A New Non-ergodic Endogenous Growth Model

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Abstract

A new non-ergodic, stochastic multi-sector endogenous growth model is presented. The coefficients of variations of the number of total sectors, and number of sectors of a given size all remain positive as the model size grow unboundedly.

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Intorduction

This paper discusses a new class of simple stochastic multi-sector growth models that are non-ergodic. As the time passes and the model size tend to infinity,[1] the coefficient of variation of the number of sectors in the model does not converge to zero, but remain positive. This indicates that the model is influenced by history, and is non self-averaging in the language of statistical physics. We show that the class of one-parameter Poisson-Dirichlet models, also known as Ewens models in population genetics is ergodic or self-averaging, but its extension to two-parameter Poisson-Dirichlet models by Pitman (1999) is not ergodic, i.e., non self-averaging. Feng and Hoppe (1998) has a model of similar structure. Their focus, however, is not on the limiting behavior of their model. They do not discuss how the coefficient of variation of the number of sectors behave as the model sizes approach infinity, which is the main topic of this correspondence.

The model

Consider an economy composed of several sectors. Different sectors are made up of different type of agents or productive units. The model sectors are thus heterogeneous. Counting the sizes of sectors in some basic units, when the economy is of size n, there are K_n sectors, that is K_n types of agents or productive units in the economy. The number K_n as well as the sizes of individual sectors, n_i, $i = 1, \ldots, K_n$, are random variables.

Time runs continuously. Over time, one of the existing sectors grows by one unit at rate which is proportional to $(n_i - \alpha)/(n + \theta)$, $i = 1, \ldots K_n$, where α is a parameter between 0 and 1, and θ is another parameter, $\theta + \alpha > 0$. The rate at which a new sector emerges in the economy is equal to $1 - \sum_{i=1}^{k}(n_i - \alpha)(n + \theta) = (\theta + k\alpha)(n + \theta)$. [2] The probability that a new sector emerges is expressible then as as

$$q_{\alpha,\theta}(n+1, k) = \frac{n - k\alpha}{n + \theta} q_{\alpha,\theta}(n, k) + \frac{\theta + (k-1)\alpha}{n + \theta} q_{\alpha,\theta}(n, k-1). \tag{1}$$

where $q_{\alpha,\theta}(n, k) := \Pr(K_n = k)$.

Eq. (1) states that the economy composed of k sectors increases in size by one unit either by one of the existing sectors growing by one unit, or by a new sector of size one emerging. We assume that a new sector always begins its life with a single unit. We can restate it as

$$\Pr(K_{n+1} = k+1 | K_n = k) = \frac{k\alpha + \theta}{n + \theta}, \text{and} \Pr(K_{n+1} = k | K_n = k) = \frac{n - k\alpha}{n + \theta}. \tag{2}$$

Note that more new sectors are likely to emerge in the economy as the numbers of sectors grow,.

[1] Thus, this model is different from those models which are inhabited by an infinite number of agents from the beginning.

[2] Our θ is $\beta - \alpha$ in Feng and Hoppe (1998).

Asymptotic Properties of the Number of Sectors

We next examine how the number of sectors behave as the size of the model grow unboudedly. We know that how it behaves when α is zero. It involves Stirling number of first kind, see Hoppe (1984) or Aoki (2002, 184). With positive α, the generalized Stirling number of the first kind, $c(n,k;\alpha)$, is involved in its expression. Dropping the subscripts α, θ from Pr, we write

$$\Pr(K_n = k) = \frac{\theta^{[k,\alpha]}}{\alpha^k \theta^{[n]}} c(n, k : \alpha), \tag{3}$$

where $\theta^{[k,\alpha]} := \theta(\theta+\alpha)\cdots(\theta+(k-1)\alpha)$, and $\theta^{[n]}; = \theta^{[n,1]} = \theta(\theta+1)\cdots(\theta+n-1)$. See Charalambides (2002) or Sibuya (2005) for their properties.

Define $S_\alpha(n,k) = c(n,k;\alpha)/\alpha^k$. It satisfies a recursion equation

$$S_\alpha(n,k) = (n - k\alpha) S_\alpha(n,k) + S_\alpha(n,k-1). \tag{4}$$

This function generalizes the power-series relation for $\theta^{[n]}$ $\theta^{[n]} = \sum_1^n c(n,k)\theta^k$, to $\theta^{[n]} = \sum S_\alpha(n,k)\theta^{[k,\alpha]}$. See Aoki (2002, 185) for example.

The Coefficients of Variaton

The number of sectors

Yamato and Sibuya (2000) have calculated moments of K_n^r , $r = 1, 2, \ldots$ recursively. For example they derive a recursion relation

$$E(K_{n+1}) = \frac{\theta}{n+\theta} + (1 + \frac{\alpha}{n+\theta}) E(K_n)$$

from which they obtain

$$E[\frac{K_n}{n^\alpha}] \sim \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)} \tag{5}$$

by applying the asymptotic expression for the Gamma function

$$\frac{\Gamma(n+a)}{\Gamma(n)} \sim n^a. \tag{6}$$

They also obtain the expression for the variance of K_n/n^α as

$$var(K_n/n^\alpha) \sim \frac{\Gamma(\theta+1}{\alpha^2}\gamma(\alpha,\theta), \tag{7}$$

where $\gamma(\alpha,\theta) := (\theta+\alpha)/(\Gamma(\theta+\alpha)) - \Gamma(\theta+1)/[\Gamma(\theta+\alpha)]^2$.

The expression for the coefficient of variation of K_n normalized by n^α then is given by

$$limC.V.(K_n/n^\alpha) = \frac{\Gamma(\theta+\alpha)}{\Gamma(\theta+1)}\sqrt{\gamma(\alpha,\theta)}. \tag{8}$$

Note that the expression $\gamma(\alpha,\theta)$ is zero when α is zero, and positive otherwise. We state this result as

Proposition The limit of the coefficient of varition is positive with positive α, and it is zero only with $\alpha = 0$.

In other words, models with $o < \alpha < 1$ are non self-averaging. Past events influence the path of the growth of this model, i.e., the model experiences non ergodic growth path.

The numbers of sectors of specified size

Let $a_j(n)$ be the number of sectors of size j when the size of the economy is n. From the definitions, note that $K_n = \sum_j a_j(n)$, and $\sum_j j a_j(n)_n$, where j ranges from 1 to n.

The results in Yamato and Sibuya can be used to show that the limit of the coefficient of variation of $a_j(n)/n^\alpha$ as n goes to infinity has the same limiting behavior as K_n/n^α, i.e., zero for $\alpha = 0$, and positive for $0 < \alpha < 1$.

Discussions

This short note shows that one-parameter Poisson-Dirichlet model, known as Ewens model in the population genetics literature, is ergodic, but its extension, two-parameter Poisson-Dirichlet models are not ergodic. The behavior of the latter models are history or sample-path dependent.

We discuss how this type of models is important in macroeconomics and finace modeling.

The two-parameter models are significant because their moments are related to those of the Mittag-Leffler distribution in a simple way, and as Darling-Kac theorem implies, Darling and Kac (1957), any analysis involving first passages, occupation times, waiting time distributions and the like are bound to involve the Mittag-Leffler functions. In other words, Mittag-Leffler functions are generic in examining model behaviors as the model sizes grow unboundedly.

One straightforward way to link the moments of K_n/n^α to the generalized Mittag-Leffler function $g_{\alpha,\theta}(x) := \frac{\Gamma(\theta+1)}{\Gamma(\mu+1)} x^\mu g_\alpha(x)$, where $\mu := \theta/\alpha$, and where g_α is a probability density with moments

$$\int_0^\infty x^p g_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)}, \tag{9}$$

for $p = 0, 1, \ldots,$. is to apply the method of moments, Breiman (1992 181-182).

Using the Laplace transform of Mittag-Leffler function, Mainardi and his associate and colleagues have discussed fractional calculus, and fractional master equations, with applications to financial problems in mind, Mainardi et al. For example see Scalas (2006).

The class of models in this letter may thus turns out to be important not only in finance but also in macroeconomics.

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