables $t = s + \tilde{\delta}$, the second uses (61), and the third cancels identical terms $\int_{\delta'}^{\Delta} x_{s+\tilde{\delta}}(s) d\tilde{\delta}$. In the limit

$$\frac{\partial}{\partial s} \int_s^{s+\Delta} x_t(s) dt = \lim_{\delta \rightarrow 0} \frac{\delta'}{\delta} (x_{s+\Delta}(s) - x_s(s)) = \frac{f(x_s(s))}{g(x_s(s))} (x_{s+\Delta}(s) - x_s(s)),$$

where we used that at first-order $\delta' g(x_s(s)) = \delta f(x_s(s))$.

If f and g have different signs, we let $\delta' > \delta$ solve $x_{s+\delta'}(s+\delta) = x_s(s)$, so $\delta f(s) + (\delta' - \delta)g(s) = 0$, as illustrated in Figure 9(right). Analogous arguments as above then show

$$\frac{\partial}{\partial s} \int_s^{s+\Delta} x_t(s) dt = \lim_{\delta \rightarrow 0} \frac{\delta' - \delta}{\delta} (x_s(s) - x_{s+\Delta}(s)) = \frac{f(x_s(s))}{g(x_s(s))} (x_{s+\Delta}(s) - x_s(s)).$$

as required □

Case 2c: $\rho = 1$. While Lemmas 10 and 11 and most other substantive intermediate results remain true for $\rho = 1$, their proofs divide by $1 - \rho$, and sometimes invoke that $L \rightarrow \infty$. Instead of re-proving everything, we provide a separate analysis, solely based on the function Ψ and its derivatives, (52-53), and the ODE (56). Specifically we will show that

$$\Psi(\Upsilon_k, \Upsilon_\ell) \leq \psi_{k, \tau_k} \leq 0 = \psi_{\ell, \tau_\ell} = \Psi(\Upsilon_k + \Upsilon_\ell, 0) \quad (63)$$

Together with (53), this implies $\Upsilon_\ell = 0$, so the inequalities in (63) must hold with equality. In particular $0 = \Psi(\Upsilon_k, 0) = P^\emptyset(\Upsilon_k)(x+y) - c$, so total information is as in the clique (or the single-agent problem) $\Upsilon_k + \Upsilon_\ell = \Upsilon_k = \bar{\tau}$.

We now show (63). The middle inequality and equality reflect (the limits of) peripherals' indifference and core agents weak shirking incentives at their respective cutoffs. The first inequality takes the limit of the strict inequality $\Psi(\Upsilon_k^I, \Upsilon_\ell^I) < \psi_{k, \tau_k^I}^I$, which reflects that core agents' observe post-cutoff information Υ_ℓ^I with a delay.

Only the last equality in (63), which states that peripherals' learning is entirely pre-cutoff, requires a novel argument and the assumption $\rho = 1$. Intuitively, information transmission by K^I core agents is infinitely faster than generation by L^I peripherals.

Formally, we will show that peripherals' aggregate post-cutoff learning vanishes

$$\frac{K^I}{I} \int_{I\tau_\ell^I}^\infty \alpha_t^I dt \rightarrow 0. \quad (64)$$

By (64), peripherals pre-cutoff learning $B_{\ell, \tau_\ell^I}^I$ converges to total information $\Upsilon_k + \Upsilon_\ell$, implying the last equality in (63).

To see (64) we first argue that $\alpha_t^I \rightarrow 0$ for all t . By line one of (56), $\alpha_t^I \leq L^I t / I \leq L^I \bar{\tau} / I \rightarrow 0$ for all $t < I\tau_k^I < \bar{\tau}$; at $t > I\tau_k^I$, lines two and three of (56) imply $\dot{\alpha}_t^I < 0$ when $\alpha_t^I \geq L^I / K^I \rightarrow (1 - \rho) / \rho = 0$. All told, $\alpha_t^I \rightarrow 0$ for all t . Turning to the aggregate in (64), line three of (56) states that α_t^I decays exponentially at rate $(1 - \alpha_t^I)K^I / I$. Since this rate converges to 1, we have $\int_{I\tau_\ell^I}^\infty \alpha_t^I dt - \alpha_{I\tau_\ell^I}^I \rightarrow 0$. Together with $\alpha_{I\tau_\ell^I}^I \rightarrow 0$, this implies (64).

C Appendix: Proofs from Section 5

C.1 Proof of Theorem 3 (Comparisons of Trees)

We first show the comparisons between directed and undirected trees

$$\bar{\tau}^{(n+1)} < \bar{\tau}^{(n)} < \bar{\tau}^{(n)} \quad \text{and} \quad \bar{V}^{(n+1)} > \bar{V}^{(n)} > \bar{V}^{(n)} \quad (65)$$

and then comment how the same arguments imply the comparison between undirected trees and triangle trees.

Emphasizing the role of degree n and cutoff τ , we write the neighbor's expected time- t effort in directed and undirected tree as $\bar{a}_t^{(n)}(\tau), \bar{a}_t^{(n)}(\tau)$. These equal 1 for $t < \tau$, and solve (17) and (20) for $t > \tau$.

We first show $\bar{\tau}^{(n)} < \bar{\tau}^{(n)}$. For a given cutoff $\tau > 0$, we have $\bar{a}_t^{(n)}(\tau) > \bar{a}_t^{(n)}(\tau)$ for all $t \geq \tau$: At the cutoff $\bar{a}_\tau^{(n)}(\tau) = 1 - e^{-n\tau} > 1 - e^{-(n-1)\tau} = \bar{a}_\tau^{(n)}(\tau)$, and this ranking prevails for $t > \tau$ since the RHS of (17) exceeds the RHS of (20). Additionally, $\bar{a}_t^{(n)}(\tau), \bar{a}_t^{(n)}(\tau)$ rise in τ , strictly for $\tau < t$. By Lemma 6(b), for any $\tau \leq \bar{\tau}^{(n)}$

$$0 = \psi_{\bar{\tau}^{(n)}}(\{n\bar{a}_t^{(n)}(\bar{\tau}^{(n)})\}) < \psi_\tau(\{n\bar{a}_t^{(n)}(\tau)\}),$$

so in equilibrium we must instead have $\bar{\tau}^{(n)} > \bar{\tau}^{(n)}$, as desired. Agents in the directed network then also have lower pre-cutoff social learning $n\bar{\tau}^{(n)} < n\bar{\tau}^{(n)}$ and hence higher welfare $\bar{V}^{(n)} = \mathcal{V}(\bar{\tau}^{(n)}, n\bar{\tau}^{(n)}) > \mathcal{V}(\bar{\tau}^{(n)}, n\bar{\tau}^{(n)}) = \bar{V}^{(n)}$ by Lemma 2.

We next show $\bar{\tau}^{(n+1)} < \bar{\tau}^{(n)}$. For a given cutoff τ , the degree difference exactly offsets the difference in the laws-of-motion (17) and (20), so at the level of i 's random neighbor j , social learning coincides $\bar{a}_t^{(n+1)}(\tau) = \bar{a}_t^{(n)}(\tau)$. But then total social learning is higher in the undirected network $(n+1)\bar{a}_t^{(n+1)}(\tau) > n\bar{a}_t^{(n)}(\tau)$. By Lemma 6(a,b), for any $\tau \leq \bar{\tau}^{(n+1)}$ we have $0 = \psi_{\bar{\tau}^{(n+1)}}(\{(n+1)\bar{a}_t^{(n+1)}(\bar{\tau}^{(n+1)})\}) < \psi_\tau(\{n\bar{a}_t^{(n)}(\tau)\})$, so in equilibrium we must instead have $\bar{\tau}^{(n)} > \bar{\tau}^{(n+1)}$, as desired.

For the associated welfare ranking we will show more strongly that

$$\bar{\tau}^{(n+1)} < \tau' := \frac{n+1}{n+2} \bar{\tau}^{(n)} \quad (66)$$

for then, by Lemma 2,

$$\bar{V}^{(n+1)} = \mathcal{V}(\bar{\tau}^{(n+1)}, (n+2)\bar{\tau}^{(n+1)} - \bar{\tau}^{(n+1)}) > \mathcal{V}(\bar{\tau}^{(n+1)}, (n+1)\bar{\tau}^{(n)} - \bar{\tau}^{(n+1)}) > \mathcal{V}(\bar{\tau}^{(n)}, n\bar{\tau}^{(n)}) = \bar{V}^{(n)},$$

where the first inequality uses (66), and the second that adding $\bar{\tau}^{(n)} - \bar{\tau}^{(n+1)} > 0$ to the first argument of \mathcal{V} and subtracting it from the second argument decreases \mathcal{V} , since $\partial_\tau \mathcal{V} < \partial_B \mathcal{V} < 0$, as shown in (31).

To see (66), we compare a neighbor's expected experimentation $\delta \geq 0$ after the cutoff for the directed n -tree with equilibrium cutoff $\bar{\tau}^{(n)}$, $\bar{a}_\delta := \bar{a}_{\bar{\tau}^{(n)} + \delta}^{(n)}(\bar{\tau}^{(n)})$, and the undirected $(n+1)$ -tree with non-equilibrium cutoff τ' from (66), $\bar{a}'_\delta := \bar{a}_{\tau' + \delta}^{(n+1)}(\tau')$. We will show that

$$n\bar{a}_\delta < (n+1)\bar{a}'_\delta. \quad (67)$$

Since pre-cutoff learning coincides, $\bar{p}_{\bar{\tau}^{(n)}}^{(n)} = P^\emptyset((n+1)\bar{\tau}^{(n)}) = P^\emptyset((n+2)\tau') = \bar{p}_{\tau'}^{(n+1)}$, we get

$$\begin{aligned} 0 &= \psi_{\bar{\tau}^{(n)}}(\{n\bar{a}_t^{(n)}(\bar{\tau}^{(n)})\}) = \bar{p}_{\bar{\tau}^{(n)}}^{(n)}(x + ry e^{-\int_0^\infty (r+n\bar{a}_\delta)d\delta}) - c \\ &> \bar{p}_{\tau'}^{(n+1)}(x + ry e^{-\int_0^\infty (r+(n+1)\bar{a}'_\delta)d\delta}) - c = \psi_{\tau'}(\{(n+1)\bar{a}_t^{(n+1)}(\tau')\}) \end{aligned}$$

and hence in equilibrium $\bar{\tau}^{(n+1)} < \tau'$, which is (66).

We now argue (67). We first claim a partial converse, $\bar{a}'_\delta < \bar{a}_\delta$. This follows by $\bar{a}'_0 = 1 - \exp(-n\tau') < 1 - \exp(-n\bar{\tau}^{(n)}) = \bar{a}_0$ and the fact that \bar{a}'_δ and \bar{a}_δ follow the same law-of-motion $\dot{a} = (n-1)a(1-a)$ and hence can't cross. Now (67) follows for $\delta = 0$ as follows

$$n\bar{a}_0 = n(1 - \exp(-n\bar{\tau}^{(n)})) < (n+1)(1 - \exp(-n\frac{n}{n+1}\bar{\tau}^{(n)})) < (n+1)(1 - \exp(-n\tau')) = (n+1)\bar{a}'_0 \quad (68)$$

where the first inequality uses that $(1 - \exp(-x))/x$ falls in x , and the second inequality that $\frac{n}{n+1}\bar{\tau}^{(n)} < \frac{n+1}{n+2}\bar{\tau}^{(n)} = \tau'$. We show (67) for all $\delta > 0$ by arguing that $n\bar{a}_\delta$ cannot cross

$(n+1)\bar{a}_\delta$ from below: Assume $n\vec{a}_\delta = (n+1)\bar{a}_\delta$. Then, using $\vec{a}_\delta > \bar{a}'_\delta$

$$n\dot{\vec{a}}_\delta = n(n-1)\vec{a}_\delta(1-\vec{a}_\delta) < (n+1)(n-1)\bar{a}'_\delta(1-\bar{a}'_\delta) = (n+1)\dot{\bar{a}}'_\delta.$$

We have thus proven the comparison between directed and undirected trees (65).

The comparison between undirected and triangle trees

$$\hat{\tau}^{(n+1)} < \bar{\tau}^{(n)} < \hat{\tau}^{(n)} \quad \text{and} \quad \hat{V}^{(n+1)} > \bar{V}^{(n)} > \hat{V}^{(n)}$$

follows analogously: For equal degree n , we use the fact that $\bar{a}_t^{(n)}(\tau) > \hat{a}_t^{(n)}(\tau)$ to argue $\bar{\tau}^{(n)} < \hat{\tau}^{(n)}$ and consequently $\bar{V}^{(n)} > \hat{V}^{(n)}$. For triangle networks with degree $n+1$, we use the fact that $\hat{a}_t^{(n+1)}(\tau) = \bar{a}_t^{(n)}(\tau)$ to argue $\hat{\tau}^{(n+1)} < \bar{\tau}^{(n)}$; the argument that $\hat{V}^{(n+1)} > \bar{V}^{(n)}$ relies again on showing that incentives in the triangle network are strictly positive at $\tau' := \frac{n+1}{n+2}\bar{\tau}^{(n)}$. The only difference is that now $\hat{a}_t^{(n+1)}(\tau) = \bar{a}_t^{(n)}(\tau)$ equals $1 - e^{-(n-1)t}$ for $t = \tau$ and then evolves according to $\dot{a} = (n-2)a(1-a)$, while in the comparison of the undirected $n+1$ -tree to the directed n -tree the corresponding values were to $1 - e^{-nt}$ for $t = \tau$ and then $\dot{a} = (n-1)a(1-a)$, so in the analogue of (68) we must now use the fact that $\frac{n-1}{n} < \frac{n+1}{n+2}$.

C.2 Proof of Proposition 4 (Convergence)

We lead with the proof of part (b) for undirected networks (which are also used in Section 4.2), dropping the “upper bar”, say on \bar{a}_t , to ease notation. Subsequently, we discuss how to adapt the proof for directed and triangular networks in parts (a) and (c) of Proposition 4.

Notation and conventions. For an arbitrary cutoff $\tau \in [0, \bar{\tau}]$, write the social learning curve in the I -agent network as $\mathcal{B}^I(\tau) = \{\mathcal{B}_t^I(\tau)\}_t$; in the unique equilibrium, $b_t^I = \mathcal{B}_t^I(\tau^I)$ and $\psi_{\tau^I}(b^I) = 0$. Analogously, in the infinite regular n -tree \mathcal{T} , define $\mathcal{A}(\tau) = \{\mathcal{A}_t(\tau)\}_t$ as follows: For $t < \tau$, $\mathcal{A}_t(\tau) := 1$; for $t > \tau$, it is the solution of (20), $\dot{a} = (n-2)a(1-a)$ with boundary condition $a_\tau = 1 - e^{-(n-1)\tau}$. In equilibrium, $(\tau^*, a^* = \{a_t^*\})$ uniquely solve $a^* = \mathcal{A}(\tau^*)$ and $\psi_{\tau^*}(na^*) = 0$. Convergence of sequences of functions $b^I = \{b_t^I\}$ as $I \rightarrow \infty$ is always point-wise for all but at most one t , namely the cutoff $t = \bar{\tau}^*$.

We will prove that $\tau^I \rightarrow \tau^*$ and that $b_t^I \rightarrow na_t^*$ for all $t \neq \tau^*$. We restrict attention to a subsequence where τ^I converges to some τ^∞ . The triangle inequality implies that for all I

$$\begin{aligned}
|\psi_{\tau^\infty}(n\mathcal{A}(\tau^\infty))| &\leq |\psi_{\tau^\infty}(n\mathcal{A}(\tau^\infty)) - \psi_{\tau^\infty}(n\mathcal{A}(\tau^I))| + \\
&\quad |\psi_{\tau^\infty}(n\mathcal{A}(\tau^I)) - \psi_{\tau^\infty}(\mathcal{B}^I(\tau^I))| + \\
&\quad |\psi_{\tau^\infty}(\mathcal{B}^I(\tau^I)) - \psi_{\tau^I}(\mathcal{B}^I(\tau^I))| + |\psi_{\tau^I}(\mathcal{B}^I(\tau^I))|
\end{aligned}$$

As $I \rightarrow \infty$, the first term vanishes by continuity of $\mathcal{A}_t(\tau)$ in τ for all $t \neq \tau^\infty$, and continuity of $\psi_{\tau^\infty}(b)$ in $b = \{b_t\}$. The second term vanishes by continuity of $\psi_{\tau^\infty}(b)$ in $b = \{b_t\}$ and because for all $t \geq 0$

$$\lim_{I \rightarrow \infty} \sup_{\tau \in [0, \bar{\tau}]} |\mathcal{B}_t^I(\tau) - n\mathcal{A}_t(\tau)| = 0 \quad (69)$$

as we show below. The third term vanishes by Lemma 6(b). The fourth term is 0 for all I since τ^I is the equilibrium cutoff of \mathcal{G}^I

Thus, $\psi_{\tau^\infty}(n\mathcal{A}(\tau^\infty)) = 0$. Since τ^* is the unique solution of this equation, we have $\tau^\infty = \tau^*$. Since the subsequence of τ^I that converges to τ^∞ was arbitrary, the entire sequence τ^I converges to τ^* as desired. The triangle inequality then implies $|b_t^I - na_t^*| = |\mathcal{B}_t^I(\tau^I) - n\mathcal{A}_t(\tau^*)| \leq |\mathcal{B}_t^I(\tau^I) - n\mathcal{A}_t(\tau^I)| + n|\mathcal{A}_t(\tau^I) - \mathcal{A}_t(\tau^*)| \rightarrow 0$ for all $t \neq \tau^*$.

Proof of (69); social learning converges for fixed τ . Social learning converges for fixed τ . Fix an agent i , and consider times $t > \tau$. Let $G_i^{I,r}$ be the event that, i has n neighbors, $n(n-1)$ second neighbors, ..., $n(n-1)^{r-1}$ agents at distance r , and all of these agents are distinct. For all fixed r and t , $\lim_{I \rightarrow \infty} \Pr(G_i^{I,r}) \rightarrow 1$. For the upcoming arguments, we state that this convergence also conditional on the event $\{\theta = H, t < T_i, S_i\}$

$$\lim_{I \rightarrow \infty} \Pr^H(G_i^{I,r} | t < T_i, S_i) \rightarrow 1. \quad (70)$$

We will now define upper and lower bounds $\underline{a}_t^r(\tau), \bar{a}_t^r(\tau)$ for the expected effort of i 's neighbors j in both the network $a_t^{I,r}(\tau) := E^H[A_{j,t}^I | G_i^{I,r}, t < T_i, S_i]$ and in the infinite tree $\mathcal{A}_t(\tau)$. We show below that

$$\lim_{r \rightarrow \infty} \sup_{\tau \in [0, \bar{\tau}]} |\bar{a}_t^r(\tau) - \underline{a}_t^r(\tau)| = 0. \quad (71)$$

Then, by the triangle inequality

$$\begin{aligned}
|\mathcal{B}_{i,t}^I(\tau) - n\mathcal{A}_t(\tau)| &\leq |\mathcal{B}_t^I(\tau) - na_t^{I,r}(\tau)| + |na_t^{I,r}(\tau) - n\mathcal{A}_t(\tau)| \\
&\leq n(1 - \Pr^H(G_i^{I,r} | t < T_i, S_i)) + n|\bar{a}_t^r(\tau) - \underline{a}_t^r(\tau)|
\end{aligned}$$

and so (70) and (71) imply

$$\lim_{r \rightarrow \infty} \lim_{I \rightarrow \infty} \sup_{\tau \in [0, \bar{\tau}]} |\mathcal{B}_{i,t}^I(\tau) - n\mathcal{A}_t(\tau)| \leq \lim_{r \rightarrow \infty} \lim_{I \rightarrow \infty} n(1 - \Pr^H(G_i^{I,r} | t < T_i, S_i)) + \lim_{r \rightarrow \infty} \sup_{\tau \in [0, \bar{\tau}]} n|\bar{a}_t^r(\tau) - \underline{a}_t^r(\tau)| = 0$$

which is (69), since the LHS does not depend on r . \square

Proof of (71); construction of the bounds $\underline{a}_t^r, \bar{a}_t^r$ and their convergence. We define the bounds $\underline{a}_t^r, \bar{a}_t^r$ (dropping τ for a moment to ease notation) as i 's expectation over neighbor j 's effort conditional on pessimistic/optimistic assumptions about successes of distant agents. Specifically, we define expectations $\underline{E}^{-i,r}, \bar{E}^{-i,r}$ over the first success times T_i of all agents k with distance $1, \dots, r$ from i , both of which condition on $G_i^{I,r}, \theta = H$ and the fact that i 's neighbors j have not seen i succeed. Additionally, $\underline{E}^{-i,r}$ conditions on no “leaf agent” ℓ with distance r from i having observed a success from an “outside” agent at distance $r+1$ from i ; conversely, $\bar{E}^{-i,r}$ conditions on *every* “leaf agent” ℓ having observed a success from an “outside” agent. We then set $\underline{a}_t^r := \underline{E}^{-i,r}[A_{j,t} | t < T_j]$ and $\bar{a}_t^r := \bar{E}^{-i,r}[A_{j,t} | t < T_j]$.

We proceed by induction over r . For $r = 1$, this means $\underline{a}_t^1 \equiv 0, \bar{a}_t^1 \equiv 1$ for $t > \tau$. More generally, for $r > 1$, i 's neighbor j shirks at $t > \tau$ iff none of his $n-1$ other neighbors $k \in N_j(G) \setminus \{i\}$ have succeeded.

$$1 - \underline{a}_t^r = \frac{\Pr^{-i}(t < T_j, t < T_k \forall k \in N_j \setminus \{i\})}{\Pr^{-i}(t < T_j)} = \frac{\exp\left(-n\tau - (n-1) \int_{\tau}^t \underline{a}_s^{r-1} ds\right)}{\exp\left(-\tau - \int_{\tau}^t \underline{a}_s^r ds\right)} \quad (72)$$

The last equality is analogous to the undirected line in Example 4: The denominator follows because the hazard rate of T_j equals 1 before τ and \underline{a}_s^r after. In turn the event in the numerator has hazard rate n when all agents experiment before τ ; after τ , having observed no success j shirks, while the expected effort of each of his $n-1$ neighbors k equals \underline{a}_s^{r-1} since the event $G_i^{I,r}$ implies $G_j^{I,r-1}$. We rewrite (72) as an ODE

$$\dot{\underline{a}}^r = ((n-1)\underline{a}^{r-1} - \underline{a}^r)(1 - \underline{a}^r) \quad (73)$$

with $\underline{a}_{\tau}^r = 1 - e^{-(n-1)\tau}$. The upper bounds \bar{a}_t^r also obey (73) with anchor $\bar{a}_t^1 \equiv 1$.

Since successes outside $G_i^{I,r}$ only affect j 's expected effort via the leaf agents, and the solution of (73) is monotone in \underline{a}^{r-1} , the so-defined functions indeed bound expected effort, $\underline{a}_t^r < a_t^{I,r}, \mathcal{A}_t < \bar{a}_t^r$. Moreover, the monotonicity of (73) together with $\underline{a}^1 \equiv 0$ implies that \underline{a}^r increases in r and so converges to some $\underline{a}^{\infty} = \{\underline{a}_t^{\infty}(\tau)\}_t$ which must then solve (20), so $\underline{a}_t^{\infty}(\tau) = \mathcal{A}_t(\tau)$ for all t . Similarly, $\bar{a}^r(\tau) \rightarrow \mathcal{A}(\tau)$. Since $\underline{a}_t^r(\tau), \bar{a}_t^r(\tau), \mathcal{A}_t(\tau)$ are all increasing and equi-Lipschitz in τ , the convergence is uniform in $\tau \in [0, \bar{\tau}]$, so we have proven (71). \square

Proof of parts (a) and (c). The only difference is the number of neighbors in (72) and (73). For n -regular directed network, we define $G_i^{I,r}$ as the event that i has n neighbors, n^2 second neighbors, ..., and n^r agents with distance r . Since i 's neighbor j has n additional neighbors k , (73) becomes $\dot{a}^r = (n\underline{a}^{r-1} - \underline{a}^r)(1 - \underline{a}^r)$ with boundary condition $\underline{a}_\tau^r = 1 - e^{-n\tau}$, so as $r \rightarrow \infty$, we obtain (17).

For triangular networks, i 's neighbor j shares one more, triangular neighbor j' with i , as well as $n - 2$ other neighbors k with distance two from i . Thus, (72) becomes

$$1 - \underline{a}_t^r = \frac{\Pr^{-i}(t < T_j, T_{j'}, t < T_k \forall k \in N_j \setminus \{i, j'\})}{\Pr^{-i}(t < T_j, T_{j'})} = \frac{\exp\left(-n\tau - \int_\tau^t ((n-2)\underline{a}_s^{r-1} + \underline{a}_s^r) ds\right)}{\exp\left(-2\tau - 2 \int_\tau^t \underline{a}_s^r ds\right)}.$$

To understand the integral in the numerator, after τ expected effort is 0 for j , \underline{a}_s^r for neighbor j' , and \underline{a}_s^{r-1} for each of the $n - 2$ second neighbors k . Thus, (73) becomes $\dot{a}^r = ((n-2)\underline{a}^{r-1} - \underline{a}^r)(1 - \underline{a}^r)$ with boundary condition $\underline{a}_\tau^r = 1 - e^{-(n-2)\tau}$, so as $r \rightarrow \infty$, we obtain (21).