Long-run Behavior of Macroeconomy with Several Types of Agents: Non-self averaging phenomena in Macroeconomics

Nasanao Aoki*
Department of Economics, Univ. of California, Los Angeles
Los Angeles Cal. 90095-1477
e-mail aoki@econ.ucla.edu, Fax 1-310-825-9528

January, 2006

Abstract

Macroeconomic literature generally does not pay attention to fluctuations of macroeocnomic variables in the long-run, the presumption of mainstream macroeconomicsts being that variances of whatever macroeconomic variables under consideration tend to zero as the size of the economy becomes large.

The notion of non-self averaging in physics is relevant here. If the coefficient of variation of some variable does not approach zero as some parameter related to the size of the model goes to infinity, then that variable is called not self-averaging. It implies that sample behavior of a non-self averaging variable is sufficiently random even with a very large number of samples.

In this paper we interpret the model of Feng and Hoppe (1998) as a model in macroeconomics, and show existence of non-self averaging variables. This model grows with a stochastic stream of entries of agents, some of whom initiate new sectors in economy. Sectors grow at a rate proportional to the number of sectors in the economy. We interpret their analysis of the model and show that the model exhibit non-self averaging behavior. In particular we show that a suitably normalized number of sectors in the economy, and also a suitably normalized size of sectors (number of agents in sectors of certain size) are non-self averaging.

We also show the existence of power-law behavior of the model.

Short Description of the Model

The model of Feng and Hoppe (1998) is in essence as follows. Assume that a stream, $I(t)$, of agents (resosurces in some basic units) arrive stochastically. It is modeled as a pure birth process with infinitesimal birth rate given by

$$lim\frac{1}{h}\Pr\{I(t+h)-I(t)=1|I(t)=k\} := \lambda_k = \alpha(k-1)+\beta,$$

*The author is grateful to M. Sibuya for his helps and for making available to me some of his unpublished notes.

where we define $\theta = \beta - \alpha$. New types of entrant start new sectors, while entrants of existing types join the cluster of agents (entrants) of the same type. A newly established sector grows also as a pure birth process. This is denoted by $x(t)$, $x(0) = 1$.

The total size of the economy, $N(t)$, is

$$N(t) = \sum_{i=1}^{I(t)} x_i(t - T_i),$$

where $0 < T_1 < T_2 < \cdots$ are the random arrival times of entrants. Here $x_i(t - T_i)$ is the size at time $t > T_i$ of the ith sector initiated by the ith arrival at time T_i with the infinitesimal growth rate

$$l_n = n - \alpha,$$

for $n = 1, 2, \ldots$. This imposes a constraint $\alpha < 1$.

Hence the size $N(t)$ is growing at the infinitesimal rate

$$\rho_n = \alpha(k-1) + \beta + \sum_{j=1}^{k}(n_j - \alpha) = n + \theta.$$

We note that θ is zero when $\alpha = \beta$. With α zero, the infinitesimal arrival rate of innovation is constant $\theta = \beta$. These are important special cases.

Short Summary of Analysis

Define K_n as the number of sectors when there are n agents of several types in the model. In other words, n agents form K_n random clusters.

We have an urn model with

$$P(K_{n+1} = k | K_n = k) = \frac{n - k\alpha}{n + \theta},$$

and

$$P(K_{n+1} = k + 1 | K_n = k) = \frac{\theta + k\alpha}{n + \theta}.$$

We can write a recurrence equation for $P(K_n = k)$ using these conditional probability expressions, and boundary conditions. The result is

$$P(K_n = k) = \frac{\theta^{[k,\alpha]}}{\theta^{[n]}} S_\alpha(n, k),$$

where

$$\theta^{[n]} = \theta(\theta + 1) \cdots (\theta + n - 1) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)},$$

and

$$\theta^{[k,\alpha]} = \theta(\theta + \alpha) \cdots (\theta + \alpha(n - 1)).$$

Note that these conditional probabilities reduce to those associated with the Ewens sampling formla with $\alpha = 0$. See Hoppe (1984) or Aoki (2002, Sec.10.6, A5) for example. Alternatively we can assume the form for $P(K_n =$

k) and obtain a recursion equation for $S_\alpha(n, k)$. By summing this probability over k we obtain

$$\theta^{[n]} = \sum_{k=1}^{n} S_\alpha, k)\theta^{[k,\alpha]}.$$

This generalizes

$$\theta^{[n]} = \sum_k c(n, k)\theta^k,$$

where $c(n, k)$ is unsigned Stirling numbers of the first kind to which the above expression reduces when α is set to zero.

This expression generalizes that of the case with $\alpha = 0$, that is the Ewens distribution, which is

$$P_\theta = \frac{\theta^k}{\theta^{[n}} c(n, k),$$

where $c(n, k)$ is the unsigned Stirling number of the first kind, see Aoki (2002,p.176).

That is why the expression $S_\alpha(n, k)$ is called a generalized Stirling number of the first kind.

Asymptotic Expressions

Feng and Hoppe simplified the proof of Pitman to establish

$$lim_{n\to\infty} \frac{K_n}{n^\alpha} = L, a.s.$$

where $0 < \alpha < 1$ and $\theta > -\alpha$, and where L is a random variable such that $\Pr(0 < L < 1) = 1$.

Yamato and Sibuya (2000) derived an expression for the expected value

$$E(K_n = \frac{\theta}{\alpha}[\frac{(\theta + \alpha)^{[n]}}{\theta^{[n]}} - 1].$$

Using the approximate expression for the ratio of gamma functions

$$\frac{\Gamma(\theta + \alpha + n)}{\Gamma(\theta)} \approx n^\alpha.$$

They also have the result

$$C.V(K_n/n^\alpha) = \sqrt{[\Gamma(\theta + 1 + \alpha) - \Gamma(\theta + 1)]/\Gamma(\theta + 1)}.$$

In other words K_n/n^α is non-self averaging.

In a similar vein, we can show that

$$E(\frac{ja_j}{n^\alpha}) \to^d L,$$

that is this expression is also non-self averaging.

There are results on power-laws, and local limit theorem that involve Mittag-Leffler probability densities. See Pitman (2002) and Yamato and Sibuya (2000).

Conclusion

We have exhibited a macroeconomic growth model in which several types of agents or sectors co-exist in the long-run. Iwai (1998) have considered long-run profits. Our approach gives firm analytical foundation to his results. Aoki, Nakano, Yoshida, and Ono (2005) have an example in the same vein in which two sectors composed of technical advanced and not so advanced firms interact and produce non-self averaging behavior in the long run.

References

Aoki, M.(2002). *Modeling Aggregate Behavior and Fluctuations in Economics: Stochastic Views of Interacting Agents* Cambridge Univ. Press, New york

—–, T. Nakano, G. Yoshida, T.Ono (2005), "Long-run patterns of interactions of two types of firms," (under preparation)

Hoppe, F. M. (1984). Pólya -like urns and the Ewens sampling formula. *J. Math. Biol.* **20**, 9-94.

Iwai, K. 1998)."Towards a disequilibrium Theory of Long-run Profits: Schumpeterian Perspective.

Pitman, J. (2002) *Combinatorial Stochastic Processes /}, lecture notes, summer school of probvability, St. Flour, France.*

Yamato, H., and M. Sibuya (2000), "Moment Relations of Stirling Numbers", Inform. and Cybernetics , **37***,xx-xx.*