

TESTING REGRESSION MONOTONICITY IN ECONOMETRIC MODELS

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ABSTRACT. Monotonicity is a key qualitative prediction of a wide array of economic models derived via robust comparative statics. It is therefore important to design effective and practical econometric methods for testing this prediction in empirical analysis. This paper develops a general nonparametric framework for testing monotonicity of a regression function. Using this framework, a broad class of new tests is introduced, which gives an empirical researcher a lot of flexibility to incorporate ex ante information she might have. The paper also develops new methods for simulating critical values, which are based on the combination of a bootstrap procedure and new selection algorithms. These methods yield tests that have correct asymptotic size and are asymptotically nonconservative. It is also shown how to obtain an adaptive rate optimal test that has the best attainable rate of uniform consistency against models whose regression function has Lipschitz-continuous first-order derivatives and that automatically adapts to the unknown smoothness of the regression function. Simulations show that the power of the new tests in many cases significantly exceeds that of some prior tests, e.g. that of Ghosal, Sen, and Van der Vaart (2000). An application of the developed procedures to the dataset of Ellison and Ellison (2011) shows that there is some evidence of strategic entry deterrence in pharmaceutical industry where incumbents may use strategic investment to prevent generic entries when their patents expire.

1. INTRODUCTION

The concept of monotonicity plays an important role in economics. For example, monotone comparative statics has been a popular research topic in economic theory for many years; see, in particular, the seminal work on this topic by Milgrom and Shannon (1994) and Athey (2002). Matzkin (1994) mentions monotonicity as one of the most important implications of economic theory that can be used in econometric analysis. Given importance of monotonicity in economic theory, the natural question is whether we observe monotonicity in the data. Although there do exist some methods for testing monotonicity in statistics, there is no general theory that would suffice for empirical analysis in economics. For example, I am not aware of any test of monotonicity that would allow for multiple covariates. In addition, there are currently no

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published results on testing monotonicity that would allow for endogeneity of covariates. Such a theory is provided in this paper. In particular, this paper provides a general nonparametric framework for testing monotonicity of a regression function. Tests of monotonicity developed in this paper can be used to evaluate assumptions and implications of economic theory concerning monotonicity. In addition, as was recently noticed by Ellison and Ellison (2011), these tests can also be used to provide evidence of existence of certain phenomena related to strategic behavior of economic agents that are difficult to detect otherwise. Several motivating examples are presented in the next section.

I start with the model

$$Y = f(X) + \varepsilon \tag{1}$$

where Y is a scalar dependent random variable, X a scalar independent random variable, $f(\cdot)$ an unknown function, and ε an unobserved scalar random variable satisfying $E[\varepsilon|X] = 0$ almost surely. Later in the paper, I extend the analysis to cover models with multivariate and endogenous X 's. I am interested in testing the null hypothesis, \mathcal{H}_0 , that $f(\cdot)$ is nondecreasing against the alternative, \mathcal{H}_a , that there are x_1 and x_2 such that $x_1 < x_2$ but $f(x_1) > f(x_2)$. The decision is to be made based on the i.i.d. sample of size n , $\{X_i, Y_i\}_{1 \leq i \leq n}$ from the distribution of (X, Y) . I assume that $f(\cdot)$ is smooth but do not impose any parametric structure on it. I derive a theory that yields tests with the correct asymptotic size. I also show how to obtain consistent tests and how to obtain a test with the optimal rate of uniform consistency against classes of functions with Lipschitz-continuous first order derivatives. Moreover, the rate optimal test constructed in this paper is adaptive in the sense that it automatically adapts to the unknown smoothness of $f(\cdot)$.

This paper makes several contributions. First, I introduce a general framework for testing monotonicity. This framework allows me to develop a broad class of new tests, which also includes some existing tests as special cases. This gives a researcher a lot of flexibility to incorporate ex ante information she might have. Second, I develop new methods to simulate the critical values for these tests that in many cases yield higher power than that of existing methods. Third, I consider the problem of testing monotonicity in models with multiple covariates for the first time in the literature. As will be explained in the paper, these models are more difficult to analyze and require a different treatment in comparison with the case of univariate X . Finally, I consider models with endogenous X that are identified via instrumental variables, and I consider models with sample selection.

Providing a *general* framework for testing monotonicity is a difficult problem. The problem arises because different test statistics studied in this paper have different limit distributions and require different normalizations. Some of the test statistics have $N(0, 1)$ limit distribution, and some others have an extreme value limit distribution. Importantly, there are also many

test statistics that are “in between”, so that their distributions are far both from $N(0, 1)$ and from extreme value distributions, and so their asymptotic approximations are difficult to obtain. Moreover, and equally important, the limit distribution of the statistic that leads to the rate optimal and adaptive test is *unknown*. The main difficulty here is that the processes underlying the test statistic do not have an asymptotic equicontinuity property, and so classical functional central limit theorems, as presented for example in Van der Vaart and Wellner (1996) and Dudley (1999), do not apply. This paper addresses these issues and provide bootstrap critical values that are valid uniformly over a large class of different test statistics and different data generating processes. Two previous papers, Hall and Heckman (2000) and Ghosal, Sen, and van der Vaart (2000), used specific techniques to prove validity of their tests of monotonicity but it is difficult to generalize their techniques to make them applicable for other tests of monotonicity. In contrast, in this paper, I introduce a general approach that can be used to prove validity of many different tests of monotonicity. Other shape restrictions, such as concavity and super-modularity, can be tested by procedures similar to those developed in this paper.

Another problem is that test statistics studied in this paper have some asymptotic distribution when $f(\cdot)$ is constant but diverge if $f(\cdot)$ is strictly increasing. This discontinuity implies that for some sequences of models $f(\cdot) = f_n(\cdot)$, the limit distribution depends on the local slope function, which is an unknown infinite-dimensional nuisance parameter that can not be estimated consistently from the data. A common approach in the literature to solve this problem is to calibrate the critical value using the case when the type I error is maximized (the least favorable model), i.e. the model with constant $f(\cdot)$.¹ In contrast, I develop two selection procedures that estimate the set where $f(\cdot)$ is not strictly increasing, and then adjust the critical value to account for this set. The estimation is conducted so that no violation of the asymptotic size occurs. The critical values obtained using these selection procedures yield important power improvements in comparison with other tests if $f(\cdot)$ is strictly increasing over some subsets of the support of X . The first selection procedure, which is based on the one-step approach, is related to those developed in Chernozhukov, Lee, and Rosen (2013), Andrews and Shi (2010), and Chetverikov (2012), all of which deal with the problem of testing conditional moment inequalities. The second selection procedure is novel and is based on the step-down approach. It is somewhat related to methods developed in Romano and Wolf (2005a) and Romano and Shaikh (2010) but the details are rather different.

Further, an important issue that applies to nonparametric testing in general is how to choose a smoothing parameter for the test. In theory, the optimal smoothing parameter can be derived for many smoothness classes of functions $f(\cdot)$. In practice, however, the smoothness class that $f(\cdot)$ belongs to is usually unknown. I deal with this problem by employing the adaptive testing

¹The exception is Wang and Meyer (2011) who use the model with an isotonic estimate of $f(\cdot)$ to simulate the critical value. They do not prove whether their test maintains the required size, however.

approach. This allows me to obtain tests with good power properties when the information about smoothness of the function $f(\cdot)$ possessed by the researcher is absent or limited. More precisely, I construct a test statistic using many different weighting functions that correspond to many different values of the smoothing parameter so that the distribution of the test statistic is mainly determined by the optimal weighting function. I provide a basic set of weighting functions that yields a rate optimal and adaptive test and show how the researcher can change this set in order to incorporate ex ante information. Importantly, the approach taken in this paper does not require “under-smoothing”. This feature of my approach is important because, to the best of my knowledge, all procedures in the literature to achieve “under-smoothing” are ad hoc and do not have a sound theoretical justification.

The literature on testing monotonicity of a nonparametric regression function is quite large. The tests of Gijbels et. al. (2000) and Ghosal, Sen, and van der Vaart (2000) (from now on, GHJK and GSV, respectively) are based on the signs of $(Y_{i+k} - Y_i)(X_{i+k} - X_i)$. Hall and Heckman (2000) (from now on, HH) developed a test based on the slopes of local linear estimates of $f(\cdot)$. The list of other papers includes Schlee (1982), Bowman, Jones, and Gijbels (1998), Dumbgen and Spokoiny (2001), Durot (2003), Beraud, Huet, and Laurent (2005), and Wang and Meyer (2011). In a contemporaneous work, Lee, Song, and Whang (2011b) derive another approach to testing monotonicity based on L_p -functionals. The results in this paper complement the results of that paper. An advantage of their method is that the asymptotic distribution of their test statistic in the least favorable model under \mathcal{H}_0 turns out to be $N(0, 1)$, so that obtaining a critical value for their test is computationally very simple. A disadvantage of their method, however, is that their test is not adaptive. Results in this paper are also different from those in Romano and Wolf (2011) who also consider the problem of testing monotonicity. In particular, they assume that X is non-stochastic and discrete, which makes their problem semi-parametric and substantially simplifies proving validity of critical values, and they test the null hypothesis that $f(\cdot)$ is *not* weakly increasing against the alternative that it is weakly increasing. Lee, Linton, and Whang (2009) and Delgado and Escanciano (2010) derived tests of stochastic monotonicity, which is a related but different problem. Specifically, stochastic monotonicity means that the conditional cdf of Y given X , $F_{Y|X}(y, x)$, is (weakly) decreasing in x for any fixed y .

As an empirical application of the results developed in this paper, I consider the problem of detecting strategic entry deterrence in the pharmaceutical industry. In that industry, incumbents whose drug patents are about to expire can change their investment behavior in order to prevent generic entries after the expiration of the patent. Although there are many theoretically compelling arguments as to how and why incumbents should change their investment behavior (see, for example, Tirole (1988)), the empirical evidence is rather limited. Ellison and Ellison (2011) showed that, under certain conditions, the dependence of investment on market size should be monotone if no strategic entry deterrence is present. In addition, they noted that the entry

deterrence motive should be important in intermediate-sized markets and less important in small and large markets. Therefore, strategic entry deterrence might result in the non-monotonicity of the relation between market size and investment. Hence, rejecting the null hypothesis of monotonicity provides the evidence in favor of the existence of strategic entry deterrence. I apply the tests developed in this paper to Ellison and Ellison’s dataset and show that there is some evidence of non-monotonicity in the data. The evidence is rather weak, though.

The rest of the paper is organized as follows. Section 2 provides motivating examples. Section 3 describes the general test statistic and gives several methods to simulate the critical value. Section 4 contains the main results under high-level conditions when there are no additional covariates. Since in most practically relevant cases, the model also contains some additional covariates, Section 5 studies the cases of fully nonparametric and partially linear models with multiple covariates. Section 6 extends the analysis to cover the case where X is endogenous and identification is achieved via instrumental variables. Section 7 briefly explains how to test monotonicity in sample selection models. Section 8 presents a small Monte Carlo simulation study. Section 9 describes the empirical application. Section 10 concludes. All proofs are contained in the Appendix. In addition, Appendix A contains implementation details, and Appendix B is devoted to the verification of high-level conditions under primitive assumptions.

Notation. Throughout this paper, let $\{\epsilon_i\}$ denote a sequence of independent $N(0, 1)$ random variables that are independent of the data. The sequence $\{\epsilon_i\}$ will be used in bootstrapping critical values. The notation $i = \overline{1, n}$ is a shorthand for $i \in \{1, \dots, n\}$. For any set \mathcal{S} , I denote the number of elements in this set by $|\mathcal{S}|$.

2. MOTIVATING EXAMPLES

Many testable implications of economic theory are concerned with comparative statics analysis. These implications most often take the form of qualitative statements like “Increasing factor X will positively (negatively) affect response variable Y ”. The common approach to test such implications on the data is to look at the corresponding coefficient in the linear (or other parametric) regression. Relying on these strong parametric assumptions, however, can lead to highly misleading results. For example, the test based on the linear regression will not be consistent and the test based on the quadratic regression may severely over-reject if the model is misspecified. In contrast, this paper provides a class of tests that are valid without these strong parametric assumptions. The purpose of this section is to give three examples from the literature where tests developed in this paper can be applied.

1. Detecting strategic effects. Certain strategic effects, the existence of which is difficult to prove otherwise, can be detected by testing for monotonicity. An example on strategic entry deterrence in the pharmaceutical industry is described in the Introduction and is analyzed in

Section 9. Below I provide another example concerned with the problem of debt pricing. This example is based on Morris and Shin (2003). Consider a model where investors hold a collateralized debt. The debt will yield a fixed payment, say 1, in the future if it is rolled over and an underlying project is successful. Otherwise the debt will yield nothing (0). Alternatively, all investors have an option of not rolling over and getting the value of the collateral, $\kappa \in (0, 1)$, immediately. The probability that the project turns out to be successful depends on the fundamentals, θ , and on how many investors roll over. Specifically, assume that the project is successful if θ exceeds the proportion of investors who roll over. Under *global game reasoning*, if private information possessed by investors is sufficiently accurate, the project will succeed if and only if $\theta \geq \kappa$; see Morris and Shin (2003) for details. Then ex ante value of the debt is given by

$$V(\kappa) = \kappa \cdot P(\theta < \kappa) + 1 \cdot P(\theta \geq \kappa),$$

and the derivative of the ex ante debt value with respect to the collateral value is

$$\frac{dV(\kappa)}{d\kappa} = P(\theta < \kappa) - (1 - \kappa) \frac{dP(\theta < \kappa)}{d\kappa}$$

The first and second terms on the right hand side of this equation represent direct and strategic effects, respectively. The strategic effect represents coordination failure among investors. It arises because high value of the collateral leads investors to believe that many other investors will not roll over, and the project will not be successful even though the project is profitable ($\kappa < 1$). Morris and Shin (2004) argue that this effect is important for understanding anomalies in empirical implementation of the standard debt pricing theory of Merton (1974). A natural question is how to prove existence of this effect in the data. Note that in the absence of strategic effect, the relation between value of the debt and value of the collateral will be monotonically increasing. If strategic effect is sufficiently strong, however, it can cause non-monotonicity in this relation. Therefore, one can detect the existence of the strategic effect and coordination failure by testing whether conditional mean of the price of the debt given the value of the collateral is a monotonically increasing function. Rejecting the null hypothesis of monotonicity provides evidence in favor of the existence of the strategic effect and coordination failure.

2. Testing assumptions of treatment effect models. Monotonicity is often assumed in the econometrics literature on estimating treatment effects. A widely used econometric model in this literature is as follows. Suppose that we observe a sample of individuals, $i = \overline{1, n}$. Each individual has a random response function $y_i(t)$ that gives her response for each level of treatment $t \in T$. Let z_i and $y_i = y_i(z_i)$ denote the realized level of the treatment and the realized response, respectively (both are observable). The problem is how to derive inference on $E[y_i(t)]$. To address this problem, Manski and Pepper (2000) introduced assumptions of monotone treatment response, which imposes that $y_i(t_2) \geq y_i(t_1)$ whenever $t_2 \geq t_1$, and monotone treatment selection, which imposes that $E[y_i(t)|z_i = v]$ is increasing in v for all $t \in T$. The combination of these

assumptions yields a testable prediction. Indeed, for all $v_2 \geq v_1$,

$$\begin{aligned} E[y_i|z_i = v_2] &= E[y_i(v_2)|z_i = v_2] \\ &\geq E[y_i(v_1)|z_i = v_2] \geq E[y_i(v_1)|z_i = v_1] = E[y_i|z_i = v_1]. \end{aligned}$$

Since both z_i and y_i are observed, this prediction can be tested by the procedures developed in this paper. Note that the tests of stochastic monotonicity as described in the Introduction do not apply here since the testable prediction is monotonicity of the conditional mean function.

3. Testing the theory of the firm. A classical paper Holmstrom and Milgrom (1994) on the theory of the firm is built around the observation that in multi-task problems different incentive instruments are expected to be complementary to each other. Indeed, increasing an incentive for one task may lead the agent to spend too much time on that task ignoring other responsibilities. This can be avoided if incentives on different tasks are balanced with each other. To derive testable implications of the theory, Holmstrom and Milgrom study a model of industrial selling introduced in Anderson and Schmittlein (1984) where a firm chooses between an in-house agent and an independent representative who divide their time into four tasks: (i) direct sales, (ii) investing in future sales to customers, (iii) non-sale activities, such as helping other agents, and (iv) selling the products of other manufacturers. Proposition 4 in their paper states that under certain conditions, the conditional probability of having an in-house agent is a (weakly) increasing function of the marginal cost of evaluating performance and is a (weakly) increasing function of the importance of non-selling activities. These are hypotheses that can be directly tested on the data by procedures developed in this paper. This would be an important extension of linear regression analysis performed, for example, in Anderson and Schmittlein (1984) and Poppo and Zenger (1998). Again, note that the tests of stochastic monotonicity as described in the Introduction do not apply here.

3. THE TEST

3.1. The General Test Statistic. Recall that I consider a model given in equation (1), and the test should be based on the i.i.d. sample $\{X_i, Y_i\}_{1 \leq i \leq n}$ of n observations from the distribution of (X, Y) where X and Y are independent and dependent random variables, respectively. In this section and in Section 4, I assume that X is a scalar and there are no additional covariates Z . The case where additional covariates Z are present is considered in Section 5.

Let $Q(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a weighting function satisfying $Q(x_1, x_2) = Q(x_2, x_1)$ and $Q(x_1, x_2) \geq 0$ for all $x_1, x_2 \in \mathbb{R}$, and let

$$b = b(\{X_i, Y_i\}) = (1/2) \sum_{1 \leq i, j \leq n} (Y_i - Y_j) \text{sign}(X_j - X_i) Q(X_i, X_j)$$

be a test function. Since $Q(X_i, X_j) \geq 0$ and $E[Y_i|X_i] = f(X_i)$, it is easy to see that under \mathcal{H}_0 , that is, when the function $f(\cdot)$ is non-decreasing, $E[b] \leq 0$. On the other hand, if \mathcal{H}_0 is violated and there exist x_1 and x_2 on the support of X such that $x_1 < x_2$ but $f(x_1) > f(x_2)$, then there exists a function $Q(\cdot, \cdot)$ such that $E[b] > 0$ if $f(\cdot)$ is smooth. Therefore, b can be used to form a test statistic if I can find an appropriate function $Q(\cdot, \cdot)$. For this purpose, I will use the adaptive testing approach developed in statistics literature. Even though this approach has attractive features, it is almost never used in econometrics. A notable exception is Horowitz and Spokoiny (2001), who used it for specification testing.

The idea behind the adaptive testing approach is to choose $Q(\cdot, \cdot)$ from a large set of potentially useful weighting functions that maximizes the studentized version of b . Formally, let \mathcal{S}_n be some general set that depends on n and is (implicitly) allowed to depend on $\{X_i\}$, and for $s \in \mathcal{S}_n$, let $Q(\cdot, \cdot, s) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be some function satisfying $Q(x_1, x_2, s) = Q(x_2, x_1, s)$ and $Q(x_1, x_2, s) \geq 0$ for all $x_1, x_2 \in \mathbb{R}$. The functions $Q(\cdot, \cdot, s)$ are also (implicitly) allowed to depend on $\{X_i\}$. In addition, let

$$b(s) = b(\{X_i, Y_i\}, s) = (1/2) \sum_{1 \leq i, j \leq n} (Y_i - Y_j) \text{sign}(X_j - X_i) Q(X_i, X_j, s) \quad (2)$$

be a test function. Conditional on $\{X_i\}$, the variance of $b(s)$ is given by

$$V(s) = V(\{X_i\}, \{\sigma_i\}, s) = \sum_{1 \leq i \leq n} \sigma_i^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \quad (3)$$

where $\sigma_i = (E[\varepsilon_i^2|X_i])^{1/2}$ and $\varepsilon_i = Y_i - f(X_i)$. In general, σ_i 's are unknown, and have to be estimated from the data. Let $\hat{\sigma}_i$ denote some (not necessarily consistent) estimator of σ_i . Available estimators are discussed later in this section. Then the estimated conditional variance of $b(s)$ is

$$\hat{V}(s) = V(\{X_i\}, \{\hat{\sigma}_i\}, s) = \sum_{1 \leq i \leq n} \hat{\sigma}_i^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2. \quad (4)$$

The general form of the test statistic that I consider in this paper is

$$T = T(\{X_i, Y_i\}, \{\hat{\sigma}_i\}, \mathcal{S}_n) = \max_{s \in \mathcal{S}_n} \frac{b(\{X_i, Y_i\}, s)}{\sqrt{\hat{V}(\{X_i\}, \{\hat{\sigma}_i\}, s)}}. \quad (5)$$

Large values of T indicate that the null hypothesis is violated. Later in this section, I will provide methods for estimating quantiles of T under \mathcal{H}_0 and for choosing a critical value for the test based on the statistic T .

The set \mathcal{S}_n determines adaptivity properties of the test, that is the ability of the test to detect many different deviations from \mathcal{H}_0 . Indeed, each weighting function $Q(\cdot, \cdot, s)$ is useful for detecting some deviation, and so the larger is the set of weighting functions \mathcal{S}_n , the larger is

the number of different deviations that can be detected, and the higher is adaptivity of the test. In this paper, I allow for *exponentially* large (in the sample size n) sets \mathcal{S}_n . This implies that the researcher can choose a huge set of weighting functions, which allows her to detect large set of different deviations from \mathcal{H}_0 . The downside of the adaptivity, however, is that expanding the set \mathcal{S}_n increases the critical value, and thus decreases the power of the test against those alternatives that can be detected by weighting functions already included in \mathcal{S}_n . Fortunately, in many cases the loss of power is relatively small. In particular, it follows from Lemma D.1 and Borell's inequality (see Proposition A.2.1 in Van der Vaart and Wellner (1996)) that the critical values for test developed below are bounded from above by a slowly growing $C(\log p)^{1/2}$ for some $C > 0$ where $p = |\mathcal{S}_n|$, the number of elements in the set \mathcal{S}_n .

3.2. Typical Weighting Functions. Let me now describe typical weighting functions. Consider some compactly supported kernel function $K : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $K(x) \geq 0$ for all $x \in \mathbb{R}$. For convenience, I will assume that the support of K is $[-1, 1]$. In addition, let $s = (x, h)$ where x is a location point and h is a bandwidth value (smoothing parameter). Finally, define

$$Q(x_1, x_2, (x, h)) = |x_1 - x_2|^k K\left(\frac{x_1 - x}{h}\right) K\left(\frac{x_2 - x}{h}\right) \quad (6)$$

for some $k \geq 0$. I refer to this Q as a *kernel weighting function*.²

Assume that a test is based on kernel weighting functions and \mathcal{S}_n consists of pairs $s = (x, h)$ with many different values of x and h . To explain why this test has good adaptivity properties, consider figure 1 that plots two regression functions. Both $f_1(\cdot)$ and $f_2(\cdot)$ violate \mathcal{H}_0 but locations where \mathcal{H}_0 is violated are different. In particular, $f_1(\cdot)$ violates \mathcal{H}_0 on the interval $[x_1, x_2]$ and $f_2(\cdot)$ violates \mathcal{H}_0 on the interval $[x_3, x_4]$. In addition, $f_1(\cdot)$ is relatively less smooth than $f_2(\cdot)$, and $[x_1, x_2]$ is shorter than $[x_3, x_4]$. To have good power against $f_1(\cdot)$, \mathcal{S}_n should contain a pair (x, h) such that $[x - h, x + h] \subset [x_1, x_2]$. Indeed, if $[x - h, x + h]$ is not contained in $[x_1, x_2]$, then positive and negative values of the summand of b will cancel out yielding a low value of b . In particular, it should be the case that $x \in [x_1, x_2]$. Similarly, to have good power against $f_2(\cdot)$, \mathcal{S}_n should contain a pair (x, h) such that $x \in [x_3, x_4]$. Therefore, using many different values of x yields a test that adapts to the location of the deviation from \mathcal{H}_0 . This is spatial adaptivity. Further, note that larger values of h yield higher signal-to-noise ratio. So, given that $[x_3, x_4]$ is longer than $[x_1, x_2]$, the optimal pair (x, h) to test against $f_2(\cdot)$ has larger value of h than that to test against $f_1(\cdot)$. Therefore, using many different values of h results in adaptivity with respect to smoothness of the function, which, in turn, determines how fast its first derivative is varying and how long the interval of non-monotonicity is.

²It is possible to extend the definition of kernel weighting functions given in (6). Specifically, the term $|x_1 - x_2|^k$ in the definition can be replaced by general function $\bar{K}(x_1, x_2)$ satisfying $\bar{K}(x_1, x_2) \geq 0$ for all x_1 and x_2 . I thank Joris Pinkse for this observation.

If no ex ante information is available, I recommend using kernel weighting functions with $\mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\}, h \in H_n\}$ where $H_n = \{h = h_{\max} u^l : h \geq h_{\min}, l = 0, 1, 2, \dots\}$ and $h_{\max} = \max_{1 \leq i, j \leq n} |X_i - X_j|/2$. I also recommend setting $u = 0.5$, $h_{\min} = 0.4h_{\max}(\log n/n)^{1/3}$, and $k = 0$ or 1 . I refer to this \mathcal{S}_n as a *basic set* of weighting functions. This choice of parameters is consistent with the theory presented in this paper and has worked well in simulations. The basic set of weighting functions yields a rate optimal and adaptive test. The value of h_{\min} is selected so that the test function $b(s)$ for any given s uses no less than approximately 15 observations when $n = 100$ and X is distributed uniformly on some interval.

If some ex ante information is available, the general framework considered here gives the researcher a lot of flexibility to incorporate this information. In particular, if the researcher expects that the function $f(\cdot)$ is rather smooth, then the researcher can restrict the set \mathcal{S}_n by considering only pairs (x, h) with large values of h since in this case deviations from \mathcal{H}_0 , if present, are more likely to happen on long intervals. Moreover, if the smoothness of the function $f(\cdot)$ is known, one can find an optimal value of the smoothing parameter $\tilde{h} = \tilde{h}_n$ corresponding to this level of smoothness, and then consider kernel weighting functions with this particular choice of the bandwidth value, that is $\mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\}, h = \tilde{h}\}$. Further, if non-monotonicity is expected at one particular point \tilde{x} , one can consider kernel weighting functions with $\mathcal{S}_n = \{(x, h) : x = \tilde{x}, h = \tilde{h}\}$ or $\mathcal{S}_n = \{(x, h) : x = \tilde{x}, h \in H_n\}$ depending on whether smoothness of $f(\cdot)$ is known or not. More broadly, if non-monotonicity is expected on some interval \mathcal{X} , one can use kernel weighting functions with $\mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\} \cap \mathcal{X}, h \in \tilde{h}\}$ or $\mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\} \cap \mathcal{X}, h \in H_n\}$ again depending on whether smoothness of $f(\cdot)$ is known or not. Note that all these modifications will increase the power of the test because smaller sets \mathcal{S}_n yield lower critical values.

Another interesting choice of the weighting functions is

$$Q(x_1, x_2, s) = \sum_{1 \leq r \leq m} |x_1 - x_2|^k K\left(\frac{x_1 - x^r}{h}\right) K\left(\frac{x_2 - x^r}{h}\right)$$

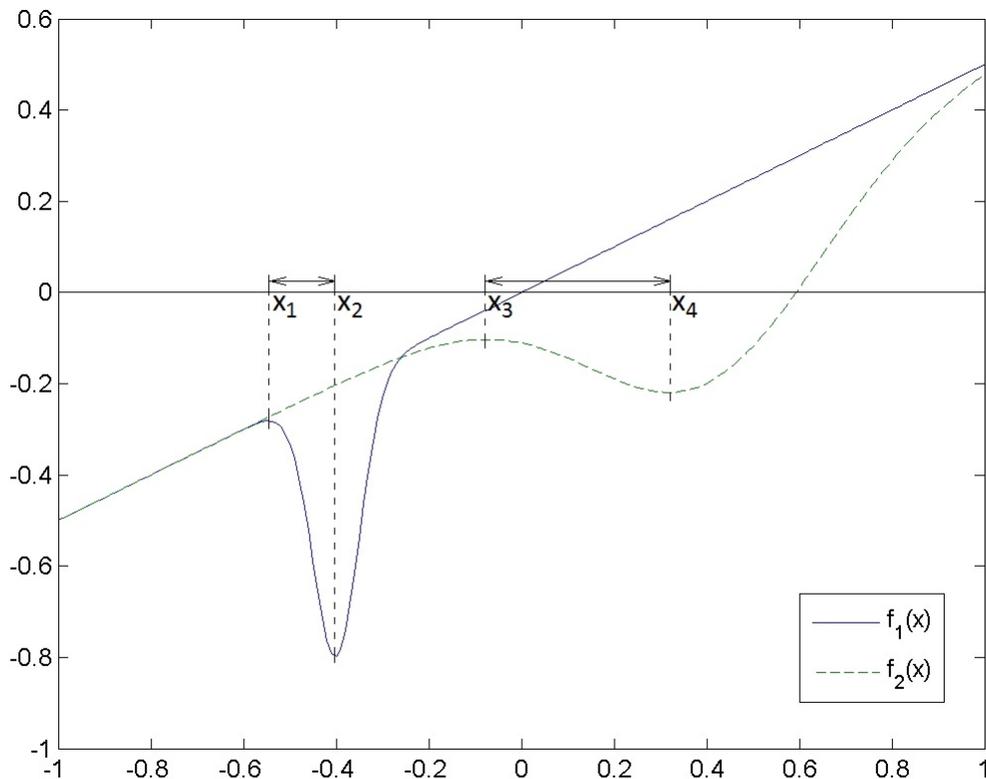
where $s = (x^1, \dots, x^m, h)$. These weighting functions are useful if the researcher expects multiple deviations from \mathcal{H}_0 .

3.3. Comparison with Other Known Tests. I will now show that the general framework described above includes the Hall and Heckman's (HH) test statistic and a slightly modified version of the Ghosal, Sen, and van der Vaart's (GSV) test statistic as special cases that correspond to different values of k in the definition of kernel weighting functions.

GSV use the following test function:

$$b(s) = (1/2) \sum_{1 \leq i, j \leq n} \text{sign}(Y_i - Y_j) \text{sign}(X_j - X_i) K\left(\frac{X_i - x}{h}\right) K\left(\frac{X_j - x}{h}\right),$$

FIGURE 1. Regression Functions Illustrating Different Deviations from \mathcal{H}_0



whereas setting $k = 0$ in equation (6) yields

$$b(s) = (1/2) \sum_{1 \leq i, j \leq n} (Y_i - Y_j) \text{sign}(X_j - X_i) \left(\frac{X_i - x}{h} \right) K \left(\frac{X_j - x}{h} \right), \quad (7)$$

and so the only difference is that I include the term $(Y_i - Y_j)$ whereas they use $\text{sign}(Y_i - Y_j)$. It will be shown in the next section that my test is consistent. On the other hand, I claim that GSV test is not consistent under the presence of conditional heteroscedasticity. Indeed, assume that $f(X_i) = -X_i$, and that ε_i is $-2X_i$ or $2X_i$ with equal probabilities. Then $(Y_i - Y_j)(X_j - X_i) > 0$ if and only if $(\varepsilon_i - \varepsilon_j)(X_j - X_i) > 0$, and so the probability of rejecting \mathcal{H}_0 for the GSV test is numerically equal to that in the model with $f(X_i) = 0$ for $i = \overline{1, n}$. But the latter probability does not exceed the size of the test. This implies that the GSV test is not consistent since it maintains the required size asymptotically. Moreover, they consider a unique non-stochastic value of h , which means that the GSV test is nonadaptive with respect to the smoothness of the function $f(\cdot)$.

Let me now consider the HH test. The idea of this test is to make use of local linear estimates of the slope of the function $f(\cdot)$. Using well-known formulas for the OLS regression, it is easy to show that the slope estimate of the function $f(\cdot)$ given the data $\{X_i, Y_i\}_{i=s_1+1}^{s_2}$ with $s_1 < s_2$ where $\{X_i\}_{i=1}^n$ is an increasing sequence is given by

$$b(s) = \frac{\sum_{s_1 < i \leq s_2} Y_i \sum_{s_1 < j \leq s_2} (X_i - X_j)}{(s_2 - s_1) \sum_{s_1 < i \leq s_2} X_i^2 - (\sum_{s_1 < i \leq s_2} X_i)^2}, \quad (8)$$

where $s = (s_1, s_2)$. Note that the denominator of (8) depends only on X_i 's, and so it disappears after studentization. In addition, simple rearrangements show that the numerator in (8) is up to the sign is equal to

$$(1/2) \sum_{1 \leq i, j \leq n} (Y_i - Y_j)(X_j - X_i) 1\{x - h \leq X_i \leq x + h\} 1\{x - h \leq X_j \leq x + h\} \quad (9)$$

for some x and h . On the other hand, setting $k = 1$ in equation (6) yields

$$b(s) = (1/2) \sum_{1 \leq i, j \leq n} (Y_i - Y_j)(X_j - X_i) K\left(\frac{X_i - x}{h}\right) K\left(\frac{X_j - x}{h}\right). \quad (10)$$

Noting that expression in (9) is proportional to that on the right hand side in (10) with $K(\cdot) = 1\{-1, +1\}(\cdot)$ implies that the HH test statistic is a special case of those studied in this paper.

3.4. Estimating σ_i . In practice, σ_i 's are usually unknown, and, hence, have to be estimated from the data. Let $\hat{\sigma}_i$ denote some estimator of σ_i . I provide results for two types of estimators. The first type of estimators is easier to implement but the second worked better in simulations.

First, σ_i can be estimated by the residual $\hat{\varepsilon}_i$. More precisely, let $\hat{f}(\cdot)$ be some uniformly consistent estimator of $f(\cdot)$ with a polynomial rate of consistency in probability, i.e. $\hat{f}(X_i) - f(X_i) = o_p(n^{-\kappa_1})$ uniformly over $i = \overline{1, n}$ for some $\kappa_1 > 0$, and let $\hat{\sigma}_i = \hat{\varepsilon}_i$ where $\hat{\varepsilon}_i = Y_i - \hat{f}(X_i)$. Note that $\hat{\sigma}_i$ can be negative. Clearly, $\hat{\sigma}_i$ is not a consistent estimator of σ_i . Nevertheless, as I will show in Section 4 that this estimator leads to valid inference. Intuitively, it works because the test statistic contains the weighted average sum of σ_i^2 over $i = \overline{1, n}$, and the estimation error averages out. To obtain a uniformly consistent estimator $\hat{f}(\cdot)$ of $f(\cdot)$, one can use a series method (see Newey (1997), theorem 1) or local polynomial method (see Tsybakov (2009), theorem 1.8). If one prefers kernel methods, it is important to use generalized kernels in order to deal with boundary effects when higher order kernels are used; see, for example, Muller (1991). Alternatively, one can choose \mathcal{S}_n so that boundary points are excluded from the test statistic. In addition, if the researcher decides to impose some parametric structure on the set of potentially possible heteroscedasticity functions, then parametric methods like OLS will typically give uniform consistency with κ_1 arbitrarily close to $1/2$.

The second way of estimating σ_i is to use a parametric or nonparametric estimator $\hat{\sigma}_i$ satisfying $\hat{\sigma}_i - \sigma_i = o_p(n^{-\kappa_2})$ uniformly over $i = \overline{1, n}$ for some $\kappa_2 > 0$. Many estimators of σ_i satisfy this

condition. Assume that the observations $\{X_i, Y_i\}_{i=1}^n$ are arranged so that $X_i \leq X_j$ whenever $i \leq j$. Then the estimator of Rice (1984), given by

$$\hat{\sigma} = \left(\frac{1}{2n} \sum_{i=1}^{n-1} (Y_{i+1} - Y_i)^2 \right)^{1/2}, \quad (11)$$

is \sqrt{n} -consistent if $\sigma_i = \sigma$ for all $i = \overline{1, n}$ and $f(\cdot)$ is piecewise Lipschitz-continuous.

The Rice estimator can be easily modified to allow for conditional heteroscedasticity. Choose a bandwidth value $b_n > 0$. For $i = \overline{1, n}$, let $J(i) = \{j = \overline{1, n} : |X_j - X_i| \leq b_n\}$. Let $|J(i)|$ denote the number of elements in $J(i)$. Then σ_i can be estimated by

$$\hat{\sigma}_i = \left(\frac{1}{2|J(i)|} \sum_{j \in J(i): j+1 \in J(i)} (Y_{j+1} - Y_j)^2 \right)^{1/2}. \quad (12)$$

I refer to (12) as a local version of Rice's estimator. An advantage of this estimator is that it is adaptive with respect to the smoothness of the function $f(\cdot)$. Proposition B.1 in Appendix B provides conditions that are sufficient for uniform consistency of this estimator with a polynomial rate. The key condition there is that $|\sigma_{j+1} - \sigma_j| \leq C|X_{j+1} - X_j|$ for some $C > 0$ and all $j = \overline{1, n-1}$. The intuition for consistency is as follows. Note that X_{j+1} is close to X_j . So, if the function f is continuous, then

$$Y_{j+1} - Y_j = f(X_{j+1}) - f(X_j) + \varepsilon_{j+1} - \varepsilon_j \approx \varepsilon_{j+1} - \varepsilon_j,$$

so that

$$\mathbb{E}[(Y_{j+1} - Y_j)^2 | \{X_i\}] \approx \sigma_{j+1}^2 + \sigma_j^2$$

since ε_{j+1} is independent of ε_j . Further, if b_n is sufficiently small, then $\sigma_{j+1}^2 + \sigma_j^2 \approx 2\sigma_i^2$ since $|X_{j+1} - X_i| \leq b_n$ and $|X_j - X_i| \leq b_n$, and so $\hat{\sigma}_i^2$ is close to σ_i^2 . Other available estimators are presented, for example, in Muller and Stadtmuller (1987), Fan and Yao (1998), Horowitz and Spokoiny (2001), Hardle and Tsybakov (2007), and Cai and Wang (2008).

3.5. Simulating the Critical Value. In this subsection, I provide three different methods for estimating quantiles of the null distribution of the test statistic T . These are plug-in, one-step, and step-down methods. All of these methods are based on the procedure known as the Wild bootstrap. The Wild bootstrap was introduced in Wu (1986) and used, among many others, by Liu (1988), Mammen (1993), Hardle and Mammen (1993), Horowitz and Spokoiny (2001), and Chetverikov (2012). See also Chernozhukov, Chetverikov, and Kato (2012). The three methods are arranged in terms of increasing power and computational complexity. The validity of all three methods is established in theorem 4.1. Recall that $\{\epsilon_i\}$ denotes a sequence of independent $N(0, 1)$ random variables that are independent of the data.

Plug-in Approach. Suppose that we want to obtain a test of level α . The plug-in approach is based on two observations. First, under \mathcal{H}_0 ,

$$b(s) = (1/2) \sum_{1 \leq i, j \leq n} (f(X_i) - f(X_j) + \varepsilon_i - \varepsilon_j) \text{sign}(X_j - X_i) Q(X_i, X_j, s) \quad (13)$$

$$\leq (1/2) \sum_{1 \leq i, j \leq n} (\varepsilon_i - \varepsilon_j) \text{sign}(X_j - X_i) Q(X_i, X_j, s) \quad (14)$$

since $Q(X_i, X_j, s) \geq 0$ and $f(X_i) \geq f(X_j)$ whenever $X_i \geq X_j$ under \mathcal{H}_0 , and so the $(1 - \alpha)$ quantile of T is bounded from above by the $(1 - \alpha)$ quantile of T in the model with $f(x) = 0$ for all $x \in \mathbb{R}$, which is the least favorable model under \mathcal{H}_0 . Second, it will be shown that the distribution of T asymptotically depends on the distribution of noise $\{\varepsilon_i\}$ only through $\{\sigma_i^2\}$. These two observations suggest that the critical value for the test can be obtained by simulating the conditional $(1 - \alpha)$ quantile of $T^* = T(\{X_i, Y_i^*\}, \{\hat{\sigma}_i\}, \mathcal{S}_n)$ given $\{X_i\}$, $\{\hat{\sigma}_i\}$, and \mathcal{S}_n where $Y_i^* = \hat{\sigma}_i \varepsilon_i$ for $i = \overline{1, n}$. This is called the plug-in critical value $c_{1-\alpha}^{PI}$. See Section A of the Appendix for detailed step-by-step instructions.

One-Step Approach. The test with the plug-in critical value is computationally rather simple. It has, however, poor power properties. Indeed, the distribution of T in general depends on $f(\cdot)$ but the plug-in approach is based on the least favorable regression function $f(x) = 0$ for all $x \in \mathbb{R}$, and so it is too conservative when $f(\cdot)$ is strictly increasing. More formally, suppose for example that kernel weighting functions are used, and that $f(\cdot)$ is strictly increasing in h -neighborhood of x_1 but is constant in h -neighborhood of x_2 . Let $s_1 = s(x_1, h)$ and $s_2 = s(x_2, h)$. Then $b(s_1)/(\widehat{V}(s_1))^{1/2}$ is no greater than $b(s_2)/(\widehat{V}(s_2))^{1/2}$ with probability approaching one. On the other hand, $b(s_1)/(\widehat{V}(s_1))^{1/2}$ is greater than $b(s_2)/(\widehat{V}(s_2))^{1/2}$ with nontrivial probability in the model with $f(x) = 0$ for all $x \in \mathbb{R}$, which is used to obtain $c_{1-\alpha}^{PI}$. Therefore, $c_{1-\alpha}^{PI}$ overestimates the corresponding quantile of T . The natural idea to overcome the conservativeness of the plug-in approach is to simulate a critical value using not all elements of \mathcal{S}_n but only those that are relevant for the given sample. In this paper, I develop two selection procedures that are used to decide what elements of \mathcal{S}_n should be used in the simulation. The main difficulty here is to make sure that the selection procedures do not distort the size of the test. The simpler of these two procedures is the one-step approach.

Let $\{\gamma_n\}$ be a sequence of positive numbers converging to zero, and let $c_{1-\gamma_n}^{PI}$ be the $(1 - \gamma_n)$ plug-in critical value. In addition, denote

$$\mathcal{S}_n^{OS} = \mathcal{S}_n^{OS}(\{X_i, Y_i\}, \{\hat{\sigma}_i\}, \mathcal{S}_n) = \{s \in \mathcal{S}_n : b(s)/(\widehat{V}(s))^{1/2} > -2c_{1-\gamma_n}^{PI}\}.$$

Then the one-step critical value $c_{1-\alpha}^{OS}$ is the conditional $(1 - \alpha)$ quantile of the simulated statistic $T^* = T(\{X_i, Y_i^*\}, \{\hat{\sigma}_i\}, \mathcal{S}_n^{OS})$ given $\{X_i\}$, $\{\hat{\sigma}_i\}$, and \mathcal{S}_n^{OS} where $Y_i^* = \hat{\sigma}_i \varepsilon_i$ for $i = \overline{1, n}$.³ Intuitively,

³If \mathcal{S}_n^{OS} turns out to be empty, assume that \mathcal{S}_n^{OS} consists of one randomly chosen element of \mathcal{S}_n .

the one-step critical value works because the weighting functions corresponding to elements of the set $\mathcal{S}_n \setminus \mathcal{S}_n^{OS}$ have an asymptotically negligible influence on the distribution of T under \mathcal{H}_0 . Indeed, it will be shown that the probability that at least one element s of \mathcal{S}_n such that

$$(1/2) \sum_{1 \leq i, j \leq n} (f(X_i) - f(X_j)) \text{sign}(X_j - X_i) Q(X_i, X_j, s) / (\widehat{V}(s))^{1/2} > -c_{1-\gamma_n}^{PI} \quad (15)$$

belongs to the set $\mathcal{S}_n \setminus \mathcal{S}_n^{OS}$ is at most $\gamma_n + o(1)$. On the other hand, the probability that at least one element s of \mathcal{S}_n such that inequality (15) does not hold for this element gives $b(s) / (\widehat{V}(s))^{1/2} > 0$ is again at most $\gamma_n + o(1)$. Since γ_n converges to zero, this suggests that the critical value can be simulated using only elements of \mathcal{S}_n^{OS} . In practice, one can set γ_n as a small fraction of α . For example, the Monte Carlo simulations presented in this paper use $\gamma_n = 0.01$ with $\alpha = 0.1$.⁴

Step-down Approach. The one-step approach, as the name suggests, uses only one step to cut out those elements of \mathcal{S}_n that have negligible influence on the distribution of T . It turns out that this step can be iterated using the step-down procedure and yielding second-order improvements in the power. The step-down procedures were developed in the literature on multiple hypothesis testing; see, in particular, Holm (1979), Romano and Wolf (2005a), Romano and Wolf (2005b), and Romano and Shaikh (2010). See also Lehmann and Romano (2005) for a textbook introduction. The use of step-down method in this paper, however, is rather different.

To explain the step-down approach, let me define the sequences $(c_{1-\gamma_n}^l)_{l=1}^\infty$ and $(\mathcal{S}_n^l)_{l=1}^\infty$. Set $c_{1-\gamma_n}^1 = c_{1-\gamma_n}^{OS}$ and $\mathcal{S}_n^1 = \mathcal{S}_n^{OS}$. Then for $l > 1$, let $c_{1-\gamma_n}^l$ be the conditional $(1 - \gamma_n)$ quantile of $T^* = T(\{X_i, Y_i^*\}, \{\widehat{\sigma}_i\}, \mathcal{S}_n^l)$ given $\{\widehat{\sigma}_i\}$ and \mathcal{S}_n^l where $Y_i^* = \widehat{\sigma}_i \epsilon_i$ for $i = \overline{1, n}$ and

$$\mathcal{S}_n^l = \mathcal{S}_n^l(\{X_i, Y_i\}, \{\widehat{\sigma}_i\}, \mathcal{S}_n) = \{s \in \mathcal{S}_n : b(s) / (\widehat{V}(s))^{1/2} > -c_{1-\gamma_n}^{PI} - c_{1-\gamma_n}^{l-1}\}.$$

It is easy to see that $(c_{1-\gamma_n}^l)_{l=1}^\infty$ is a decreasing sequence, and so $\mathcal{S}_n^l \supseteq \mathcal{S}_n^{l+1}$ for all $l \geq 1$. Since \mathcal{S}_n^1 is a finite set, $\mathcal{S}_n^{l(0)} = \mathcal{S}_n^{l(0)+1}$ for some $l(0) \geq 1$ and $\mathcal{S}_n^l = \mathcal{S}_n^{l+1}$ for all $l \geq l(0)$. Let $\mathcal{S}_n^{SD} = \mathcal{S}_n^{l(0)}$. Then the step-down critical value $c_{1-\alpha}^{SD}$ is the conditional $(1 - \alpha)$ quantile of $T^* = T(\{X_i, Y_i^*\}, \{\widehat{\sigma}_i\}, \mathcal{S}_n^{SD})$ given $\{X_i\}$, $\{\widehat{\sigma}_i\}$, and \mathcal{S}_n^{SD} where $Y_i^* = \widehat{\sigma}_i \epsilon_i$ for $i = \overline{1, n}$.

Note that $\mathcal{S}_n^{SD} \subset \mathcal{S}_n^{OS} \subset \mathcal{S}_n$, and so $c_\eta^{SD} \leq c_\eta^{OS} \leq c_\eta^{PI}$ for any $\eta \in (0, 1)$. This explains that the three methods for simulating the critical values are arranged in terms of increasing power.

⁴More formally, it is shown in the proof of Theorem 4.1 that the probability of rejecting \mathcal{H}_0 under \mathcal{H}_0 in large sample is bounded from above by $\alpha + 2\gamma_n$. This suggests that if the researcher does not agree to tolerate small size distortions, she can use the test with level $\tilde{\alpha} = \alpha - 2\gamma_n$ instead. On the other hand, I note that $\alpha + 2\gamma_n$ is only an *upper* bound on the probability of rejecting \mathcal{H}_0 , and in many cases the true probability of rejecting \mathcal{H}_0 is smaller than $\alpha + 2\gamma_n$.

4. THEORY UNDER HIGH-LEVEL CONDITIONS

This section describes the high-level assumptions used in the paper and presents the main results under these assumptions.

Let $c_1, C_1, \kappa_1, \kappa_2$, and κ_3 be strictly positive constants. The size properties of the test will be obtained under the following assumptions.

A1. $E[|\varepsilon_i|^4|X_i] \leq C_1$ and $\sigma_i \geq c_1$ for all $i = \overline{1, n}$.

This is a mild assumption on the moments of disturbances. The condition $\sigma_i \geq c_1$ for all $i = \overline{1, n}$ precludes the existence of super-efficient estimators.

Recall that the results in this paper are obtained for two types of estimators of σ_i . When $\hat{\sigma}_i = \hat{\varepsilon}_i = Y_i - \hat{f}(X_i)$ for some estimator $\hat{f}(\cdot)$ of $f(\cdot)$, I will assume

A2. (i) $\hat{\sigma}_i = Y_i - \hat{f}(X_i)$ for all $i = \overline{1, n}$ and (ii) $\hat{f}(X_i) - f(X_i) = o_p(n^{-\kappa_1})$ uniformly over $i = \overline{1, n}$.

This assumption is satisfied for many parametric and nonparametric estimators of $f(\cdot)$; see, in particular, Subsection 3.4. When $\hat{\sigma}_i$ is some consistent estimator of σ_i , I will assume

A3. $\hat{\sigma}_i - \sigma_i = o_p(n^{-\kappa_2})$ uniformly over $i = \overline{1, n}$.

See Subsection 3.4 for different available estimators. See also Proposition B.1 in Appendix B where Assumption A3 is proven for the local version of Rice's estimator.

A4. $(\hat{V}(s)/V(s))^{1/2} - 1 = o_p(n^{-\kappa_3})$ and $(V(s)/\hat{V}(s))^{1/2} - 1 = o_p(n^{-\kappa_3})$ uniformly over $s \in \mathcal{S}_n$.

This is a high-level assumption that is verified for kernel weighting functions under primitive conditions in Appendix B (Proposition B.2).

Let

$$A_n = \max_{s \in \mathcal{S}_n} \max_{1 \leq i \leq n} \left| \sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) / (V(s))^{1/2} \right|. \quad (16)$$

I refer to A_n as a sensitivity parameter. It provides an upper bound on how much any test function depends on a particular observation. Intuitively, approximation of the distribution of the test statistic is possible only if A_n is sufficiently small.

A5. (i) $nA_n^4(\log(pn))^7 = o_p(1)$ where $p = |\mathcal{S}_n|$, the number of elements in the set \mathcal{S}_n ; (ii) if A2 holds, then $\log p/n^{(1/4) \wedge \kappa_1 \wedge \kappa_3} = o_p(1)$, and if A3 holds, then $\log p/n^{\kappa_2 \wedge \kappa_3} = o_p(1)$.

This is a key growth assumption that restricts the choice of the weighting functions and, hence, the set \mathcal{S}_n . Note that this condition includes p only through $\log p$, and so it allows an exponentially large (in the sample size n) number of weighting functions. Proposition B.2 in Appendix

B provides an upper bound on A_n for kernel weighting functions, allowing me to verify this assumption under primitive conditions for the basic set of weighting functions.

Let \mathcal{M} be a class of models given by equation (1), regression function $f(\cdot)$, joint distribution of X and ε such that $E[\varepsilon|X] = 0$ almost surely, weighting functions $Q(\cdot, \cdot, s)$ for $s \in \mathcal{S}_n$, and estimators $\{\hat{\sigma}_i\}$ such that uniformly over this class, (i) Assumptions A1, A4, and A5 are satisfied, and (ii) either Assumption A2 or A3 is satisfied.⁵ For $M \in \mathcal{M}$, let $P_M(\cdot)$ denote the probability measure generated by the model M .

Theorem 4.1 (Size properties of the test). *Let $P = PI, OS, \text{ or } SD$. Let \mathcal{M}_0 denote the set of all models $M \in \mathcal{M}$ satisfying \mathcal{H}_0 . Then*

$$\inf_{M \in \mathcal{M}_0} P_M(T \leq c_{1-\alpha}^P) \geq 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

In addition, let \mathcal{M}_{00} denote the set of all models $M \in \mathcal{M}_0$ such that $f(x) = C$ for some constant C and all $x \in \mathbb{R}$. Then

$$\sup_{M \in \mathcal{M}_{00}} P(T \leq c_{1-\alpha}^P) = 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

Comment 4.1. (i) This theorem states that the Wild Bootstrap combined with the selection procedures developed in this paper yields valid critical values. Moreover, critical values are valid uniformly over the class of models \mathcal{M}_0 . The second part of the theorem states that the test is nonconservative in the sense that its level converges to the nominal level α .

(ii) The proof technique used in this theorem is based on *finite sample* approximations that are built on the results of Chernozhukov, Chetverikov, and Kato (2012) and Chernozhukov, Chetverikov, and Kato (2011). In particular, the validity of the bootstrap is established *without* referring to the asymptotic distribution of the test statistic.

(iii) The standard techniques from empirical process theory as presented, for example, in Van der Vaart and Wellner (1996) can not be used to prove the results of Theorem 4.1. The problem is that it is not possible to embed the process $\{b(s)/(V(s))^{1/2} : s \in \mathcal{S}_n\}$ into asymptotically equicontinuous process since, for example, when the basic set of kernel weighting functions is used, random variables $b(x_1, h)/(V(x_1, h))^{1/2}$ and $b(x_2, h)/(V(x_2, h))^{1/2}$ for fixed $x_1 < x_2$ become asymptotically independent as $h \rightarrow 0$.

(iv) To better understand the importance of the finite sample approximations used in the proof of this theorem, suppose that one would like to use *asymptotic* approximations based on the limit distribution of the test statistic T instead. The difficulty with this approach would be

⁵Assumptions A2, A3, A4, and A5 contain statements of the form $Z = o_p(n^{-\kappa})$ for some random variable Z and $\kappa > 0$. I say that these assumptions hold uniformly over a class of models if for any $C > 0$, $P(|Z| > Cn^{-\kappa}) = o(1)$ uniformly over this class. Note that this notion of uniformity is weaker than uniform convergence in probability; in particular, it applies to random variables defined on different probability spaces.

to *derive* the limit distribution of T . Indeed, note that the class of test statistics considered in this theorem is large and different limit distributions are possible. For example, if \mathcal{S}_n is a singleton with $Q(\cdot, \cdot, s)$ being a kernel weighting function, so that $s = (x, h)$, and x being fixed and $h = h_n$ converging to zero with an appropriate rate (this statistic is useful if the researcher wants to test monotonicity in a neighborhood of a particular point x), then $T \Rightarrow N(0, 1)$ where \Rightarrow denotes weak convergence. On the other hand, if kernel weighting functions are used but $\mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\}, h = h_n\}$ with $h = h_n$ converging to zero with an appropriate rate (this statistic is useful if the researcher wants to test monotonicity on the whole support of X but the smoothness of the function $f(\cdot)$ is known from the ex ante considerations), then it is possible to show that $a_n(T - b_n) \Rightarrow \mathcal{G}$ where \mathcal{G} is the Gumbel distribution for some a_n and b_n that are of order $(\log n)^{1/2}$. This implies that not only limit distributions vary among test statistics in the studied class but also appropriate normalizations are different. Further, note that the theorem also covers test statistics that are “in between” the two statistics described above (these are statistics with $\mathcal{S}_n = \{(x, h) : x \in \{X_{l(1)}, \dots, X_{l(k)}\}, h = h_n\}$ where $l(j) \in \{1, \dots, n\}$, $j = \overline{1, k}$, $k \in \{1, \dots, n\}$; these statistics are useful if the researcher wants to test monotonicity on some subset of the support of X). The distribution of these statistics can be far both from $N(0, 1)$ and from \mathcal{G} in finite samples for *any* sample size n . Finally, if the basic set of weighting functions is used, the limit distribution of the corresponding test statistic T is *unknown* in the literature, and one can guess that this distribution is quite complicated. In contrast, Theorem 4.1 shows that the critical values suggested in this paper are valid uniformly over the whole class of test statistics under consideration.

(v) Note that T asymptotically has a form of U-statistic. The analysis of such statistics typically requires a preliminary Hoeffding projection. An advantage of the approximation method used in this paper is that it applies directly to the test statistic with no need for the Hoeffding projection, which greatly simplifies the analysis.

(vi) To obtain a particular application of the general result presented in this theorem, assume that the basic set of weighting functions introduced in Subsection 3.2 is used. Suppose that Assumption A1 holds. In addition, suppose that either Assumption A2 or Assumption A3 holds. Then Assumptions A4 and A5 hold by Proposition B.2 in Appendix B (under mild conditions on $K(\cdot)$ stated in Proposition B.2), and so the result of Theorem 4.1 applies. Therefore, the basic set of weighting functions yields a test with the correct asymptotic size, and so it can be used for testing monotonicity. An advantage of this set is that, as will follow from Theorems 4.4 and 4.5, it gives an adaptive test with the best attainable rate of uniform consistency in the minimax sense against alternatives with regression functions that have Lipschitz-continuous first order derivatives.

(vii) The theorem can be extended to cover more general versions of the test statistic T . In particular, the theorem continues to hold if the test statistic T is replaced by

$$\tilde{T} = \tilde{T}(\{X_i, Y_i\}, \{\hat{\sigma}_i\}, \mathcal{S}_n) = \max_{s \in \mathcal{S}_n} \frac{w(\{X_i\}, s)b(\{X_i, Y_i\}, s)}{\sqrt{\hat{V}(\{X_i\}, \{\hat{\sigma}_i\}, s)}}$$

where for some $c, C > 0$, $w(\{X_i\}, \cdot) : \mathcal{S}_n \rightarrow [c, C]$ is the function that can be used to give unequal weights to different weighting functions (if \tilde{T} is used, then critical values should be calculated using $\tilde{T}(\cdot, \cdot, \cdot)$ as well). This gives the researcher additional flexibility to incorporate information on what weighting functions are more important ex ante.

(viii) Finally, the theorem can be extended by equicontinuity arguments to cover certain cases with infinite \mathcal{S}_n where maximum over infinite set of weighting functions in the test statistic T can be well approximated by maximum over some finite set of weighting functions. Note that since p in the theorem can be large, approximation is typically easy to achieve. In particular, if \mathcal{X} is the support of X , \mathcal{X} has positive Lebesgue measure, the distribution of X is absolutely continuous with respect to Lebesgue measure on \mathcal{X} , and the density of X is bounded below from zero and from above on \mathcal{X} , then the theorem holds with kernel weighting functions and $\mathcal{S}_n = \{(x, h) : x \in \mathcal{X}, h \in H_n\}$ where $H_n = \{h = h_{\max} u^l : h \geq h_{\min}, l = 0, 1, 2, \dots\}$, $h_{\max} = C$ and $h_{\min} = ch_{\max}(\log n/n)^{1/3}$ for some $c, C > 0$. I omit the proof for brevity. Note, however, that using this set of weighting functions would require knowledge of the support \mathcal{X} (or some method to estimate it). In addition, taking maximum in the test statistic T would require using some optimization methods whereas for the basic set of weighting functions, where the maximum is taken only over $Cn \log n$ points for some $C > 0$, calculating the test statistic T is computationally straightforward. \square

Let $s_l = \inf\{\text{support of } X\}$ and $s_r = \sup\{\text{support of } X\}$. To prove consistency of the test and to derive the rate of consistency against one-dimensional alternatives, I will also incorporate the following assumptions.

A6. For any interval $[x_1, x_2] \subset [s_l, s_r]$, $P(X \in [x_1, x_2]) > 0$.

This is a mild assumption stating that the support of X is the interval $[s_l, s_r]$ and that for any subinterval of the support, there is a positive probability that X belongs to this interval. Let c_2 and C_2 be strictly positive constants.

A7. For any interval $[x_1, x_2] \subset [s_l, s_r]$, w.p.a.1, there exists $s \in \mathcal{S}_n$ satisfying (i) the support of $Q(\cdot, \cdot, s)$ is contained in $[x_1, x_2]^2$, (ii) $Q(\cdot, \cdot, s)$ is bounded from above by C_2 , and (iii) there exist non-intersecting subintervals $[x_{l1}, x_{r1}]$ and $[x_{l2}, x_{r2}]$ of $[x_1, x_2]$ such that $Q(y_1, y_2, s) \geq c_2$ whenever $y_1 \in [x_{l1}, x_{r1}]$ and $y_2 \in [x_{l2}, x_{r2}]$.

This assumption is satisfied, for example, if Assumption A6 holds, kernel functions are used, $\mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\}, h \in H_n\}$ where $H_n = \{h = h_{\max} u_l : h \geq h_{\min}, l = 0, 1, 2, \dots\}$, h_{\max} is bounded below from zero, and $h_{\min} \rightarrow 0$.

Let \mathcal{M}_1 be a subset of \mathcal{M} consisting of all models satisfying Assumptions A6 and A7. Then

Theorem 4.2 (Consistency against fixed alternatives). *Let $P = PI, OS, \text{ or } SD$. Then for any model M from the class \mathcal{M}_1 such that $f(\cdot)$ is continuously differentiable and there exist $x_1, x_2 \in [s_l, s_r]$ such that $x_1 < x_2$ but $f(x_1) > f(x_2)$ (\mathcal{H}_0 is false),*

$$P_M(T \leq c_{1-\alpha}^P) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Comment 4.2. (i) This theorem shows that the test is consistent against any fixed continuously differentiable alternative.

(ii) To compare the critical values based on the selection procedures developed in this paper with the plug-in approach (no selection procedure), assume that $f(\cdot)$ is continuously differentiable and strictly increasing (\mathcal{H}_0 holds). Then an argument like that used in the proof of Theorem 2 shows that \mathcal{S}_n^{OS} and \mathcal{S}_n^{SD} will be singletons w.p.a.1, which means that $P\{c_{1-\alpha}^{OS} \leq C\} \rightarrow 1$ and $P\{c_{1-\alpha}^{SD} \leq C\} \rightarrow 1$ for some $C > 0$. On the other hand, it follows from the Sudakov-Chevet Theorem (see, for example, Theorem 2.3.5 in Dudley (1999)) that $P(c_{1-\alpha}^{PI} > C) \rightarrow 1$ for any $C > 0$. Finally, under Assumption A9, which is stated below, it follows from the proof of lemma 2.3.15 in Dudley (1999) that $P\{c_{1-\alpha}^{PI} > c\sqrt{\log n}\} \rightarrow 1$ for some $c > 0$. This implies that there exist sequences of alternatives against which the tests with the one-step and step-down critical values are consistent but the test with the plug-in critical value is not. \square

Theorem 4.3 (Consistency against one-dimensional alternatives). *Let $P = PI, OS, \text{ or } SD$. Consider any model M from the class \mathcal{M}_1 such that $f(\cdot)$ is continuously differentiable and there exist $x_1, x_2 \in [s_l, s_r]$ such that $x_1 < x_2$ but $f(x_1) > f(x_2)$ (\mathcal{H}_0 is false). Assume that for every sample size n , the true model M_n coincides with M except that the regression function has the form $f_n(x) = l_n f(x)$ for all $x \in \mathbb{R}$ and for some sequence $\{l_n\}$ of positive numbers converging to zero. Then*

$$P_{M_n}(T \leq c_{1-\alpha}^P) \rightarrow 0 \text{ as } n \rightarrow \infty$$

as long as $\log p = o_p(l_n^2 n)$.

Comment 4.3. (i) This theorem establishes the consistency of the test against one-dimensional local alternatives, which are often used in the literature to investigate the power of the test; see, for example, Andrews and Shi (2010), Lee, Song, and Whang (2011), and the discussion in Horowitz and Spokoiny (2001).

(ii) Suppose that the basic set of weighting functions is used. Then $\log p \leq C \log n$ for some $C > 0$, and so the test is consistent against one-dimensional local alternatives if $(\log n/n)^{1/2} = o(l_n)$.

(iii) Now suppose that kernel weighting functions are used but \mathcal{S}_n is a maximal subset of the basic set such that for any x_1, x_2, h satisfying $(x_1, h) \in \mathcal{S}_n$ and $(x_2, h) \in \mathcal{S}_n$, $|x_2 - x_1| > 2h$, and instead of $h_{\min} = 0.4h_{\max}(\log n/n)^{1/3}$ we have $h_{\min} \rightarrow 0$ arbitrarily slowly. Then the test is consistent against one-dimensional local alternatives if $n^{-1/2} = o(l_n)$ (more precisely, for any sequence l_n such that $n^{-1/2} = o(l_n)$, one can choose a sequence $h_{\min} = h_{\min,n}$ satisfying $h_{\min} \rightarrow 0$ sufficiently slowly so that the test is consistent). In words, this test is \sqrt{n} -consistent against such alternatives. I note however, that the practical value of this \sqrt{n} -consistency is limited because there is no guarantee that for any given sample size n and given deviation from \mathcal{H}_0 , weighting functions suitable for detecting this deviation are already included in the test statistic. In contrast, it will follow from Theorem 4.4 that the test based on the basic set of weighting functions does provide this guarantee. \square

Let c_3, C_3, c_4, C_4, c_5 be strictly positive constants. In addition, let $L > 0$, $\beta \in (0, 1]$, $k \geq 0$, and $h_n = (\log p/n)^{1/(2\beta+3)}$. To derive the uniform consistency rate against the classes of alternatives with Lipschitz derivatives, Assumptions A6 and A7 will be replaced by the following (stronger) conditions.

A8. For any interval $[x_1, x_2] \subset [s_l, s_r]$, $c_3(x_2 - x_1) \leq \mathbb{P}(X \in [x_1, x_2]) \leq C_3(x_2 - x_1)$.

This assumption is stronger than A6 in that it bounds the probabilities from above in addition to bounding probabilities from below and excludes mass points but is still often imposed in the literature.

A9. *W.p.a.1*, for all $[x_1, x_2] \subset [s_l, s_r]$ with $x_2 - x_1 = h_n$, there exists $s \in \mathcal{S}_n$ satisfying (i) the support of $Q(\cdot, \cdot, s)$ is contained in $[x_1, x_2]^2$, (ii) $Q(\cdot, \cdot, s)$ is bounded from above by $C_4 h_n^k$, and (iii) there exist non-intersecting subintervals $[x_{l1}, x_{r1}]$ and $[x_{l2}, x_{r2}]$ of $[x_1, x_2]$ such that $x_{r1} - x_{l1} \geq c_5 h_n$, $x_{r2} - x_{l2} \geq c_5 h_n$, $x_{l2} - x_{r1} \geq c_5 h_n$, and $Q(y_1, y_2, s) \geq c_4 h_n^k$ whenever $y_1 \in [x_{l1}, x_{r1}]$ and $y_2 \in [x_{l2}, x_{r2}]$.

This condition is stronger than Assumption A7; specifically, in Assumption A9, the qualifier “w.p.a.1” applies uniformly over all $[x_1, x_2] \subset [s_l, s_r]$ with $x_2 - x_1 = h_n$. Proposition B.3 shows that Assumption A9 is satisfied under mild conditions on the kernel $K(\cdot)$ if Assumption A8 holds and the basic set of weighting functions is used.

Let $f^{(1)}(\cdot)$ denote the first derivative of $f(\cdot)$.

A10. For any $x_1, x_2 \in [s_l, s_r]$, $|f^{(1)}(x_1) - f^{(1)}(x_2)| \leq L|x_1 - x_2|^\beta$.

This is a smoothness condition that requires that the regression function is sufficiently well-behaved.

Let \mathcal{M}_2 be the subset of \mathcal{M} consisting of all models satisfying Assumptions A8, A9, and A10. The following theorem gives the uniform rate of consistency.

Theorem 4.4 (Uniform consistency rate). *Let $P = PI, OS, \text{ or } SD$. Consider any sequence of positive numbers $\{l_n\}$ such that $l_n \rightarrow \infty$, and let \mathcal{M}_{2n} denote the subset of \mathcal{M}_2 consisting of all models such that the regression function f satisfies $\inf_{x \in [s_l, s_r]} f^{(1)}(x) < -l_n(\log p/n)^{\beta/(2\beta+3)}$. Then*

$$\sup_{M \in \mathcal{M}_{2n}} \mathbb{P}_M(T \leq c_{1-\alpha}^P) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Comment 4.4. (i) Theorem 4.4 gives the rate of uniform consistency of the test against classes of functions with Lipschitz-continuous first order derivative with Lipschitz constant L and order β . Importance of *uniform* consistency against sufficiently large classes of alternatives like those considered here was previously emphasized in Horowitz and Spokoiny (2001). Intuitively, it guarantees that there are no reasonable alternatives against which the test has low power if the sample size is sufficiently large.

(ii) Theorem 4.4 shows that plug-in, one-step, and step-down critical values yield tests with the same rate of uniform consistency. Nonetheless, it does not mean that the selection procedures used in one-step and step-down critical values yield no power improvement in comparison with plug-in critical value. Specifically, it was shown in Comment 4.2 that there exist sequences of alternatives against which tests with one-step and step-down critical values are consistent but the test with the plug-in critical value is not.

(iii) Suppose that \mathcal{S}_n consists of the basic set of weighting functions. In addition, suppose that Assumptions A1 and A8 are satisfied. Further, suppose that either Assumption A2 or A3 is satisfied. Then Assumptions A4 and A5 hold by Proposition B.2 and Assumption A9 holds by Proposition B.3 under mild conditions on the kernel $K(\cdot)$. Then Theorem 4.4 implies that the test with this set of weighting functions is uniformly consistent against classes of functions with Lipschitz-continuous first order derivative with Lipschitz order β whenever $\inf_{x \in [s_l, s_r]} f_n^{(1)}(x) < -l_n(\log n/n)^{\beta/(2\beta+3)}$ for some $l_n \rightarrow \infty$. On the other hand, it will be shown in Theorem 4.5 that no test can be uniformly consistent against models with $\inf_{x \in [s_l, s_r]} f_n^{(1)}(x) > -C(\log n/n)^{\beta/(2\beta+3)}$ for some sufficiently small $C > 0$ if it controls size, at least asymptotically. Therefore, the test based on the basic set of weighting functions is rate optimal in the minimax sense.

(iv) Note that the test is rate optimal in the minimax sense simultaneously against classes of functions with Lipschitz-continuous first order derivative with Lipschitz order β for all $\beta \in (0, 1]$. In addition, implementing the test does not require the knowledge of β . For these reasons, the test with the basic set of weighting functions is called adaptive and rate optimal. \square

To conclude this section, I present a theorem that gives a lower bound on the possible rates of uniform consistency against the class \mathcal{M}_2 so that no test that maintains asymptotic size can have a faster rate of uniform consistency. Let $\psi = \psi(\{X_i, Y_i\})$ be a generic test. In other words,

$\psi(\{X_i, Y_i\})$ is the probability that the test rejects upon observing the data $\{X_i, Y_i\}$. Note that for any deterministic test $\psi = 0$ or 1 .

Theorem 4.5 (Lower bound on possible consistency rates). *For any test ψ satisfying $E_M[\psi] \leq \alpha + o(1)$ as $n \rightarrow \infty$ for all models $M \in \mathcal{M}_2$ such that \mathcal{H}_0 holds, there exists a sequence of models $M = M_n$ belonging to the class \mathcal{M}_2 such that $f(\cdot) = f_n(\cdot)$ satisfies $\inf_{x \in [s_l, s_r]} f_n^{(1)}(x) < -C(\log n/n)^{\beta/(2\beta+3)}$ for some sufficiently small constant $C > 0$ and $E_{M_n}[\psi] \leq \alpha + o(1)$ as $n \rightarrow \infty$. Here $E_{M_n}[\cdot]$ denotes the expectation under the distributions of the model M_n .*

Comment 4.5. This theorem shows that no test can be uniformly consistent against models with $\inf_{x \in [s_l, s_r]} f_n^{(1)}(x) > -C(\log n/n)^{\beta/(2\beta+3)}$ for some sufficiently small $C > 0$ if it controls size, at least asymptotically. \square

5. MODELS WITH MULTIPLE COVARIATES

Most empirical studies contain additional covariates that should be controlled for. In this section, I extend the results presented in Section 4 to allow for this possibility. I consider cases of both nonparametric and partially linear models. For brevity, I will only consider the results concerning size properties of the test. The power properties can be obtained using the arguments closely related to those used in Theorems 4.2, 4.3, and 4.4.

5.1. Multivariate Nonparametric Model. In this subsection, I consider a general nonparametric regression model, so that the model is given by

$$Y = f(X, Z) + \varepsilon \tag{17}$$

where Y is a scalar dependent random variable, X a scalar independent random variable, Z a vector in \mathbb{R}^d of additional independent random variables that should be controlled for, $f(\cdot)$ an unknown function, and ε an unobserved scalar random variable satisfying $E[\varepsilon|X, Z] = 0$ almost surely.

Let S_z be some subset of \mathbb{R}^d . The null hypothesis, \mathcal{H}_0 , to be tested is that for any $x_1, x_2 \in \mathbb{R}$ and $z \in S_z$, $f(x_1, z) \leq f(x_2, z)$ whenever $x_1 \leq x_2$. The alternative, \mathcal{H}_a , is that there are $x_1, x_2 \in \mathbb{R}$ and $z \in S_z$ such that $x_1 < x_2$ but $f(x_1, z) > f(x_2, z)$. The decision is to be made based on the i.i.d. sample of size n , $\{X_i, Z_i, Y_i\}_{1 \leq i \leq n}$ from the distribution of (X, Z, Y) .

The choice of the set S_z is up to the researcher and has to be made depending on the hypothesis to be tested. For example, if $S_z = \mathbb{R}^d$, then \mathcal{H}_0 means that the function $f(\cdot)$ is increasing in the first argument for all values of the second argument. If the researcher is interested in one particular value, say z_0 , then she can set $S_z = z_0$, which will mean that under \mathcal{H}_0 , the function $f(\cdot)$ is increasing in the first argument when the second argument equals z_0 .

The advantage of the nonparametric model studied in this subsection is that it is fully flexible and, in particular, allows for heterogeneous effects of X on Y . On the other hand, the nonparametric model suffers from the curse of dimensionality and may result in tests with low power if the researcher has many additional covariates. In this case, it might be better to consider the partially linear model studied below.

To define the test statistic, let \mathcal{S}_n be some general set that depends on n and is (implicitly) allowed to depend on $\{X_i, Z_i\}$. In addition, let $z : \mathcal{S}_n \rightarrow S_z$ and $\ell : \mathcal{S}_n \rightarrow (0, \infty)$ be some functions, and for $s \in \mathcal{S}_n$, let $Q(\cdot, \cdot, \cdot, \cdot, s) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be weighting functions satisfying

$$Q(x_1, x_2, z_1, z_2, s) = \bar{Q}(x_1, x_2, s) \bar{K} \left(\frac{z_1 - z(s)}{\ell(s)} \right) \bar{K} \left(\frac{z_2 - z(s)}{\ell(s)} \right)$$

for all x_1, x_2, z_1 , and z_2 where the functions $\bar{Q}(\cdot, \cdot, s)$ satisfy $\bar{Q}(x_1, x_2, s) = \bar{Q}(x_2, x_1, s)$ and $\bar{Q}(x_1, x_2, s) \geq 0$ for all x_1 and x_2 . For example, $\bar{Q}(\cdot, \cdot, s)$ can be a kernel weighting function. The functions $Q(\cdot, \cdot, \cdot, \cdot, s)$ are also (implicitly) allowed to depend on $\{X_i, Z_i\}$. Here $\bar{K} : \mathbb{R}^d \rightarrow \mathbb{R}$ is some positive compactly supported auxiliary kernel function, and $\ell(s), s \in \mathcal{S}_n$, are auxiliary bandwidth values. Intuitively, $Q(\cdot, \cdot, \cdot, \cdot, s)$ are local-in- $z(s)$ weighting functions. It is important here that the auxiliary bandwidth values $\ell(s)$ depend on s . For example, if kernel weighting functions are used, so that $s = (x, h, z, \ell)$, then one has to choose $h = h(s)$ and $\ell = \ell(s)$ so that $nh\ell^{d+2} = o_p(1/\log p)$ and $1/(nh\ell^d) \leq Cn^{-c}$ w.p.a.1 for some $c, C > 0$ uniformly over all $s = (x, h, z, \ell) \in \mathcal{S}_n$; see discussion below.

Further, let

$$b(s) = b(\{X_i, Z_i, Y_i\}, s) = (1/2) \sum_{1 \leq i, j \leq n} (Y_i - Y_j) \text{sign}(X_j - X_i) Q(X_i, X_j, Z_i, Z_j, s) \quad (18)$$

be a test function. Conditional on $\{X_i, Z_i\}$, the variance of $b(s)$ is given by

$$V(s) = V(\{X_i, Z_i\}, \{\sigma_i\}, s) = \sum_{1 \leq i \leq n} \sigma_i^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, Z_i, Z_j, s) \right)^2 \quad (19)$$

where $\sigma_i = (\mathbb{E}[\varepsilon_i^2 | X_i, Z_i])^{1/2}$, and estimated variance is

$$\hat{V}(s) = V(\{X_i, Z_i\}, \{\hat{\sigma}_i\}, s) = \sum_{1 \leq i \leq n} \hat{\sigma}_i^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, Z_i, Z_j, s) \right)^2 \quad (20)$$

where $\hat{\sigma}_i$ is some estimator of σ_i . Then the test statistic is

$$T = T(\{X_i, Z_i, Y_i\}, \{\hat{\sigma}_i\}, \mathcal{S}_n) = \max_{s \in \mathcal{S}_n} \frac{b(\{X_i, Z_i, Y_i\}, s)}{\sqrt{\hat{V}(\{X_i, Z_i\}, \{\hat{\sigma}_i\}, s)}}.$$

Large values of T indicate that \mathcal{H}_0 is violated. The critical value for the test can be calculated using any of the methods described in Section 3. For example, the plug-in critical value is defined

as the conditional $(1 - \alpha)$ quantile of $T^* = T(\{X_i, Z_i, Y_i^*\}, \{\widehat{\sigma}_i\}, \mathcal{S}_n)$ given $\{X_i, Z_i\}$, $\{\widehat{\sigma}_i\}$, and \mathcal{S}_n where $Y_i^* = \widehat{\sigma}_i \epsilon_i$ for $i = \overline{1, n}$.

Let $c_{1-\alpha}^{PI}$, $c_{1-\alpha}^{OS}$, and $c_{1-\alpha}^{SD}$ denote the plug-in, one-step, and step-down critical values, respectively. In addition, let

$$A_n = \max_{s \in \mathcal{S}_n} \max_{1 \leq i \leq n} \left| \sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, Z_i, Z_j, s) / (V(s))^{1/2} \right| \quad (21)$$

be a sensitivity parameter. Recall that $p = |\mathcal{S}_n|$. To prove the result concerning multivariate nonparametric model, I will impose the following condition.

A11. (i) $\ell(s) \sum_{1 \leq i, j \leq n} Q(X_i, X_j, Z_i, Z_j, s) / (V(s))^{1/2} = o_p(1/\sqrt{\log p})$ uniformly over $s \in \mathcal{S}_n$, and (ii) the regression function f has uniformly bounded first order partial derivatives.

Discussion of this assumption is given below. In addition, I will need to modify Assumptions A1, A2, and A5:

A1' (i) $P(|\varepsilon| \geq u | X, Z) \leq \exp(-u/C_1)$ for all $u > 0$ and $\sigma_i \geq c_1$ for all $i = \overline{1, n}$.

A2' (i) $\widehat{\sigma}_i = Y_i - \widehat{f}(X_i, Z_i)$ for all $i = \overline{1, n}$ and (ii) $\widehat{f}(X_i, Z_i) - f(X_i, Z_i) = o_p(n^{-\kappa_1})$ uniformly over $i = \overline{1, n}$.

A5' (i) $A_n (\log(pn))^{7/2} = o_p(1)$, (ii) if A2' holds, then $\log p / n^{(1/4) \wedge \kappa_1 \wedge \kappa_3} = o_p(1)$, and if A3 holds, then $\log p / n^{\kappa_2 \wedge \kappa_3} = o_p(1)$.

Assumption A1' imposes the restriction that ε_i 's have sub-exponential tails, which is stronger than Assumption A1. It holds, for example, if ε_i 's have normal distribution or ε_i 's are uniformly bounded in absolute value. Assumption A2' is a simple extension of Assumption A2 to account for the vector of additional covariates Z . Further, assume that $\bar{Q}(\cdot, \cdot, s)$ is a kernel weighting function for all $s \in \mathcal{S}_n$, so that $s = (x, h, z, \ell)$, the joint density of X and Z is bounded below from zero and from above on its support, and $h \geq ((\log n)^2 / (Cn))^{1/(d+1)}$ and $\ell \geq ((\log n)^2 / (Cn))^{1/(d+1)}$ w.p.a.1 for some $C > 0$ uniformly over all $s = (x, h, z, \ell) \in \mathcal{S}_n$. Then it follows as in the proof of Proposition B.2, that A11-i holds if $nh\ell^{d+2} = o_p(1/\log p)$ uniformly over all $s = (x, h, z, \ell) \in \mathcal{S}_n$ and that A5'-i holds if $1/(nh\ell^d) \leq Cn^{-c}$ w.p.a.1 for some $c, C > 0$ uniformly over all $s = (x, h, z, \ell) \in \mathcal{S}_n$.

The key difference between the multivariate case studied in this section and univariate case studied in Section 4 is that now it is not necessarily the case that $E[b(s)] \leq 0$ under \mathcal{H}_0 . The reason is that it can be the case under \mathcal{H}_0 that $f(x_1, z_1) > f(x_2, z_2)$ even if $x_1 < x_2$ unless $z_1 = z_2$. This yields a bias term in the test statistic. Assumption A11 ensures that this bias is asymptotically negligible relative to the concentration rate of the test statistic. The difficulty, however, is that this assumption contradicts the condition $nA_n^4 (\log(pn))^{7/2} = o_p(1)$ imposed in Assumption A5 and

used in the theory for the case when Z is absent. Indeed, under the assumptions specified in the paragraph above, the condition $nA_n^4(\log(pn))^7 = o_p(1)$ requires $1/(nh^2\ell^{2d}) = o_p(1)$ uniformly over all $s = (x, h, z, \ell) \in \mathcal{S}_n$, which is impossible if $nh\ell^{d+2} = o_p(1/\log p)$ uniformly over all $s = (x, h, z, \ell) \in \mathcal{S}_n$ and $d \geq 2$. To deal with this problem, I have to relax Assumption A5. This in turn requires imposing stronger conditions on the moments of ε . For these reasons, I replace Assumption A1 by A1'. This allows me to apply a powerful method developed in Chernozhukov, Chetverikov, and Kato (2012) and replace Assumption A5 by A5'.

Let \mathcal{M}_{NP} denote the set of models given by equation (17), function $f(\cdot)$, joint distribution of X , Z , and ε satisfying $E[\varepsilon|X, Z] = 0$ almost surely, weighting functions $Q(\cdot, \cdot, \cdot, \cdot, s)$ for $s \in \mathcal{S}_n$ (possibly depending on X_i 's and Z_i 's), and estimators $\{\hat{\sigma}_i\}$ such that uniformly over this class, (i) Assumptions A1', A4, A5', and A11 are satisfied (where $V(s)$, $\hat{V}(s)$, and A_n are defined in (19), (20), and (21), respectively), and (ii) either Assumption A2' or A3 is satisfied. The following theorem shows that the test in the multivariate nonparametric model controls asymptotic size.

Theorem 5.1 (Size properties in the multivariate nonparametric model). *Let $P = PI, OS, \text{ or } SD$. Let $\mathcal{M}_{NP,0}$ denote the set of all models $M \in \mathcal{M}_{NP}$ satisfying \mathcal{H}_0 . Then*

$$\inf_{M \in \mathcal{M}_{NP,0}} \mathbb{P}_M(T \leq c_{1-\alpha}^P) \geq 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

In addition, let $\mathcal{M}_{NP,00}$ denote the set of all models $M \in \mathcal{M}_{NP,0}$ such that $f(x) = C$ for some constant C and all $x \in \mathbb{R}$. Then

$$\sup_{M \in \mathcal{M}_{NP,00}} \mathbb{P}_M(T \leq c_{1-\alpha}^P) = 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

Comment 5.1. (i) The result of this theorem is new and I am not aware of any similar or related result in the literature. Here I briefly comment on difficulties arising if one tries to obtain a result like that in Theorem 5.1 by applying proof techniques that were previously used in the literature for the model where Z is absent. The approach of Ghosal, Sen, and van der Vaart (2000) consists of first providing a Gaussian coupling (strong approximation) for the whole process $\{b(s)/\hat{V}(s) : s \in \mathcal{S}_n\}$ and then employing results of the extreme value theory of Gaussian processes with one-dimensional domain (see, for example, Leadbetter, Lindgren, and Rootzen (1983) for a detailed description of these results) to derive a limit distribution of suprema of these processes. When Z is present, however, one has to apply the results of the extreme value theory for Gaussian processes with multi-dimensional domain (these processes are referred in the literature as Gaussian random fields); see Piterbarg (1996). Although there do exist important applications of this theory in econometrics (see, for example, Lee, Linton, and Whang (2009)), it is not clear how to apply it in the setting like that studied in this section where the covariance structure of the process $\{b(s)/\hat{V}(s) : s \in \mathcal{S}_n\}$ is rather complicated, which is the case when kernel weighting functions are used with many different bandwidth values. Hall and Heckman (2000) take a different approach: they first provide a Gaussian coupling and then use integration by

parts of stochastic integrals to show validity of their critical values. Even when Z is absent, they proved their results only when X_i 's are equidistant on some interval, but, more importantly, it is also unclear how to generalize their techniques based on integration by parts to multi-dimensional setting.

(ii) There are other possible notions of monotonicity in the multivariate model (17). Assume, for simplicity, that both X and Z are scalars. Then one might want to test the null hypothesis, \mathcal{H}_0 , that $f(x_2, z_2) \geq f(x_1, z_1)$ for all x_1, x_2, z_1 , and z_2 satisfying $x_2 \geq x_1$ and $z_2 \geq z_1$ against the alternative, \mathcal{H}_a , that there exist x_1, x_2, z_1 , and z_2 such that $x_2 \geq x_1$ and $z_2 \geq z_1$ but $f(x_2, z_2) < f(x_1, z_1)$. To test this \mathcal{H}_0 , one can consider the following test functions:

$$b(s) = (1/2) \sum_{1 \leq i, j \leq n} (Y_i - Y_j) \text{sign}(X_j - X_i) I\{Z_j \geq Z_i\} Q(X_i, X_j, Z_i, Z_j, s)$$

where the function Q satisfies $Q(x_1, x_2, z_1, z_2, s) \geq 0$ for all x_1, x_2, z_1, z_2 , and s . Note that there is no bias term present in the test function, and so one does not have to consider local-in- z weighting functions. The theory for this problem can be obtained along the same line as that used in Section 4.

(iii) The same techniques that I use to test monotonicity in this paper can also be used to test other hypotheses that are of interest in economic theory. For example, one can test that the function $f(\cdot, \cdot)$ has *increasing differences* or *super-modularity* property. It is said that the function $f(\cdot, \cdot)$ has increasing differences or super-modularity property if for all $x_1 < x_2$, $f(x_2, z) - f(x_1, z)$ is increasing in z . This property plays an important role in Robust comparative statics. Specifically, one can obtain a testing procedure based on the following test functions:

$$b(s) = \sum_{1 \leq j, k, l, m \leq n} ((Y_j - Y_k) - (Y_l - Y_m)) Q(X_j, X_k, X_l, X_m, Z_j, Z_k, Z_l, Z_m, s)$$

where the function Q can take the form of kernel weighting function:

$$\begin{aligned} & Q(x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4, s) \\ &= K\left(\frac{x_1 - x_r(s)}{h(s)}\right) K\left(\frac{x_2 - x_l(s)}{h(s)}\right) K\left(\frac{x_3 - x_r(s)}{h(s)}\right) K\left(\frac{x_4 - x_l(s)}{h(s)}\right) \\ &\times K\left(\frac{z_1 - z_r(s)}{h(s)}\right) K\left(\frac{z_2 - z_r(s)}{h(s)}\right) K\left(\frac{z_3 - z_l(s)}{h(s)}\right) K\left(\frac{z_4 - z_l(s)}{h(s)}\right) \end{aligned}$$

and $s = (x_l, x_r, z_l, z_r, h)$ with $x_r > x_l$ and $z_r > z_l$ and some bandwidth value h . \square

5.2. Partially Linear Model. In this model, additional covariates enter the regression function as additively separable linear form. Specifically, the model is given by

$$Y = f(X) + Z^T \beta + \varepsilon \tag{22}$$

where Y is a scalar dependent random variable, X a scalar independent random variable, Z a vector in \mathbb{R}^d of additional independent random variables that should be controlled for, $f(\cdot)$ an

unknown function, β a vector in \mathbb{R}^d of unknown parameters, and ε an unobserved scalar random variable satisfying $E[\varepsilon|X, Z] = 0$ almost surely. As in Section 4, the problem is to test the null hypothesis, \mathcal{H}_0 , that $f(\cdot)$ is nondecreasing against the alternative, \mathcal{H}_a , that there are x_1 and x_2 such that $x_1 < x_2$ but $f(x_1) > f(x_2)$. The decision is to be made based on the i.i.d. sample of size n , $\{X_i, Z_i, Y_i\}_{1 \leq i \leq n}$ from the distribution of (X, Z, Y) . The regression function $f(\cdot)$ is assumed to be smooth but I do not impose any parametric structure on it.

An advantage of the partially linear model over the fully nonparametric model is that it does not suffer from the curse of dimensionality, which decreases the power of the test and may be a severe problem if the researcher has many additional covariates to control for. On the other hand, the partially linear model does not allow for heterogeneous effects of the factor X , which might be restrictive in some applications. It is necessary to have in mind that the test obtained for the partially linear model will be inconsistent if this model is misspecified.

Let me now describe the test. The main idea is to estimate β by some $\hat{\beta}_n$ and to apply the methods described in Section 3 for the dataset $\{X_i, Y_i - Z_i^T \hat{\beta}_n\}$. For example, one can take an estimator of Robinson (1988), which is given by

$$\hat{\beta}_n = \left(\sum_{i=1}^n \hat{Z}_i \hat{Z}_i^T \right)^{-1} \left(\sum_{i=1}^n \hat{Z}_i \hat{Y}_i \right)$$

where $\hat{Z}_i = Z_i - \hat{E}[Z|X = X_i]$, $\hat{Y}_i = Y_i - \hat{E}[Y|X = X_i]$, and $\hat{E}[Z|X = X_i]$ and $\hat{E}[Y|X = X_i]$ are nonparametric estimators of $E[Z|X = X_i]$ and $E[Y|X = X_i]$, respectively. Define $\tilde{Y}_i = Y_i - Z_i^T \hat{\beta}_n$, and let the test statistic be $T = T(\{X_i, \tilde{Y}_i\}, \{\hat{\sigma}_i\}, \mathcal{S}_n)$ where $T(\cdot, \cdot, \cdot)$ is defined in (5) and estimators $\hat{\sigma}_i$ of $\sigma_i = (E[\varepsilon_i^2|X_i])^{1/2}$ (here $\varepsilon_i = Y_i - f(X_i) - Z_i^T \beta$) satisfy either $\hat{\sigma}_i = \hat{\varepsilon}_i = Y_i - \hat{f}(X_i) - Z_i^T \hat{\beta}_n$ (here $\hat{f}(X_i)$ is some estimator of $f(X_i)$, which is uniformly consistent over $i = \overline{1, n}$) or $\hat{\sigma}_i$ is some uniformly consistent estimator of σ_i . The critical value for the test is simulated by one of the methods (plug-in, one-step, or step-down) described in Section 3 using the data $\{X_i, \tilde{Y}_i\}$, estimators $\{\hat{\sigma}_i\}$, and the set of weighting functions \mathcal{S}_n . As in Section 3, let $c_{1-\alpha}^{PI}$, $c_{1-\alpha}^{OS}$, and $c_{1-\alpha}^{SD}$ denote the plug-in, one-step, and step-down critical values, respectively.

Let $C_5 > 0$ be some constant. To obtain results for partially linear models, I will impose the following condition.

A12. (i) $\|Z\| \leq C_5$ almost surely, (ii) $\|\hat{\beta}_n - \beta\| = O_p(n^{-1/2})$, and (iii) uniformly over all $s \in \mathcal{S}_n$, $\sum_{1 \leq i, j \leq n} Q(X_i, X_j, s)/V(s)^{1/2} = o_p(\sqrt{n/\log p})$.

The assumption that the vector Z is bounded is a mild regularity condition and can be easily relaxed. Further, Horowitz (2009) provides a set of conditions so that $\|\hat{\beta}_n - \beta\| = O_p(n^{-1/2})$ when $\hat{\beta}_n$ is the Robinson's estimator. Finally, it follows from the proof of Proposition B.2 that under Assumption A8, Assumption A12-iii is satisfied if kernel weighting functions are used as long as $h_{\max} = o_p(1/\log p)$ and $h_{\min} \geq (\log n)^2/(Cn)$ w.p.a.1. for some $C > 0$. In addition, I will

need to modify Assumption A2 to account for the presence of the vector of additional covariates Z :

A2'' (i) $\hat{\sigma}_i = Y_i - \hat{f}(X_i) - Z_i^T \hat{\beta}_n$ for all $i = \overline{1, n}$ and (ii) $\hat{f}(X_i) - f(X_i) = o_p(n^{-\kappa_1})$ uniformly over $i = \overline{1, n}$.

Let \mathcal{M}_{PL} be a class of models given by equation (22), function $f(\cdot)$, parameter β , joint distribution of X , Z , and ε satisfying $E[\varepsilon|X, Z] = 0$ almost surely, weighting functions $Q(\cdot, \cdot, s)$ for $s \in \mathcal{S}_n$ (possibly depending on X_i 's), an estimator $\hat{\beta}_n$, and estimators $\{\hat{\sigma}_i\}$ such that uniformly over this class, (i) Assumptions A1, A4, A5, and A12 are satisfied (where $V(s)$, $\hat{V}(s)$, and A_n are defined in (3), (4), and (16), respectively), and (ii) either Assumption A2'' or A3 is satisfied. The size properties of the test are given in the following theorem.

Theorem 5.2 (Size properties in the partially linear model). *Let $P = PI, OS, \text{ or } SD$. Let $\mathcal{M}_{PL,0}$ denote the set of all models $M \in \mathcal{M}_{PL}$ satisfying \mathcal{H}_0 . Then*

$$\inf_{M \in \mathcal{M}_{PL,0}} \mathbb{P}_M(T \leq c_{1-\alpha}^P) \geq 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

In addition, let $\mathcal{M}_{PL,00}$ denote the set of all models $M \in \mathcal{M}_{PL,0}$ such that $f(x) = C$ for some constant C and all $x \in \mathbb{R}$. Then

$$\sup_{M \in \mathcal{M}_{PL,00}} \mathbb{P}_M(T \leq c_{1-\alpha}^P) = 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

Comment 5.2. The result of Theorem 5.2 can be extended to cover a general separately additive models. Specifically, assume that the data come from the model

$$Y = f(X) + g(Z) + \varepsilon \tag{23}$$

where $g(\cdot)$ is some unknown smooth function and all other notation is the same as above. Then one can test \mathcal{H}_0 that $f(\cdot)$ is nondecreasing against the alternative \mathcal{H}_a that there are x_1 and x_2 such that $x_1 < x_2$ but $f(x_1) > f(x_2)$ using a method similar to that used above with the exception that now $\tilde{Y}_i = Y_i - \hat{g}(Z_i)$ where $\hat{g}(Z_i)$ is some nonparametric estimator of $g(Z_i)$. Specifically, consider the following conditions:

A2''' (i) $\hat{\sigma}_i = Y_i - \hat{f}(X_i) - \hat{g}(Z_i)$ for all $i = \overline{1, n}$ and (ii) $\hat{f}(X_i) - f(X_i) = o_p(n^{-\kappa_1})$ uniformly over $i = \overline{1, n}$.

A12' (i) $\max_{1 \leq i \leq n} |\hat{g}(Z_i) - g(Z_i)| = O_p(\psi_n^{-1})$ for some sequence $\psi_n \rightarrow \infty$, and (ii) uniformly over all $s \in \mathcal{S}_n$, $\sum_{1 \leq i, j \leq n} Q(X_i, X_j, s)/V(s)^{1/2} = o_p(\sqrt{\psi_n/\log p})$.

Let \mathcal{M}_{SA} , $\mathcal{M}_{SA,0}$, and $\mathcal{M}_{SA,00}$ denote sets of models that are defined as \mathcal{M}_{PL} , $\mathcal{M}_{PL,0}$, and $\mathcal{M}_{PL,00}$ with (22) and Assumptions A12 and A2'' replaced by (23) and Assumptions A12' and A2'''. Then Theorem 5.2 applies with \mathcal{M}_{PL} , $\mathcal{M}_{PL,0}$, and $\mathcal{M}_{PL,00}$ replaced by \mathcal{M}_{SA} , $\mathcal{M}_{SA,0}$, and

$\mathcal{M}_{SA,00}$. This result can be proven from the argument similar to that used to prove Theorem 5.2. \square

6. MODELS WITH ENDOGENOUS COVARIATES

Empirical studies in economics often contain endogenous covariates. Therefore, it is important to extend the results on testing monotonicity obtained in this paper to cover this possibility. Allowing for endogenous covariates in nonparametric settings like the one considered here is challenging and several approaches have been proposed in the literature. In this paper, I consider the approach proposed in Newey, Powell, and Vella (1999). In their model,

$$Y = f(X) + W, \quad (24)$$

$$X = \lambda(U) + Z \quad (25)$$

where Y is a scalar dependent variable, X a scalar covariate, U a vector in \mathbb{R}^d of instruments, $f(\cdot)$ and $\lambda(\cdot)$ unknown functions, and W and Z unobserved scalar random variables satisfying

$$\mathbb{E}[W|U, Z] = \mathbb{E}[W|Z] \text{ and } \mathbb{E}[Z|U] = 0 \quad (26)$$

almost surely. In this setting, X is endogenous in the sense that equation $\mathbb{E}[W|X] = 0$ almost surely does not necessarily hold. The problem is to test the null hypothesis, \mathcal{H}_0 , that $f(\cdot)$ is nondecreasing against the alternative, \mathcal{H}_a , that there are x_1 and x_2 such that $x_1 < x_2$ but $f(x_1) > f(x_2)$. The decision is to be made based on the i.i.d. sample of size n , $\{X_i, U_i, Y_i\}_{1 \leq i \leq n}$ from the distribution of (X, U, Y) .

It is possible to consider a more general setting where the function f of interest depends on X and U_1 , that is $Y = f(X, U_1) + \varepsilon$ and $U = (U_1, U_2)$ but I refrain from including additional covariates for brevity.

An alternative to the model defined in (24)-(26) is the model defined by (24) and

$$\mathbb{E}[W|U] = 0 \quad (27)$$

almost surely. Newey, Powell, and Vella (1999) noted that neither model is more general than the other one. The reason is that it does not follow from (26) that (27) holds, and it does not follow from (27) that (26) holds. Both models have been used in the empirical studies. The latter model have been studied in Newey and Powell (2003), Hall and Horowitz (2005), Blundell, Chen, and Kristensen (2007), and Darolles et. al. (2011), among many others.

Let me now get back to the model defined by (24)-(26). A key observation of Newey, Powell, and Vella (1999) is that in this model,

$$\mathbb{E}[Y|U, Z] = \mathbb{E}[f(X)|U, Z] + \mathbb{E}[W|U, Z] = \mathbb{E}[f(X)|U, X] + \mathbb{E}[W|Z] = f(X) + g(Z)$$

where $g(z) = E[W|Z = z]$. This implies that

$$E[Y|X, Z] = f(X) + g(Z). \quad (28)$$

Therefore, the regression function of Y on X and Z is separately additive in X and Z , and so, if $Z_i = X_i - \lambda(U_i)$'s were known, both $f(\cdot)$ and $g(\cdot)$ could be estimated by one of the nonparametric methods suitable for estimating separately additive models. One particularly convenient method to estimate such models is a series estimator. Further, note that even though Z_i 's are unknown, they can be consistently estimated from the data as residuals from the nonparametric regression of X on U . Then one can estimate both $f(\cdot)$ and $g(\cdot)$ by employing a nonparametric method for estimating separately additive models with Z_i 's replaced by their estimates. Specifically, let $\widehat{Z}_i = X_i - \widehat{E}[X|U = U_i]$ where $\widehat{E}[X|U = U_i]$ is a consistent nonparametric estimator of $E[X|U = U_i]$. Further, for an integer $L > 0$, let $r^{L,f}(x) = (r_{1L}^f(x), \dots, r_{LL}^f(x))'$ and $r^{L,g}(z) = (r_{1L}^g(z), \dots, r_{LL}^g(z))'$ be vectors of approximating functions for $f(\cdot)$ and $g(\cdot)$, respectively, so that

$$f(x) \approx r^{L,f}(x)' \beta_f \text{ and } g(z) \approx r^{L,g}(z)' \beta_g$$

for L large enough where β_f and β_g are vectors in \mathbb{R}^L of coefficients. In addition, let $r^L(x, z) = (r^{L,f}(x)', r^{L,g}(z)')'$ and $R = (r^L(X_1, \widehat{Z}_1), \dots, r^L(X_n, \widehat{Z}_n))$. Then the OLS estimator of β_f and β_g is

$$\begin{pmatrix} \widehat{\beta}_f \\ \widehat{\beta}_g \end{pmatrix} = (R'R)^{-1} (R'Y_1^n)$$

where $Y_1^n = (Y_1, \dots, Y_n)'$, and the series estimators $\widehat{f}(x)$ and $\widehat{g}(z)$ of $f(x)$ and $g(z)$ are $r^{L,f}(x)' \widehat{\beta}_f$ and $r^{L,g}(z)' \widehat{\beta}_g$ for all x and z , respectively.

Now, it follows from (28) that

$$Y = f(X) + g(Z) + \varepsilon$$

where $E[\varepsilon|X, Z] = 0$ almost surely. This is exactly the model discussed in Comment 5.2, and since I have an estimator of $g(\cdot)$, it is possible to test \mathcal{H}_0 using the same ideas as in Comment 5.2. Specifically, let $\widetilde{Y}_i = Y - \widehat{g}(\widehat{Z}_i)$ and apply a test described in Section 3 with the data $\{X_i, \widetilde{Y}_i\}_{1 \leq i \leq n}$; that is, let the test statistic be $T = T(\{X_i, \widetilde{Y}_i\}, \{\widehat{\sigma}_i\}, \mathcal{S}_n)$ where the function $T(\cdot, \cdot, \cdot)$ is defined in (5) and estimators $\widehat{\sigma}_i$ of $\sigma_i = (E[\varepsilon_i^2|X_i])^{1/2}$ (here $\varepsilon_i = Y_i - f(X_i) - g(Z_i)$) satisfy either $\widehat{\sigma}_i = \widehat{\varepsilon}_i = Y_i - \widehat{f}(X_i) - \widehat{g}(\widehat{Z}_i)$ or $\widehat{\sigma}_i$ is some uniformly consistent estimator of σ_i . Let $c_{1-\alpha}^{PI}$, $c_{1-\alpha}^{OS}$, and $c_{1-\alpha}^{SD}$ denote the plug-in, one-step, and step-down critical values, respectively, that are obtained as in Section 3 using the data $\{X_i, \widetilde{Y}_i\}$, estimators $\{\widehat{\sigma}_i\}$, and the set of weighting functions \mathcal{S}_n .

To obtain results for this testing procedure, I will use the following modifications of Assumptions A2''' and A12':

A2'''' (i) $\widehat{\sigma}_i = Y_i - \widehat{f}(X_i) - \widehat{g}(\widehat{Z}_i)$ for all $i = \overline{1, n}$ and (ii) $\widehat{f}(X_i) - f(X_i) = o_p(n^{-\kappa_1})$ uniformly over $i = \overline{1, n}$.

A12'' (i) $\max_{1 \leq i \leq n} |\widehat{g}(\widehat{Z}_i) - g(\widehat{Z}_i)| = O_p(\psi_n^{-1})$ for some sequence $\psi_n \rightarrow \infty$, and (ii) uniformly over all $s \in \mathcal{S}_n$, $\sum_{1 \leq i, j \leq n} Q(X_i, X_j, s)/V(s)^{1/2} = o_p(\sqrt{\psi_n/\log p})$.

Assumption A2''' is a simple extension of Assumption A2'' that takes into account that Z_i 's have to be estimated by \widehat{Z}_i 's. Further, Assumption A12'' requires that $\max_{1 \leq i \leq n} |\widehat{g}(\widehat{Z}_i) - g(\widehat{Z}_i)| = O_p(\psi_n^{-1})$ for some sequence $\psi_n \rightarrow \infty$. For the estimator $\widehat{g}(\cdot)$ of $g(\cdot)$ described above, this requirement follows from Lemma 4.1 and Theorem 4.3 in Newey, Powell, and Vella (1999) who also provide certain primitive conditions for this requirement to hold. Specific choices of ψ_n depend on how smooth the functions $f(\cdot)$ and $g(\cdot)$ are and how the number of series terms, L , is chosen.

Let \mathcal{M}_{EC} be a class of models given by equations (24)-(26), functions $f(\cdot)$ and $\lambda(\cdot)$, joint distribution of X , U , W , and Z , weighting functions $Q(\cdot, \cdot, s)$ for $s \in \mathcal{S}_n$ (possibly depending on X_i 's), an estimator $\widehat{g}(\cdot)$ of $g(\cdot)$, and estimators $\{\widehat{\sigma}_i\}$ of $\{\sigma_i\}$ such that uniformly over this class, (i) Assumptions A1, A4, A5, and A12'' are satisfied (where $V(s)$, $\widehat{V}(s)$, and A_n are defined in (3), (4), and (16), respectively), and (ii) either Assumption A2''' or A3 is satisfied.

Theorem 6.1 (Size properties in the model with endogenous covariates). *Let $P = PI, OS$, or SD . Let $\mathcal{M}_{EC,0}$ denote the set of all models $M \in \mathcal{M}_{EC}$ satisfying \mathcal{H}_0 . Then*

$$\inf_{M \in \mathcal{M}_{EC,0}} \mathbb{P}_M(T \leq c_{1-\alpha}^P) \geq 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

In addition, let $\mathcal{M}_{EC,00}$ denote the set of all models $M \in \mathcal{M}_{EC,0}$ such that $f \equiv C$ for some constant C . Then

$$\sup_{M \in \mathcal{M}_{EC,00}} \mathbb{P}_M(T \leq c_{1-\alpha}^P) = 1 - \alpha + o(1) \text{ as } n \rightarrow \infty.$$

Comment 6.1. Around the same time this paper became publicly available, Gutknecht (2013) obtained a test of monotonicity of the function $f(\cdot)$ in the same model (24)-(26). The test in that paper is a special case of the class of tests derived in this paper. Specifically, Gutknecht (2013) obtained a test based on test functions (7). The major difference, however, is that his test is not adaptive because it is based on one non-stochastic bandwidth value. \square

7. SAMPLE SELECTION MODELS

It is widely recognized in econometrics literature that sample selection problems can result in highly misleading inference in otherwise standard regression models. The purpose of this section is to briefly show that the same techniques that are used in the last section to deal with endogeneity issues can also be used to deal with sample selection issues. For concreteness, I consider a nonparametric version of the classical Heckman's sample selection model; see Heckman (1979). The nonparametric version of the Heckman's model was previously analyzed in great

generality in Das, Newey, and Vella (2003). Specifically, I consider the model

$$Y^* = f(X) + W, \tag{29}$$

$$Y = DY^* \tag{30}$$

where Y^* is an unobserved scalar dependent random variable, Y a scalar random variable, X a covariate, D a binary selection indicator, W unobserved scalar random variable. The problem in this model arises when W is not independent of D . To deal with this problem, let Z denote a vector of random variables that affect selection D , and let $p(x, z) = E[D|X = x, Z = z]$ be the propensity score. Further, assume that Z is such that

$$E[W|X, Z, D = 1] = \lambda(p(X, Z)) \tag{31}$$

where $\lambda(\cdot)$ is some unknown function. This condition is reasonable in many settings; see, in particular, discussion on p.35 of Das, Newey, and Vella (2003). I am interested in testing the null hypothesis that the function $f(\cdot)$ in the model (29)-(31) is weakly increasing against the alternative that it is not weakly increasing based on the random sample $\{X_i, Z_i, Y_i\}_{1 \leq i \leq n}$ from the distribution of (X, Z, Y) .

A key observation of Das, Newey, and Vella (2003) is that under (31), equations (29) and (30) imply that

$$E[Y|X, Z, D = 1] = f(X) + \lambda(p(X, Z)),$$

and so denoting $P = p(X, Z)$,

$$E[Y|X, P, D = 1] = f(X) + \lambda(P). \tag{32}$$

Therefore, if $P_i = p(X_i, Z_i)$'s were known, one could estimate both $f(\cdot)$ and $\lambda(\cdot)$ by one of the nonparametric methods suitable for estimating separately additive models applied to the regression of Y on X and P based on the subsample of observations with $D_i = 1$. Further, note that even though P_i 's are unknown, they can be estimated consistently from the data as the predicted values in the nonparametric regression of D on X and Z . Hence, one can estimate both $f(\cdot)$ and $\lambda(\cdot)$ by one of the non-parametric methods suitable for estimating additive models applied to the regression of Y on X and P with P_i 's replaced by \hat{P}_i 's where $\hat{P}_i = \hat{p}(X_i, Z_i)$ and $\hat{p}(\cdot, \cdot)$ is a nonparametric estimator of $p(\cdot, \cdot)$.

Now, (32) implies that

$$Y = f(X) + \lambda(P) + \varepsilon$$

where $E[\varepsilon|X, P, D = 1] = 0$ almost surely. This is exactly the model discussed in Comment 5.2, and since an estimator $\hat{\lambda}(\cdot)$ of $\lambda(\cdot)$ is available, one can test monotonicity of $f(\cdot)$ by applying the results in Section 3 with the data $\{X_i, \tilde{Y}_i\}_{1 \leq i \leq n}$ where $\tilde{Y}_i = Y - \hat{\lambda}(\hat{P}_i)$. For this test, one can state the theorem that is analogous to Theorem 6.1.

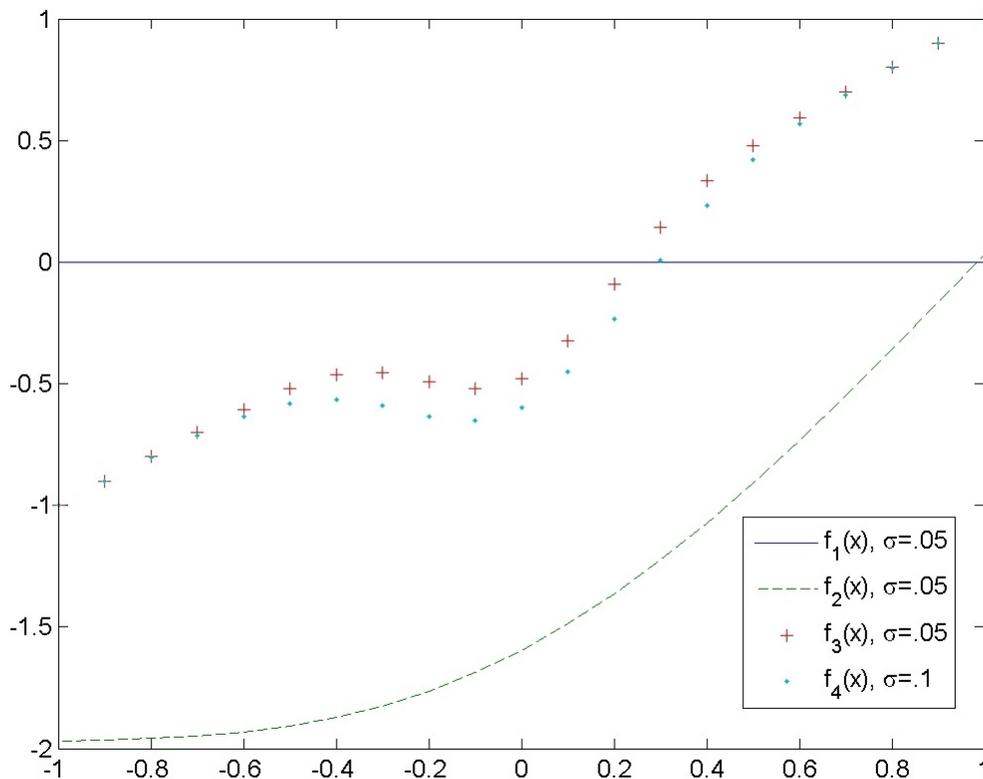
8. MONTE CARLO SIMULATIONS

In this section, I provide results of a small simulation study. The aim of the simulation study is to shed some light on the size properties of the test in finite samples and to compare its power with that of other tests developed in the literature. In particular, I consider the tests of Gijbels et. al. (2000) (GHJK), Ghosal, Sen, and van der Vaart (2000) (GSV), and Hall and Heckman (2000) (HH).

I consider samples of size $n = 100, 200,$ and 500 with X_i 's uniformly distributed on the $[-1, 1]$ interval, and regression functions of the form $f(x) = c_1x - c_2\phi(c_3x)$ where $c_1, c_2, c_3 \geq 0$ and $\phi(\cdot)$ is the pdf of the standard normal distribution. I assume that ε_i is a zero-mean random variable that is independent of X_i and has the standard deviation σ . Depending on the experiment, ε_i has either normal or continuous uniform distribution. Four combinations of parameters are studied: (1) $c_1 = c_2 = c_3 = 0$ and $\sigma = 0.05$; (2) $c_1 = c_3 = 1, c_2 = 4,$ and $\sigma = 0.05$; (3) $c_1 = 1, c_2 = 1.2, c_3 = 5,$ and $\sigma = 0.05$; (4) $c_1 = 1, c_2 = 1.5, c_3 = 4,$ and $\sigma = 0.1$. Cases 1 and 2 satisfy \mathcal{H}_0 whereas cases 3 and 4 do not. In case 1, the regression function is flat corresponding to the maximum of the type I error. In case 2, the regression function is strictly increasing. Cases 3 and 4 give examples of the regression functions that are mostly increasing but violate \mathcal{H}_0 in the small neighborhood near 0. All functions are plotted in figure 2. The parameters were chosen so that to have nontrivial rejection probability in most cases (that is, bounded from zero and from one).

Let me describe the tuning parameters for all tests that are used in the simulations. For the tests of GSV, GHJK, and HH, I tried to follow their instructions as closely as possible. For the test developed in this paper, I use kernel weighting functions with $k = 0, \mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\}, h \in H_n\}$, and the kernel $K(x) = 0.75(1 - x^2)$ for $x \in (-1; +1)$ and 0 otherwise. I use the set of bandwidth values $H_n = \{h_{\max}u^l : h \geq h_{\min}, l = 0, 1, 2, \dots\}, u = 0.5, h_{\max} = 1, h_{\min} = 0.4h_{\max}(\log n/n)^{1/3}$, and the truncation parameter $\gamma = 0.01$. For the test of GSV, I use the same kernel K with the bandwidth value $h_n = n^{-1/5}$, which was suggested in their paper, and I consider their sup-statistic. For the test of GHJK, I use their run statistic maximized over $k \in \{10(j - 1) + 1 : j = 1, 2, \dots, 0.2n\}$ (see the original paper for the explanation of the notation). For the test of HH, local polynomial estimates are calculated over $r \in nH_n$ at every design point X_i . The set nH_n is chosen so that to make the results comparable with those for the test developed in this paper. Finally, I consider two versions of the test developed in this paper depending on how σ_i is estimated. More precisely, I consider the test with σ_i estimated by the Rice's method (see equation (11)), which I refer to in the table below as CS (consistent sigma), and the test with $\hat{\sigma}_i = \hat{\varepsilon}_i$ where $\hat{\varepsilon}_i$ is obtained as the residual from estimating f using the series method with polynomials of order 5, 6 and 8 whenever the sample size n is 100, 200, and 500, respectively, which I refer to in the table below as IS (inconsistent sigma).

FIGURE 2. Regression Functions Used in Simulations



The rejection probabilities corresponding to nominal level $\alpha = 0.1$ for all tests are presented in table 1. The results are based on 1000 simulations with 500 bootstrap repetitions in all cases excluding the test of GSV where the asymptotic critical value is used.

The results of the simulations can be summarized as follows. First, the results for normal and uniform disturbances are rather similar. The test developed in this paper with σ_i estimated using the Rice's method maintains the required size quite well (given the nonparametric structure of the problem) and yields size comparable with that of the GSV, GHJK, and HH tests. On the other hand, the test with $\hat{\sigma}_i = \hat{\varepsilon}_i$ does pretty well in terms of size only when the sample size is as large as 500. When the null hypothesis does not hold, the CS test with the step-down critical value yields the highest proportion of rejections in all cases. Moreover, in case 3 with the sample size $n = 200$, this test has much higher power than that of GSV, GHJK, and HH. The CS test also has higher power than that of the IS test. Finally, the table shows that the one-step critical value gives a notable improvement in terms of power in comparison with plug-in critical value. For example, in case 3 with the sample size $n = 200$, the one-step critical value gives additional 190 rejections out 1000 simulations in comparison with the plug-in critical value

TABLE 1. Results of Monte Carlo Experiments

Noise	Case	Sample	Proportion of Rejections for								
			GSV	GHJK	HH	CS-PI	CS-OS	CS-SD	IS-PI	IS-OS	IS-SD
normal	1	100	.118	.078	.123	.128	.128	.128	.164	.164	.164
		200	.091	.051	.108	.114	.114	.114	.149	.149	.149
		500	.086	.078	.105	.114	.114	.114	.133	.133	.133
normal	2	100	0	.001	0	.001	.008	.008	.008	.024	.024
		200	0	.002	0	.001	.010	.010	.007	.017	.017
		500	0	.001	0	.002	.007	.007	.005	.016	.016
normal	3	100	0	.148	.033	.259	.436	.433	0	0	0
		200	.010	.284	.169	.665	.855	.861	.308	.633	.650
		500	.841	.654	.947	.982	.995	.997	.975	.995	.995
normal	4	100	.037	.084	.135	.163	.220	.223	.023	.042	.043
		200	.254	.133	.347	.373	.499	.506	.362	.499	.500
		500	.810	.290	.789	.776	.825	.826	.771	.822	.822
uniform	1	100	.109	.079	.121	.122	.122	.122	.201	.201	.201
		200	.097	.063	.109	.121	.121	.121	.160	.160	.160
		500	.077	.084	.107	.092	.092	.092	.117	.117	.117
uniform	2	100	.001	.001	0	0	.006	.007	.017	.032	.033
		200	0	0	0	.001	.010	.010	.012	.022	.024
		500	0	.003	0	.003	.011	.011	.011	.021	.021
uniform	3	100	0	.151	.038	.244	.438	.449	0	0	0
		200	.009	.233	.140	.637	.822	.839	.290	.607	.617
		500	.811	.582	.947	.978	.994	.994	.975	.990	.990
uniform	4	100	.034	.084	.137	.155	.215	.217	.024	.045	.046
		200	.197	.116	.326	.357	.473	.478	.323	.452	.456
		500	.803	.265	.789	.785	.844	.846	.782	.847	.848

Nominal Size is 0.1. GSV, GHJK, and HH stand for the tests of Ghosal, Sen, and van der Vaart (2000), Gijbels et. al. (2000), and Hall and Heckman (2000) respectively. CS-PI, CS-OS, and CS-SD refer to the test developed in this paper with σ_i estimated using Rice's formula and plug-in, one-step, and step-down critical values respectively. Finally, IS-PI, IS-OS, and IS-SD refer to the test developed in this paper with σ_i estimated by $\hat{\sigma}_i = \hat{\varepsilon}_i$ and plug-in, one-step, and step-down critical values respectively.

for the CS test and additional 325 rejections for the IS test. On the other hand, the step-down approach gives only minor improvements over the one-step approach. Overall, the results of the simulations are consistent with the theoretical findings in this paper. In particular, selection

procedures yielding one-step and step-down critical values improve power with no size distortions. Additional simulation results are presented in the supplementary Appendix.

9. EMPIRICAL APPLICATION

In this section, I review the arguments of Ellison and Ellison (2011) on how strategic entry deterrence might yield a non-monotone relation between market size and investment in the pharmaceutical industry and then apply the testing procedures developed in this paper to their dataset. I start with describing their theory. Then I provide the details of the dataset. Finally, I present the results.

In the pharmaceutical industry, incumbents whose patents are about to expire can use investments strategically to prevent generic entries after the expiration of the patent. In order to understand how this strategic entry deterrence influences the relation between market size and investment levels, Ellison and Ellison (2011) developed two models for an incumbent's investment. In the first model, potential entrants do not observe the incumbent's investment but they do in the second one. So, a strategic entry deterrence motive is absent in the former model but is present in the latter one. Therefore, the difference in incumbent's investment between two models is explained by the strategic entry deterrence. Ellison and Ellison showed that in the former model, the investment-market size relation is determined by a combination of direct and competition effects. The direct effect is positive if increasing the market size (holding entry probabilities fixed) raises the marginal benefit from the investment more than it raises the marginal cost of the investment. The competition effect is positive if the marginal benefit of the investment is larger when the incumbent is engaged in duopoly competition than it is when the incumbent is a monopolist. The equilibrium investment is increasing in market size if and only if the sum of two effects is positive. Therefore, a sufficient condition for the monotonicity of investment-market size relation is that both effects are of the same sign.⁶ In the latter model, there is also a strategic entry deterrence effect. The authors noted that this effect should be relatively less important in small and large markets than it is in markets of intermediate size. In small markets, there are not enough profits for potential entrants, and there is no need to prevent entry. In large markets, profits are so large that no reasonable investment levels will be enough to prevent entries. As a result, strategic entry deterrence might yield a non-monotonic relation between market size and investment no matter whether the relation in the model with no strategic entry deterrence is increasing or decreasing.

Ellison and Ellison studied three types of investment: detail advertising, journal advertising, and presentation proliferation. Detail advertising, measured as per-consumer expenditures, refers to sending representatives to doctors' offices. Since both revenues and cost of detail advertising

⁶An interested reader can find a more detailed discussion in the original paper.

are likely to be linear in the market size, it can be shown that the direct effect for detail advertising is zero. The competition effect is likely to be negative because detail advertising will benefit competitors as well. Therefore, it is expected that detail advertising is a decreasing function of the market size in the absence of strategic distortions. Strategic entry deterrence should decrease detail advertising for markets of intermediate size. Journal advertising is the placement of advertisements in medical journals. Journal advertising is also measured as per-consumer expenditures. The competition effect for journal advertising is expected to be negative for the same reason as for detail advertising. The direct effect, however, may be positive because the cost per potential patient is probably a decreasing function of the market size. Opposite directions of these effects make journal advertising less attractive for detecting strategic entry deterrence in comparison with detail advertising. Nevertheless, following the original paper, I assume that journal advertising is a decreasing function of the market size in the absence of strategic distortions. Presentation proliferation is selling a drug in many different forms. Since the benefits of introducing a new form is approximately proportional to the market size while the costs can be regarded as fixed, the direct effect for presentation proliferation should be positive. In addition, the competition effect is also likely to be positive because it creates a monopolistic niche for the incumbent. Therefore, presentation proliferation should be positively related to market size in the absence of strategic distortions.

The dataset consists of 63 chemical compounds, sold under 71 different brand names. All of these drugs lost their patent exclusivity between 1986 and 1992. There are four variables in the dataset: average revenue for each drug over three years before the patent expiration (this measure should be regarded as a proxy for market size), average costs of detail and journal advertising over the same time span as revenues, and a Herfindahl-style measure of the degree to which revenues are concentrated in a small number of presentations (this measure should be regarded as the inverse of presentation proliferation meaning that higher values of the measure indicate lower presentation proliferation).

Clearly, the results will depend on how I define both dependent and independent variables for the test. Following the strategy adopted in the original paper, I use log of revenues as the independent variable in all cases, and the ratio of advertising costs to revenues for detail and journal advertising and the Herfindahl-style measure for presentation proliferation as the dependent variable. The null hypothesis is that the corresponding conditional mean function is decreasing.⁷

⁷In the original paper, Ellison and Ellison (2011) test the null hypothesis consisting of the union of monotonically increasing and monotonically decreasing regression functions. The motivation for this modification is that increasing regression functions contradict the theory developed in the paper and, hence, should not be considered as evidence of the existence of strategic entry deterrence. On the other hand, increasing regression functions might

I consider the test with kernel weighting functions with $k = 0$ or 1 and the kernel $K(x) = 0.75(1 - x^2)$ for $x \in (-1, 1)$ and 0 otherwise. I use the set of bandwidth values $H_n = \{0.5; 1\}$ and the set of weighting functions $\mathcal{S}_n = \{(x, h) : x \in \{X_1, \dots, X_n\}, h \in H_n\}$. Implementing the test requires estimating σ_i^2 for all $i = 1, \dots, n$. Since the test based on Rice's method outperformed that with $\hat{\sigma}_i = \hat{\varepsilon}_i$ in the Monte Carlo simulations, I use this method in the benchmark procedure. I also check robustness of the results using the following two-step procedure. First, I obtain residuals of the OLS regression of Y on a set of transformations of X . In particular, I use polynomials in X up to the third degree (cubic polynomial). Second, squared residuals are projected onto the same polynomial in X using the OLS regression again. The resulting projections are estimators $\hat{\sigma}_i^2$ of σ_i^2 , $i = 1, \dots, n$.

The results of the test are presented in table 2. The table shows the p-value of the test for each type of investment and each method of estimating σ_i^2 . In the table, method 1 corresponds to estimating σ_i^2 using Rice's formula, and methods 2, 3, and 4 are based on polynomials of first, second, and third degrees respectively. Note that all methods yield similar numbers, which reassures the robustness of the results. All the methods with $k = 0$ reject the null hypothesis that journal advertising is decreasing in market size with 10% confidence level. This may be regarded as evidence that pharmaceutical companies use strategic investment in the form of journal advertising to deter generic entries. On the other hand, recall that direct and competition effects probably have different signs for journal advertising, and so rejecting the null may also be due to the fact that the direct effect dominates for some values of market size. In addition, the test with $k = 1$ does not reject the null hypothesis that journal advertising is decreasing in market size at the 10% confidence level, no matter how σ_i are estimated. No method rejects the null hypothesis in the case of detail advertising and presentation proliferation. This may be (1) because firms do not use these types of investment for strategic entry deterrence, (2) because the strategic effect is too weak to yield non-monotonicity, or (3) because the sample size is not large enough. Overall, the results are consistent with those presented in Ellison and Ellison (2011).

10. CONCLUSION

In this paper, I have developed a general framework for testing monotonicity of a nonparametric regression function, and have given a broad class of new tests. A general test statistic uses many different weighting functions so that an approximately optimal weighting function is determined automatically. In this sense, the test adapts to the properties of the model. I have also obtained new methods to simulate the critical values for these tests. These are based on selection procedures. The procedures are used to estimate what counterparts of the test statistic should be used in simulating the critical value. They are constructed so that no violation of

arise if the strategic entry deterrence effect over-weighs direct and competition effects even in small and large markets, which could be considered as extreme evidence of the existence of strategic entry deterrence.

TABLE 2. Incumbent Behavior versus Market Size: Monotonicity Test p-value

Method	Investment Type					
	Detail Advertising		Journal Advertising		Presentation Proliferation	
	k=0	k=1	k=0	k=1	k=0	k=1
1	.120	.111	.056	.120	.557	.661
2	.246	.242	.088	.168	.665	.753
3	.239	.191	.099	.195	.610	.689
4	.301	.238	.098	.194	.596	.695

the asymptotic size occurs. Finally, I have given tests suitable for models with multiple and endogenous covariates for the first time in the literature.

The new methods have numerous applications in economics. In particular, they can be applied to test qualitative predictions of comparative statics analysis including those derived via robust comparative statics. In addition, they are useful for evaluating monotonicity assumptions, which are often imposed in economic and econometric models, and for classifying economic objects in those cases where classification includes the concept of monotonicity (for example, normal/inferior and luxury/necessity goods). Finally, these methods can be used to detect strategic behavior of economic agents that might cause non-monotonicity in otherwise monotone relations.

The attractive properties of the new tests are demonstrated via Monte Carlo simulations. In particular, it is shown that the rejection probability of the new tests greatly exceeds that of other tests for some simulation designs. In addition, I applied the tests developed in this paper to study entry deterrence effects in the pharmaceutical industry using the dataset of Ellison and Ellison (2011). I showed that the investment in the form of journal advertising seems to be used by incumbents in order to prevent generic entries after the expiration of patents. The evidence is rather weak, though.

APPENDIX A. IMPLEMENTATION DETAILS

In this section, I provide detailed step-by-step instructions for implementing plug-in, one-step, and step-down critical values. The instructions are given for constructing a test of level α . In all cases, let B be a large integer denoting the number of bootstrap repetitions, and let $\{\epsilon_{i,b}\}_{i=1,b=1}^{n,B}$ be a set of independent $N(0,1)$ random variables. For one-step and step-down critical values, let γ denote the truncation probability, which should be small relative to α .

A.1. Plug-in Approach.

- (1) For each $b = \overline{1, B}$ and $i = \overline{1, n}$, calculate $Y_{i,b}^* = \widehat{\sigma}_i \epsilon_{i,b}$.
- (2) For each $b = \overline{1, B}$, calculate the value T_b^* of the test statistic using the sample $\{X_i, Y_{i,b}^*\}_{i=1}^n$.

- (3) Define the plug-in critical value, $c_{1-\alpha}^{PI}$, as the $(1 - \alpha)$ sample quantile of $\{T_b^*\}_{b=1}^B$.

A.2. One-Step Approach.

- (1) For each $b = \overline{1, B}$ and $i = \overline{1, n}$, calculate $Y_{i,b}^* = \widehat{\sigma}_i \epsilon_{i,b}$.
- (2) Using the plug-in approach, simulate $c_{1-\gamma}^{PI}$.
- (3) Define \mathcal{S}_n^{OS} as the set of values $s \in \mathcal{S}_n$ such that $b(s)/(\widehat{V}(s))^{1/2} > -2c_{1-\gamma}^{PI}$.
- (4) For each $b = \overline{1, B}$, calculate the value T_b^* of the test statistic using the sample $\{X_i, Y_{i,b}^*\}_{i=1}^n$ and taking maximum only over \mathcal{S}_n^{OS} instead of \mathcal{S}_n .
- (5) Define the one-step critical value, $c_{1-\alpha}^{OS}$, as the $(1 - \alpha)$ sample quantile of $\{T_b^*\}_{b=1}^B$.

A.3. Step-down Approach.

- (1) For each $b = \overline{1, B}$ and $i = \overline{1, n}$, calculate $Y_{i,b}^* = \widehat{\sigma}_i \epsilon_{i,b}$.
- (2) Using the plug-in and one-step approaches, simulate $c_{1-\gamma}^{PI}$ and $c_{1-\gamma}^{OS}$, respectively.
- (3) Denote $\mathcal{S}_n^1 = \mathcal{S}_n^{OS}$, $c^1 = c_{1-\gamma}^{OS}$, and set $l = 1$.
- (4) For given value of $l \geq 1$, define \mathcal{S}_n^{l+1} as the set of values $s \in \mathcal{S}_n^l$ such that $b(s)/(\widehat{V}(s))^{1/2} > -c_{1-\gamma}^{PI} - c^l$.
- (5) For each $b = \overline{1, B}$, calculate the value T_b^* of the test statistic using the sample $\{X_i, Y_{i,b}^*\}_{i=1}^n$ and taking the maximum only over \mathcal{S}_n^{l+1} instead of \mathcal{S}_n .
- (6) Define c^{l+1} , as the $(1 - \gamma)$ sample quantile of $\{T_b^*\}_{b=1}^B$.
- (7) If $\mathcal{S}_n^{l+1} = \mathcal{S}_n^l$, then go to step (8). Otherwise, set $l = l + 1$ and go to step (4).
- (8) For each $b = \overline{1, B}$, calculate the value T_b^* of the test statistic using the sample $\{X_i, Y_{i,b}^*\}_{i=1}^n$ and taking the maximum only over \mathcal{S}_n^l instead of \mathcal{S}_n .
- (9) Define $c_{1-\alpha}^{SD}$, as the $(1 - \alpha)$ sample quantile of $\{T_b^*\}_{b=1}^B$.

APPENDIX B. VERIFICATION OF HIGH-LEVEL CONDITIONS

This section verifies high level assumptions used in Section 4 under primitive conditions. Assumptions used in Sections 5, 6, and 7 can be verified by similar arguments. First, I consider Assumption A3, which concerns the uniform consistency of the estimator $\widehat{\sigma}_i$ of σ_i over $i = \overline{1, n}$. Second, I verify Assumption A4 and derive a bound on the sensitivity parameter A_n when kernel weighting functions are used. This allows me to verify Assumption A5. Finally, I provide primitive conditions for Assumption A9.

Let C_6 and L' be strictly positive constants. The following proposition proves consistency of the local version of Rice's estimator.

Proposition B.1 (Verifying Assumption A3). *Suppose that $\widehat{\sigma}_i$ is the local version of Rice's estimator of σ_i given in equation (12). Suppose also that (i) Assumptions A1 and A8 hold, (ii) $b_n \geq (\log n)^2/(C_6 n)$, (iii) $|f(X_i) - f(X_j)| \leq L'|X_i - X_j|$ for all $i, j = \overline{1, n}$, and (iv) $|\sigma_i^2 - \sigma_j^2| \leq$*

$L'|X_i - X_j|$ for all $i, j = \overline{1, n}$. Then $\max_{1 \leq i \leq n} |\hat{\sigma}_i - \sigma_i| = O_p(b_n^2 + \log n / (b_n n^{1/2}))$, and so Assumption A3 holds with any κ_2 satisfying $b_n^2 + \log n / (b_n n^{1/2}) = o(n^{-\kappa_2})$.

Next, I consider restrictions on the weighting functions to ensure that Assumption A4 holds and give an upper bound on the sensitivity parameter A_n .

Proposition B.2 (Verifying Assumptions A4 and A5). *Suppose that kernel weighting functions are used. In addition, suppose that (i) Assumptions A1 and A8 hold, (ii) $K(\cdot)$ has the support $[-1, +1]$, is continuous, and strictly positive on the interior of its support, (iii) $x \in [s_l, s_r]$ for all $(x, h) \in \mathcal{S}_n$, (iv) $h_{\min} \geq (\log n)^2 / (C_6 n)$ w.p.a.1 where $h_{\min} = \min_{(x, h) \in \mathcal{S}_n} h$, and (v) $h_{\max} \leq (s_r - s_l) / 2$ where $h_{\max} = \max_{(x, h) \in \mathcal{S}_n} h$. Then (a) $A_n \leq C / (nh_{\min})^{1/2}$ w.p.a.1 for some $C > 0$, and so Assumption A5 holds for the basic set of weighting functions; (b) if Assumption A3 is satisfied, then Assumption A4 holds with $\kappa_3 = \kappa_2$; (c) if Assumption A2 is satisfied, then Assumption A4 holds with $\kappa_3 = \kappa_1$ as long as $\log p / (h_{\min} n^{1/2}) = o_p(n^{-\kappa_1})$.*

Restrictions on the kernel $K(\cdot)$ imposed in this lemma are satisfied for most commonly used kernel functions including uniform, triangular, Epanechnikov, biweight, triweight, and tricube kernels. Note, however, that these restrictions exclude higher order kernels since those are necessarily negative at some points on their supports. Assumption A8 imposes the restriction that the density of X is bounded below from zero on its support. This is needed to make sure that $V(s)$ is sufficiently well separated from zero since $V(s)$ appears in the denominator of the test statistic T . Alternatively, one can truncate $V(s)$ from below with the truncation parameter converging to zero with an appropriate rate as the sample size n increases. I do not consider this possibility for brevity of the paper. Finally, I verify Assumption A9.

Proposition B.3 (Verifying Assumption A9). *Suppose that the basic set of weighting functions is used. In addition, suppose that (i) Assumption A8 holds and (ii) $K(\cdot)$ has the support $[-1, +1]$, is continuous, and strictly positive on the interior of its support. Then Assumption A9 holds.*

APPENDIX C. ADDITIONAL NOTATION

I will use the following additional notation in Appendices C and D. Recall that $\{\epsilon_i\}$ is a sequence of independent $N(0, 1)$ random variables that are independent of the data. Denote $e_i = \sigma_i \epsilon_i$ and $\hat{e}_i = \hat{\sigma}_i \epsilon_i$ for $i = \overline{1, n}$. Let

$$w_i(s) = \sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s),$$

$$a_i(s) = w_i(s) / (V(s))^{1/2} \text{ and } \hat{a}_i(s) = w_i(s) / (\hat{V}(s))^{1/2},$$

$$e(s) = \sum_{1 \leq i \leq n} a_i(s) e_i, \text{ and } \hat{e}(s) = \sum_{1 \leq i \leq n} \hat{a}_i(s) \hat{e}_i,$$

$$\begin{aligned}\varepsilon(s) &= \sum_{1 \leq i \leq n} a_i(s) \varepsilon_i \text{ and } \widehat{\varepsilon}(s) = \sum_{1 \leq i \leq n} \widehat{a}_i(s) \varepsilon_i, \\ f(s) &= \sum_{1 \leq i \leq n} a_i(s) f(X_i) \text{ and } \widehat{f}(s) = \sum_{1 \leq i \leq n} \widehat{a}_i(s) f(X_i).\end{aligned}$$

Note that $T = \max_{s \in \mathcal{S}_n} \sum_{1 \leq i \leq n} \widehat{a}_i(s) Y_i = \max_{s \in \mathcal{S}_n} (\widehat{f}(s) + \widehat{\varepsilon}(s))$. In addition, for any $\mathcal{S} \subset \mathcal{S}_n$, which may depend on $\{X_i\}_{1 \leq i \leq n}$, and all $\eta \in (0, 1)$, let $c_\eta^\mathcal{S}$ denote the conditional η quantile of $T^* = T(\{X_i, Y_i^*\}, \{\widehat{\sigma}_i\}, \mathcal{S})$ given $\{X_i\}$, $\{\widehat{\sigma}_i\}$, and \mathcal{S} where $Y_i^* = \widehat{\sigma}_i \varepsilon_i$ for $i = \overline{1, n}$, and let $c_\eta^{\mathcal{S}, 0}$ denote the conditional η quantile of $T^* = T(\{X_i, Y_i^*\}, \{\sigma_i\}, \mathcal{S})$ given $\{X_i\}$ and \mathcal{S} where $Y_i^* = \sigma_i \varepsilon_i$ for $i = \overline{1, n}$. Further, for $\eta \leq 0$, define $c_\eta^\mathcal{S}$ and $c_\eta^{\mathcal{S}, 0}$ as $-\infty$, and for $\eta \geq 1$, define $c_\eta^\mathcal{S}$ and $c_\eta^{\mathcal{S}, 0}$ as $+\infty$.

In the next section, I prove some auxiliary lemmas. For these lemmas, it will be convenient to use the following additional notation. Recall that \mathcal{S}_n is a set that is allowed to depend on $\{X_i\}$. In addition, I will need some subsets of \mathcal{S}_n that are selected from \mathcal{S}_n according to certain mappings that also depend $\{X_i\}$, i.e. $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\}) \subset \mathcal{S}_n$. Some results will be stated to hold uniformly over all mappings $\mathcal{S}(\cdot, \cdot)$.

Let $\{\psi_n\}$ be a sequence of positive numbers converging to zero sufficiently slowly so that (i) $\log p/n^{\kappa_3} = o_p(\psi_n)$ (recall that by Assumption A5, $\log p/n^{\kappa_3} = o_p(1)$, and so such a sequence exists), (ii) uniformly over all $\mathcal{S}(\cdot, \cdot)$ and $\eta \in (0, 1)$, $P(c_{\eta+\psi_n}^{\mathcal{S}, 0} < c_\eta^\mathcal{S}) = o(1)$ and $P(c_{\eta+\psi_n}^\mathcal{S} < c_\eta^{\mathcal{S}, 0}) = o(1)$ where $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\})$ (Lemma D.5 establishes existence of such a sequence under Assumptions A1, A3, A4, and A5 and Lemma D.7 establishes existence under Assumptions A1, A2, A4, and A5). Let

$$\mathcal{S}_n^R = \{s \in \mathcal{S}_n : f(s) > -c_{1-\gamma_n-\psi_n}^{\mathcal{S}_n, 0}\}.$$

For $D = PI, OS, SD, R$, let $c_\eta^D = c_\eta^{\mathcal{S}_n^D}$ and $c_\eta^{D, 0} = c_\eta^{\mathcal{S}_n^{D, 0}}$ where $\mathcal{S}_n^{PI} = \mathcal{S}_n$. Note that $c_\eta^{PI, 0}$ and $c_\eta^{R, 0}$ are non-stochastic.

Moreover, I denote $\mathcal{V} = \max_{s \in \mathcal{S}_n} (V(s)/\widehat{V}(s))^{1/2}$. Finally, I denote the space of k -times continuously differentiable functions on \mathbb{R} by $\mathbb{C}^k(\mathbb{R}, \mathbb{R})$. For $g \in \mathbb{C}^k(\mathbb{R}, \mathbb{R})$, the symbol $g^{(r)}$ for $r \leq k$ denotes the r th derivative of g , and $\|g^{(r)}\|_\infty = \sup_{t \in \mathbb{R}} |g^{(r)}(t)|$.

APPENDIX D. PROOFS FOR SECTION 4

In this Appendix, I first prove a sequence of auxiliary lemmas (Subsection D.1). Then I present the proofs of the theorems stated in Section 4 (Subsection D.2). In Subsection D.1, all results hold uniformly over all models $M \in \mathcal{M}$. For simplicity of notation, however, I drop index M everywhere.

D.1. Auxiliary Lemmas.

Lemma D.1. $E[\max_{s \in \mathcal{S}_n} |e(s)| | \{X_i\}] \leq C(\log p)^{1/2}$ for some universal $C > 0$.

Proof. Note that by construction, conditional on $\{X_i\}$, $e(s)$ is distributed as a $N(0, 1)$ random variable, and $|\mathcal{S}_n| = p$. So, the result follows from lemma 2.2.2 in Van der Vaart and Wellner (1996). \square

Lemma D.2. For all $\mathcal{S}(\cdot, \cdot)$ and $\Delta > 0$,

$$\sup_{t \in \mathbb{R}} \mathbb{P} \left(\max_{s \in \mathcal{S}} e(s) \in [t, t + \Delta] \mid \{X_i\} \right) \leq C \Delta (\log p)^{1/2}$$

and for all $\mathcal{S}(\cdot, \cdot)$ and $(\eta, \delta) \in (0, 1)^2$,

$$c_{\eta+\delta}^{\mathcal{S}, 0} - c_{\eta}^{\mathcal{S}, 0} \geq C \delta / (\log p)^{1/2}$$

for some universal $C > 0$ where $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\})$.

Proof. Theorem 3 in Chernozhukov, Chetverikov, and Kato (2011) shows that if W_1, \dots, W_p is a sequence of $N(0, 1)$ random (not necessarily independent) variables, then for all $\Delta > 0$,

$$\sup_{t \in \mathbb{R}} \mathbb{P} \left(\max_{1 \leq j \leq p} W_j \in [t, t + \Delta] \right) \leq 4\Delta \left(\mathbb{E} \left[\max_{1 \leq j \leq p} W_j \right] + 1 \right).$$

Therefore,

$$\sup_{t \in \mathbb{R}} \mathbb{P} \left(\max_{s \in \mathcal{S}} e(s) \in [t, t + \Delta] \mid \{X_i\} \right) \leq 4\Delta \left(\mathbb{E} \left[\max_{s \in \mathcal{S}} e(s) \mid \{X_i\} \right] + 1 \right),$$

and so the first claim follows from Lemma D.1. The second claim follows from the result in the first claim. \square

Lemma D.3. For all $\mathcal{S}(\cdot, \cdot)$, $\eta \in (0, 1)$, and $t \in \mathbb{R}$,

$$c_{\eta - C|t| \log p / (1-\eta)}^{\mathcal{S}, 0} \leq c_{\eta}^{\mathcal{S}, 0} (1 + t) \leq c_{\eta + C|t| \log p / (1-\eta)}^{\mathcal{S}, 0}$$

for some universal $C > 0$ where $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\})$.

Proof. Recall that $c_{\eta}^{\mathcal{S}, 0}$ is the conditional η quantile of $\max_{s \in \mathcal{S}} e(s)$ given $\{X_i\}$, and so combining Lemma D.1 and Markov inequality shows that $c_{\eta}^{\mathcal{S}, 0} \leq C (\log p)^{1/2} / (1 - \eta)$ for some universal $C > 0$. Therefore, Lemma D.2 gives

$$c_{\eta + C|t| \log p / (1-\eta)}^{\mathcal{S}, 0} - c_{\eta}^{\mathcal{S}, 0} \geq C|t| (\log p)^{1/2} / (1 - \eta) \geq |t| c_{\eta}^{\mathcal{S}, 0} \tag{33}$$

if $C > 0$ is sufficiently large in (33). The lower bound follows similarly. \square

Lemma D.4. Under Assumptions A1 and A5, uniformly over all $\mathcal{S}(\cdot, \cdot)$ and $\eta \in (0, 1)$,

$$\mathbb{P} \left(\max_{s \in \mathcal{S}} \varepsilon(s) \leq c_{\eta}^{\mathcal{S}, 0} \right) = \eta + o(1) \text{ and } \mathbb{P} \left(\max_{s \in \mathcal{S}} (-\varepsilon(s)) \leq c_{\eta}^{\mathcal{S}, 0} \right) = \eta + o(1)$$

where $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\})$.

Proof. Note that $\sum_{1 \leq i \leq n} (a_i(s)\sigma_i)^2 = 1$. In addition, by Assumption A5, $nA_n^4(\log(pn))^7 = o_p(1)$, and so there exists a sequence $\{\omega_n\}$ of positive numbers such that $nA_n^4(\log(pn))^7 = o_p(\omega_n)$ and $\omega_n = o(1)$. Let \mathcal{A}_n denote the event that $nA_n^4(\log(pn))^7 > \omega_n$. Then $P(\mathcal{A}_n) = o(1)$. Further, note that $E[\varepsilon_i^2/\sigma_i^2|X_i] = 1$, and since $E[\varepsilon_i^4|X_i] \leq C_1$ and $\sigma_i \geq c_1$ by Assumption A1, $E[\varepsilon_i^4/\sigma_i^4|X_i] \leq C$ for some $C > 0$. Therefore, applying Lemma H.4, case (ii), with $z_{is} = \sqrt{n}a_i(s)\sigma_i$ and $u_i = \varepsilon_i/\sigma_i$ conditional on $\{X_i\}$ shows that there exists a sequence $\{\omega'_n\}$ of positive numbers converging to zero and depending only on $\{\omega_n\}$ such that outside of \mathcal{A}_n ,

$$\left| P\left(\max_{s \in \mathcal{S}} \varepsilon(s) \leq c_\eta^{\mathcal{S},0} | \{X_i\}\right) - \eta \right| \leq \omega'_n.$$

Since $P(\mathcal{A}_n) = o(1)$, this implies that

$$P\left(\max_{s \in \mathcal{S}} \varepsilon(s) \leq c_\eta^{\mathcal{S},0}\right) = \eta + o(1),$$

which gives the first asserted claim. The second claim follows by replacing $\varepsilon(s)$ by $-\varepsilon(s)$. Note that since the sequence $\{\omega'_n\}$ depends only on the sequence $\{\omega_n\}$, it follows that the result of this lemma holds uniformly over all models $M \in \mathcal{M}$. \square

Lemma D.5. *Under Assumptions A1, A3, A4, and A5, there exists a sequence $\{\psi_n\}$ of positive numbers converging to zero such that uniformly over all $\mathcal{S}(\cdot, \cdot)$ and $\eta \in (0, 1)$, $P(c_{\eta+\psi_n}^{\mathcal{S},0} < c_\eta^{\mathcal{S}}) = o(1)$ and $P(c_{\eta+\psi_n}^{\mathcal{S}} < c_\eta^{\mathcal{S},0}) = o(1)$ where $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\})$.*

Proof. Denote

$$T^{\mathcal{S}} = \max_{s \in \mathcal{S}} \widehat{e}(s) = \max_{s \in \mathcal{S}} \sum_{1 \leq i \leq n} \widehat{a}_i(s) \widehat{\sigma}_i \epsilon_i \text{ and } T^{\mathcal{S},0} = \max_{s \in \mathcal{S}} e(s) = \max_{s \in \mathcal{S}} \sum_{1 \leq i \leq n} a_i(s) \sigma_i \epsilon_i.$$

In addition, denote

$$p_1 = \max_{s \in \mathcal{S}} |e(s)| \max_{s \in \mathcal{S}} \left| 1 - (V(s)/\widehat{V}(s))^{1/2} \right|,$$

$$p_2 = \max_{s \in \mathcal{S}} \left| \sum_{1 \leq i \leq n} a_i(s) (\widehat{\sigma}_i - \sigma_i) \epsilon_i \right| \max_{s \in \mathcal{S}} (V(s)/\widehat{V}(s))^{1/2}.$$

Then $|T^{\mathcal{S}} - T^{\mathcal{S},0}| \leq p_1 + p_2$. Combining Lemma D.1 and Assumption A4 gives

$$p_1 = o_p\left((\log p)^{1/2} n^{-\kappa_3}\right).$$

Consider p_2 . Conditional on $\{\widehat{\sigma}_i\}$, $(\widehat{\sigma}_i - \sigma_i)\epsilon_i$ is distributed as a $N(0, (\widehat{\sigma}_i - \sigma_i)^2)$ random variable, and so applying the argument like that in Lemma D.1 conditional on $\{X_i, \widehat{\sigma}_i\}$ and using Assumption A3 gives

$$\max_{s \in \mathcal{S}} \left| \sum_{1 \leq i \leq n} a_i(s) (\widehat{\sigma}_i - \sigma_i) \epsilon_i \right| = o_p\left((\log p)^{1/2} n^{-\kappa_2}\right).$$

Since $\max_{s \in \mathcal{S}} (V(s)/\widehat{V}(s))^{1/2} \rightarrow_p 1$ by Assumption A4, this implies that

$$p_2 = o_p\left((\log p)^{1/2} n^{-\kappa_2}\right).$$

Therefore, $T^{\mathcal{S}} - T^{\mathcal{S},0} = o_p((\log p)^{1/2}n^{-\kappa_2 \wedge \kappa_3})$, and so there exists a sequence $\{\tilde{\psi}_n\}$ of positive numbers converging to zero such that

$$\mathbb{P}\left(|T^{\mathcal{S}} - T^{\mathcal{S},0}| > (\log p)^{1/2}n^{-\kappa_2 \wedge \kappa_3}\right) = o(\tilde{\psi}_n).$$

Hence,

$$\mathbb{P}\left(\mathbb{P}\left(|T^{\mathcal{S}} - T^{\mathcal{S},0}| > (\log p)^{1/2}n^{-\kappa_2 \wedge \kappa_3} \mid \{X_i, \hat{\sigma}_i\}\right) > \tilde{\psi}_n\right) \rightarrow 0.$$

Let \mathcal{A}_n denote the event that

$$\mathbb{P}\left(|T^{\mathcal{S}} - T^{\mathcal{S},0}| > (\log p)^{1/2}n^{-\kappa_2 \wedge \kappa_3} \mid \{X_i, \hat{\sigma}_i\}\right) \leq \tilde{\psi}_n.$$

I will take $\psi_n = \tilde{\psi}_n + C(\log p)n^{-\kappa_2 \wedge \kappa_3}$ for a constant C that is larger than that in the statement of Lemma D.2. By assumption A5, $\psi_n \rightarrow 0$. Then note that

$$\mathbb{P}(T^{\mathcal{S},0} \leq c_{\eta}^{\mathcal{S},0} \mid \{X_i, \hat{\sigma}_i\}) \geq \eta \text{ and } \mathbb{P}(T^{\mathcal{S}} \leq c_{\eta}^{\mathcal{S}} \mid \{X_i, \hat{\sigma}_i\}) \geq \eta$$

for any $\eta \in (0, 1)$. So, on \mathcal{A}_n ,

$$\begin{aligned} \eta + \tilde{\psi}_n &\leq \mathbb{P}\left(T^{\mathcal{S},0} \leq c_{\eta + \tilde{\psi}_n}^{\mathcal{S},0} \mid \{X_i, \hat{\sigma}_i\}\right) \\ &\leq \mathbb{P}\left(T^{\mathcal{S}} \leq c_{\eta + \tilde{\psi}_n}^{\mathcal{S},0} + (\log p)^{1/2}n^{-\kappa_2 \wedge \kappa_3} \mid \{X_i, \hat{\sigma}_i\}\right) + \tilde{\psi}_n \leq \mathbb{P}\left(T^{\mathcal{S}} \leq c_{\eta + \psi_n}^{\mathcal{S},0} \mid \{X_i, \hat{\sigma}_i\}\right) + \tilde{\psi}_n \end{aligned}$$

where the last inequality follows from Lemma D.2. Therefore, on \mathcal{A}_n , $c_{\eta}^{\mathcal{S}} \leq c_{\eta + \psi_n}^{\mathcal{S},0}$, i.e. $\mathbb{P}(c_{\eta + \psi_n}^{\mathcal{S},0} < c_{\eta}^{\mathcal{S}}) = o(1)$. The second claim follows similarly. \square

Lemma D.6. *Let $c_{\eta}^{\mathcal{S},1}$ denote the conditional η quantile of $T^{\mathcal{S},1} = \max_{s \in \mathcal{S}} \sum_{1 \leq i \leq n} a_i(s) \varepsilon_i \epsilon_i$ given $\{X_i, \varepsilon_i\}$. Let Assumptions A1, A2, and A5 hold. Then there exists a sequence $\{\tilde{\psi}_n\}$ of positive numbers converging to zero such that uniformly over all $\mathcal{S}(\cdot, \cdot)$ and $\eta \in (0, 1)$, $\mathbb{P}(c_{\eta + \tilde{\psi}_n}^{\mathcal{S},0} < c_{\eta}^{\mathcal{S},1}) = o(1)$ and $\mathbb{P}(c_{\eta + \tilde{\psi}_n}^{\mathcal{S},1} < c_{\eta}^{\mathcal{S},0}) = o(1)$ where $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\})$.*

Proof. Let $Z^1 = \{\sum_{1 \leq i \leq n} a_i(s) \varepsilon_i \epsilon_i\}_{s \in \mathcal{S}}$ and $Z^2 = \{\sum_{1 \leq i \leq n} a_i(s) \sigma_i \epsilon_i\}_{s \in \mathcal{S}}$. Conditional on $\{X_i, \varepsilon_i\}$, these are zero-mean Gaussian vectors in $\mathbb{R}^{|\mathcal{S}|}$ with covariances Σ^1 and Σ^2 given by

$$\Sigma_{s_1 s_2}^1 = \sum_{1 \leq i \leq n} a_i(s_1) a_i(s_2) \varepsilon_i^2 \text{ and } \Sigma_{s_1 s_2}^2 = \sum_{1 \leq i \leq n} a_i(s_1) a_i(s_2) \sigma_i^2.$$

Let $\Delta_{\Sigma} = \max_{s_1, s_2 \in \mathcal{S}} |\Sigma_{s_1 s_2}^1 - \Sigma_{s_1 s_2}^2|$. Applying Lemma H.2 shows that under Assumption A1, for some $C > 0$,

$$\mathbb{E}[\Delta_{\Sigma} \mid \{X_i\}] \leq C \left(A_n \sqrt{\log p} + A_n^2 \log p \mathbb{E} \left[\max_{1 \leq i \leq n} \varepsilon_i^4 \mid \{X_i\} \right]^{1/2} \right).$$

Further, $\mathbb{E}[\max_{1 \leq i \leq n} \varepsilon_i^4 \mid \{X_i\}] \leq \sum_{1 \leq i \leq n} \mathbb{E}[\varepsilon_i^4 \mid \{X_i\}] \leq Cn$ for some $C > 0$ by Assumption A1, and so Assumption A5 implies that

$$(\log p)^2 \Delta_{\Sigma} = o_p(1) \tag{34}$$

because $A_n^2(\log p)^3 n^{1/2} = o_p(1)$ and $A_n(\log p)^{5/2} = o_p(1)$ (to verify the latter equation, recall that $\log p/n^{1/4} = o(1)$ by Assumption A5; taking square of this equation and multiplying it by $A_n^2(\log p)^3 n^{1/2} = o_p(1)$ gives $A_n(\log p)^{5/2} = o_p(1)$). Therefore, it follows that there exists a sequence $\{\tilde{\psi}_n\}$ of positive numbers converging to zero such that

$$(\log p)^2 \Delta_\Sigma = o_p(\tilde{\psi}_n^4). \quad (35)$$

Let $g \in \mathbb{C}^2(\mathbb{R}, \mathbb{R})$ be a function satisfying $g(t) = 1$ for $t \leq 0$, $g(t) = 0$ for $t \geq 1$, and $g(t) \in [0, 1]$ for $t \in [0, 1]$, and let $g_n(t) = g((t - c_{\eta+\tilde{\psi}_n/2}^{\mathcal{S},0}) / (c_{\eta+\tilde{\psi}_n}^{\mathcal{S},0} - c_{\eta+\tilde{\psi}_n/2}^{\mathcal{S},0}))$. Then for some $C > 0$, Lemma D.2 shows that

$$\begin{aligned} \|g_n^{(1)}\|_\infty &\leq \|g^{(1)}\|_\infty / \left(c_{\eta+\tilde{\psi}_n}^{\mathcal{S},0} - c_{\eta+\tilde{\psi}_n/2}^{\mathcal{S},0} \right) \leq C(\log p)^{1/2} / \tilde{\psi}_n, \\ \|g_n^{(2)}\|_\infty &\leq \|g^{(2)}\|_\infty / \left(c_{\eta+\tilde{\psi}_n}^{\mathcal{S},0} - c_{\eta+\tilde{\psi}_n/2}^{\mathcal{S},0} \right)^2 \leq C(\log p) / \tilde{\psi}_n^2. \end{aligned}$$

Applying Lemma H.3 gives for some $C > 0$,

$$D_n = \left| \mathbb{E} \left[g_n \left(\max_{s \in \mathcal{S}} Z_s^1 \right) - g_n \left(\max_{s \in \mathcal{S}} Z_s^2 \right) \mid \{X_i, \varepsilon_i\} \right] \right| \quad (36)$$

$$\leq C \left((\log p) \Delta_\Sigma / \tilde{\psi}_n^2 + (\log p) (\Delta_\Sigma)^{1/2} / \tilde{\psi}_n \right) = o_p(\tilde{\psi}_n) \quad (37)$$

by equation (35). Note that $\max_{s \in \mathcal{S}} Z_s^1 = T^{\mathcal{S},1}$ and, using the notation in the proof of Lemma D.5, $\max_{s \in \mathcal{S}} Z_s^2 = T^{\mathcal{S},0}$. Therefore,

$$\begin{aligned} \mathbb{P} \left(T^{\mathcal{S},1} \leq c_{\eta+\tilde{\psi}_n}^{\mathcal{S},0} \mid \{X_i, \varepsilon_i\} \right) &\geq_{(1)} \mathbb{E} [g_n(T^{\mathcal{S},1}) \mid \{X_i, \varepsilon_i\}] \\ &\geq_{(2)} \mathbb{E} [g_n(T^{\mathcal{S},0}) \mid \{X_i, \varepsilon_i\}] - D_n \\ &\geq_{(3)} \mathbb{P} \left(T^{\mathcal{S},0} \leq c_{\eta+\tilde{\psi}_n/2}^{\mathcal{S},0} \mid \{X_i, \varepsilon_i\} \right) - D_n \\ &=_{(4)} \mathbb{P} \left(T^{\mathcal{S},0} \leq c_{\eta+\tilde{\psi}_n/2}^{\mathcal{S},0} \mid \{X_i\} \right) - D_n \geq \eta + \tilde{\psi}_n/2 - D_n \end{aligned} \quad (38)$$

where (1) and (3) are by construction of the function g_n , (2) is by the definition of D_n , and (4) is because $T^{\mathcal{S},0}$ and $c_{\eta+\tilde{\psi}_n/2}^{\mathcal{S},0}$ are jointly independent of $\{\varepsilon_i\}$. Finally, note that the right hand side of line (38) is bounded from below by η w.p.a.1. This implies that $\mathbb{P}(c_{\eta+\tilde{\psi}_n}^{\mathcal{S},0} < c_\eta^{\mathcal{S},1}) = o(1)$, which is the first asserted claim. The second claim of the lemma follows similarly. \square

Lemma D.7. *Under Assumptions A1, A2, A4, and A5, there exists a sequence $\{\psi_n\}$ of positive numbers converging to zero such that uniformly over all $\mathcal{S}(\cdot, \cdot)$ and $\eta \in (0, 1)$, $\mathbb{P}(c_{\eta+\psi_n}^{\mathcal{S},0} < c_\eta^{\mathcal{S}}) = o(1)$ and $\mathbb{P}(c_{\eta+\psi_n}^{\mathcal{S}} < c_\eta^{\mathcal{S},0}) = o(1)$ where $\mathcal{S} = \mathcal{S}(\mathcal{S}_n, \{X_i\})$.⁸*

Proof. Lemma D.6 established that

$$\mathbb{P}(c_{\eta+\tilde{\psi}_n}^{\mathcal{S},0} < c_\eta^{\mathcal{S},1}) = o(1) \text{ and } \mathbb{P}(c_{\eta+\tilde{\psi}_n}^{\mathcal{S},1} < c_\eta^{\mathcal{S},0}) = o(1).$$

⁸Note that Lemmas D.5 and D.7 provide the same results under two different methods for estimating σ_i .

Therefore, it suffices to show that

$$\mathbb{P}(c_{\eta+\widehat{\psi}_n}^{\mathcal{S}} < c_{\eta}^{\mathcal{S},1}) \text{ and } \mathbb{P}(c_{\eta+\widehat{\psi}_n}^{\mathcal{S},1} < c_{\eta}^{\mathcal{S}})$$

for some sequence $\{\widehat{\psi}_n\}$ of positive numbers converging to zero. Denote

$$p_1 = \max_{s \in \mathcal{S}} \left| \sum_{1 \leq i \leq n} a_i(s) \varepsilon_i \epsilon_i \right| \max_{s \in \mathcal{S}} \left| 1 - (V(s)/\widehat{V}(s))^{1/2} \right|,$$

$$p_2 = \max_{s \in \mathcal{S}} \left| \sum_{1 \leq i \leq n} a_i(s) (\widehat{\sigma}_i - \varepsilon_i) \epsilon_i \right| \max_{s \in \mathcal{S}} (V(s)/\widehat{V}(s))^{1/2}.$$

Note that $|T^{\mathcal{S}} - T^{\mathcal{S},1}| \leq p_1 + p_2$ and that by Lemmas D.1 and D.6, $\max_{s \in \mathcal{S}} |\sum_{1 \leq i \leq n} a_i(s) \varepsilon_i \epsilon_i| = O_p((\log p)^{1/2})$. Therefore, the result follows by the argument similar to that used in the proof of Lemma D.5 since $\widehat{\sigma}_i - \varepsilon_i = o_p(n^{-\kappa_1})$ by Assumption A2. \square

Lemma D.8. *Let Assumptions A1, A4, and A5 hold. In addition, let either Assumption A2 or A3 hold. Then $\mathbb{P}(\mathcal{S}_n^R \subset \mathcal{S}_n^{SD}) \geq 1 - \gamma_n + o(1)$ and $\mathbb{P}(\mathcal{S}_n^R \subset \mathcal{S}_n^{OS}) \geq 1 - \gamma_n + o(1)$.*

Proof. By the definition of ψ_n , $\log p/n^{\kappa_3} = o_p(\psi_n)$, and so there exists a sequence $\{\omega_n\}$ of positive numbers converging to zero such that $\log p/n^{\kappa_3} = o_p(\omega_n \psi_n)$. Let \mathcal{A}_n denote the event that $\log p/n^{\kappa_3} > \omega_n \psi_n$. Then $\mathbb{P}(\mathcal{A}_n) = o(1)$.

Further, suppose that $\mathcal{S}_n^R \setminus \mathcal{S}_n^{SD} \neq \emptyset$. Then there exists the smallest integer l such that $\mathcal{S}_n^R \setminus \mathcal{S}_n^l \neq \emptyset$ and $\mathcal{S}_n^R \subset \mathcal{S}_n^{l-1}$ (if $l = 1$, let $\mathcal{S}_n^0 = \mathcal{S}_n$). Therefore, $c_{1-\gamma_n}^R \leq c_{1-\gamma_n}^{l-1}$. It follows that there exists an element s of \mathcal{S}_n^R such that

$$\widehat{f}(s) + \widehat{\varepsilon}(s) \leq -c_{1-\gamma_n}^{PI} - c_{1-\gamma_n}^{l-1} \leq -c_{1-\gamma_n}^{PI} - c_{1-\gamma_n}^R,$$

and so for some $C > 0$,

$$\begin{aligned} \mathbb{P}(\mathcal{S}_n^R \setminus \mathcal{S}_n^{SD} \neq \emptyset) &\leq \mathbb{P}(\min_{s \in \mathcal{S}_n^R} (\widehat{f}(s) + \widehat{\varepsilon}(s)) \leq -c_{1-\gamma_n}^{PI} - c_{1-\gamma_n}^R) \\ &\leq_{(1)} \mathbb{P}(\min_{s \in \mathcal{S}_n^R} (f(s) + \varepsilon(s)) \mathcal{V} \leq -c_{1-\gamma_n}^{PI} - c_{1-\gamma_n}^R) \\ &\leq_{(2)} \mathbb{P}(\min_{s \in \mathcal{S}_n^R} (f(s) + \varepsilon(s)) \mathcal{V} \leq -c_{1-\gamma_n-\psi_n}^{PI,0} - c_{1-\gamma_n-\psi_n}^{R,0}) + o(1) \\ &\leq_{(3)} \mathbb{P}(\min_{s \in \mathcal{S}_n^R} (\varepsilon(s) - c_{1-\gamma_n-\psi_n}^{PI,0}) \mathcal{V} \leq -c_{1-\gamma_n-\psi_n}^{PI,0} - c_{1-\gamma_n-\psi_n}^{R,0}) + o(1) \\ &=_{(4)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} (-\varepsilon(s)) \geq c_{1-\gamma_n-\psi_n}^{PI,0} (1/\mathcal{V} - 1) + c_{1-\gamma_n-\psi_n}^{R,0}/\mathcal{V}) + o(1) \\ &\leq_{(5)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} (-\varepsilon(s)) \geq c_{1-\gamma_n-\psi_n}^{R,0}/\mathcal{V} - C(\log p)^{1/2} n^{-\kappa_3}/(\gamma_n + \psi_n)) + o(1) \\ &\leq_{(6)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} (-\varepsilon(s)) \geq c_{1-\gamma_n-\psi_n-C(\log p)n^{-\kappa_3}/(\gamma_n+\psi_n)}^{R,0}) + o(1) \\ &\leq_{(7)} \gamma_n + \psi_n + C\omega_n + \mathbb{P}(\mathcal{A}_n) + o(1) =_{(8)} \gamma_n + o(1) \end{aligned}$$

where (1) follows from the definitions of $\hat{f}(s)$ and $\hat{\varepsilon}(s)$, (2) is by the definition of ψ_n , (3) is by the definition of \mathcal{S}_n^R , (4) is rearrangement, (5) is by Lemma D.1 and Assumption A4, (6) is by Lemmas D.2 and D.3, (7) is by Lemma D.4 and definitions of ω_n and \mathcal{A}_n , and (8) follows from the definition of ψ_n again. The first asserted claim follows. The second claim follows from the fact that $\mathcal{S}_n^{SD} \subset \mathcal{S}_n^{OS}$. \square

Lemma D.9. *Let Assumptions A1, A4, and A5 hold. In addition, let either Assumption A2 or A3 hold. Then $\mathbb{P}(\max_{s \in \mathcal{S}_n \setminus \mathcal{S}_n^R} (\hat{f}(s) + \hat{\varepsilon}(s)) \leq 0) \geq 1 - \gamma_n + o(1)$.*

Proof. The result follows from

$$\begin{aligned} \mathbb{P}(\max_{s \in \mathcal{S}_n \setminus \mathcal{S}_n^R} (\hat{f}(s) + \hat{\varepsilon}(s)) \leq 0) &= \mathbb{P}(\max_{s \in \mathcal{S}_n \setminus \mathcal{S}_n^R} (f(s) + \varepsilon(s)) \leq 0) \\ &\geq_{(1)} \mathbb{P}(\max_{s \in \mathcal{S}_n \setminus \mathcal{S}_n^R} \varepsilon(s) \leq c_{1-\gamma_n-\psi_n}^{PI,0}) \\ &\geq \mathbb{P}(\max_{s \in \mathcal{S}_n} \varepsilon(s) \leq c_{1-\gamma_n-\psi_n}^{PI,0}) \\ &=_{(2)} 1 - \gamma_n - \psi_n + o(1) =_{(3)} 1 - \gamma_n + o(1) \end{aligned}$$

where (1) follows from the definition of \mathcal{S}_n^R , (2) is by Lemma D.4, and (3) is by the definition of ψ_n . \square

D.2. Proofs of Theorems.

Proof of Theorem 4.1. All inequalities presented in this proof hold uniformly over all models $M \in \mathcal{M}$ because all of them only rely on assumptions and lemmas in Subsection D.1, which hold uniformly over all models $M \in \mathcal{M}$. For brevity of notation, I drop indexing by M .

By the definition of ψ_n , $\log p/n^{\kappa_3} = o_p(\psi_n)$, and so there exists a sequence $\{\omega_n\}$ of positive numbers converging to zero such that $\log p/n^{\kappa_3} = o_p(\omega_n \psi_n)$. Let \mathcal{A}_n denote the event that $\log p/n^{\kappa_3} > \omega_n \psi_n$. Then $\mathbb{P}(\mathcal{A}_n) = o(1)$.

Further, note that for some $C > 0$,

$$\begin{aligned} \mathbb{P}(T \leq c_{1-\alpha}^P) &= \mathbb{P}(\max_{s \in \mathcal{S}_n} (\hat{f}(s) + \hat{\varepsilon}(s)) \leq c_{1-\alpha}^P) \geq_{(1)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} (\hat{f}(s) + \hat{\varepsilon}(s)) \leq c_{1-\alpha}^P) - \gamma_n + o(1) \\ &\geq_{(2)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} (\hat{f}(s) + \hat{\varepsilon}(s)) \leq c_{1-\alpha}^R) - 2\gamma_n + o(1) \geq_{(3)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} \hat{\varepsilon}(s) \leq c_{1-\alpha}^R) - 2\gamma_n + o(1) \\ &\geq_{(4)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} \varepsilon(s) \mathcal{V} \leq c_{1-\alpha-\psi_n}^{R,0}) - 2\gamma_n + o(1) =_{(5)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} \varepsilon(s) \leq c_{1-\alpha-\psi_n}^{R,0}/\mathcal{V}) - 2\gamma_n + o(1) \\ &\geq_{(6)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} \varepsilon(s) \leq c_{1-\alpha-\psi_n}^{R,0} (1 - n^{-\kappa_3})) - 2\gamma_n + o(1) \\ &\geq_{(7)} \mathbb{P}(\max_{s \in \mathcal{S}_n^R} \varepsilon(s) \leq c_{1-\alpha-\psi_n-C(\log p)n^{-\kappa_3}/(\alpha+\psi_n)}^{R,0}) - 2\gamma_n + o(1) \\ &\geq_{(8)} 1 - \alpha - \psi_n - C\omega_n - \mathbb{P}(\mathcal{A}_n) - 2\gamma_n + o(1) =_{(9)} 1 - \alpha + o(1) \end{aligned}$$

where (1) follows from Lemma D.9, (2) is by Lemma D.8, (3) is because under \mathcal{H}_0 , $\widehat{f}(s) \leq 0$, (4) follows from the definitions of $\widehat{\varepsilon}(s)$ and ψ_n , (5) is rearrangement, (6) is by Assumption A4, (7) is by Lemma D.3, (8) is by Lemma D.4 and definitions of ω_n and \mathcal{A}_n , and (9) is by the definitions of ψ_n and γ_n . The first asserted claim follows.

In addition, when $f(\cdot)$ is identically constant,

$$\begin{aligned} \mathbb{P}(T \leq c_{1-\alpha}^P) &=_{(1)} \mathbb{P}(\max_{s \in \mathcal{S}_n} \widehat{\varepsilon}(s) \leq c_{1-\alpha}^P) \leq_{(2)} \mathbb{P}(\max_{s \in \mathcal{S}_n} \widehat{\varepsilon}(s) \leq c_{1-\alpha}^{PI}) \\ &\leq_{(3)} \mathbb{P}(\max_{s \in \mathcal{S}_n} \widehat{\varepsilon}(s) \leq c_{1-\alpha+\psi_n}^{PI,0}) + o(1) \leq_{(4)} \mathbb{P}(\max_{s \in \mathcal{S}_n} \varepsilon(s) \leq c_{1-\alpha+\psi_n}^{PI,0} (1 + n^{-\kappa_3})) + o(1) \\ &\leq_{(5)} \mathbb{P}(\max_{s \in \mathcal{S}_n} \varepsilon(s) \leq c_{1-\alpha+\psi_n+C(\log p)n^{-\kappa_3}/(\alpha-\psi_n)}^{PI,0}) + o(1) \leq_{(6)} 1 - \alpha + o(1) \end{aligned}$$

where (1) follows from the fact that $\widehat{f}(s) = 0$ whenever $f(\cdot)$ is identically constant, (2) follows from $\mathcal{S}_n^P \subset \mathcal{S}_n$, (3) is by the definition of ψ_n , (4) is by Assumption A4, (5) is by Lemma D.3, and (6) is from Lemma D.4 and the definition of ψ_n . The second asserted claim follows. \square

Proof of Theorem 4.2. Let $x_1, x_2 \in [s_l, s_r]$ be such that $x_1 < x_2$ but $f(x_1) > f(x_2)$. By the mean value theorem, there exists $x_0 \in (x_1, x_2)$ satisfying

$$f'(x_0)(x_2 - x_1) = f(x_2) - f(x_1) < 0.$$

Therefore, $f'(x_0) < 0$. Since $f'(\cdot)$ is continuous, $f'(x) < f'(x_0)/2$ for any $x \in [x_0 - \Delta_x, x_0 + \Delta_x]$ for some $\Delta_x > 0$. Apply Assumption A7 to the interval $[x_0 - \Delta_x, x_0 + \Delta_x]$ to obtain $s = s_n \in \mathcal{S}_n$ and intervals $[x_{l1}, x_{r1}]$ and $[x_{l2}, x_{r2}]$ appearing in Assumption A7-iii. By Assumption A7-ii,

$$V(s) \leq Cn^3 \tag{39}$$

for some $C > 0$. In addition, by Assumption A6 and Chebyshev's inequality, there is some $C > 0$ such that $|\{i = \overline{1, n} : X_i \in [x_{l1}, x_{r1}]\}| \geq Cn$ and $|\{i = \overline{1, n} : X_i \in [x_{l2}, x_{r2}]\}| \geq Cn$ w.p.a.1. Hence, by Assumptions A7-(i) and A7-(iii),

$$\sum_{1 \leq i, j \leq n} (f(X_i) - f(X_j)) \text{sign}(X_j - X_i) Q(X_i, X_j, s) \geq Cn^2 \tag{40}$$

w.p.a.1 for some $C > 0$. Combining (39) and (40) yields $f(s) \geq Cn^{1/2}$ w.p.a.1 for some $C > 0$. Let \mathcal{A}_n denote the event that $f(s) < Cn^{1/2}$, so that $\mathbb{P}(\mathcal{A}_n) = o(1)$. Further, since $\sum_{1 \leq i \leq n} a_i(s)^2 \sigma_i^2 = 1$, Assumption A1 implies $A_n \geq C/n^{1/2}$ for some $C > 0$, and so Assumption A5 gives $\log p = o_p(n)$, and so

$$(\log p)^{1/2} = o_p(f(s)). \tag{41}$$

Therefore, there exists a sequence $\{\omega_n\}$ of positive numbers converging to zero such that $(\log p)^{1/2} = o_p(\omega_n f(s))$, and so letting \mathcal{B}_n denote the event that $(\log p)^{1/2} > \omega_n f(s)$ leads to $\mathbb{P}(\mathcal{B}_n) = o(1)$.

Hence, for some $C > 0$,

$$\begin{aligned}
\mathbb{P}(T \leq c_{1-\alpha}^P) &\leq_{(1)} \mathbb{P}(T \leq c_{1-\alpha}^{PI}) \leq_{(2)} \mathbb{P}(T \leq c_{1-\alpha+\psi_n}^{PI,0}) + o(1) \\
&\leq_{(3)} \mathbb{P}(T \leq C(\log p)^{1/2}) + o(1) \leq_{(4)} \mathbb{P}(\widehat{f}(s) + \widehat{\varepsilon}(s) \leq C(\log p)^{1/2}) + o(1) \\
&\leq_{(5)} \mathbb{P}(f(s) + \varepsilon(s) \leq C(\log p)^{1/2}(1 + n^{-\kappa_3})) + o(1) \\
&\leq_{(6)} \mathbb{P}(f(s) + \varepsilon(s) \leq 2C(\log p)^{1/2}) + o(1) \\
&\leq_{(7)} \mathbb{P}(\varepsilon(s) \leq 2C(\log p)^{1/2} - f(s)) + o(1) \\
&\leq_{(8)} \mathbb{P}(\varepsilon(s) \leq (2C\omega_n - 1)f(s)) + \mathbb{P}(\mathcal{B}_n) + o(1) \\
&\leq_{(9)} C/n + \mathbb{P}(\mathcal{A}_n) + \mathbb{P}(\mathcal{B}_n) + o(1) =_{(10)} o(1)
\end{aligned}$$

where (1) follows from $\mathcal{S}_n^P \subset \mathcal{S}_n^{PI}$, (2) is by the definition of ψ_n , (3) is by Lemma D.1, (4) is since $T = \max_{s \in \mathcal{S}_n} (\widehat{f}(s) + \widehat{\varepsilon}(s))$, (5) is by Assumption A4, (6) is because $n^{-\kappa_3} \leq 1$ for large n , (7) is rearrangement, (8) follows from definitions of ω_n and \mathcal{B}_n , (9) follows by noting that $2C\omega_n \leq 1/2$ for large n and applying Chebyshev's inequality conditional on $\{X_i\}$ with $\mathbb{E}[\varepsilon(s)^2 | \{X_i\}] = 1$, and (10) follows from $\mathbb{P}(\mathcal{A}_n) + \mathbb{P}(\mathcal{B}_n) = o(1)$. This completes the proof of the theorem. \square

Proof of Theorem 4.3. The proof follows from an argument similar to that used in the proof of Theorem 4.2 with equation (40) replaced by

$$\sum_{1 \leq i, j \leq n} (f(X_i) - f(X_j)) \text{sign}(X_j - X_i) Q(X_i, X_j, s) \geq Cl_n n^2,$$

so that $f(s) \geq Cl_n n^{1/2}$ w.p.a.1 for some $C > 0$, and condition $\log p = o_p(n)$ replaced by $\log p = o_p(l_n^2 n)$, so that $(\log p)^{1/2} = o_p(f(s))$. \square

Proof of Theorem 4.4. Since $\inf_{x \in [s_l, s_r]} f^{(1)}(x) < -l_n (\log p/n)^{\beta/(2\beta+3)}$, for sufficiently large n , there exists an interval $[x_{n,1}, x_{n,2}] \subset [s_l, s_r]$ such that $|x_{n,2} - x_{n,1}| = h_n$ and for all $x \in [x_{n,1}, x_{n,2}]$, $f^{(1)}(x) < -l_n (\log p/n)^{\beta/(2\beta+3)}/2$. Take $s = s_n \in \mathcal{S}_n$ as in Assumption A9 applied to the interval $[x_{n,1}, x_{n,2}]$. Assumptions A1, A8, and A9-(ii) yield $V(s) \leq C(nh_n)^3 h_n^{2k}$ for some $C > 0$ because by Lemma H.1, $|\{i = \overline{1, n} : X_i \in [x_{n,1}, x_{n,2}]\}| \leq Cnh_n$ w.p.a.1. In addition, combining Assumptions A8, A9-(i), and A9-(iii),

$$\sum_{1 \leq i, j \leq n} (f(X_i) - f(X_j)) \text{sign}(X_j - X_i) Q(X_i, X_j, s) \geq l_n C h_n^{1+\beta+k} (nh_n)^2$$

w.p.a.1 for some $C > 0$, and so $f(s) \geq Cl_n h_n^{1+\beta} (nh_n)^{1/2}$ w.p.a.1 by an argument similar to that used in the proof of Theorem 4.2. From this point, since $\log p = o(l_n^2 h_n^{2\beta+3} n)$, the argument like that used in the proof of Theorem 4.2 yields the result. \square

Proof of Theorem 4.5. Let $s_l = 0$ and $s_r = 1$. Let $h = h_n = c(\log n/n)^{1/(2\beta+3)}$ for sufficiently small $c > 0$. Let $L = [(s_r - s_l)/(4h)]$ where $[x]$ is the largest integer smaller or equal than x . For $l = \overline{1, L}$, let $x_l = 4h(l-1)$ and define $f_l : [s_l, s_r] \rightarrow \mathbb{R}$ by $f_l(s_l) = 0$, $f_l^{(1)}(x) = 0$ if $x \leq x_l$,

$f_l^{(1)}(x) = -L(x - x_l)^\beta$ if $x \in (x_l, x_l + h]$, $f_l^{(1)}(x) = -L(x_l + 2h - x)^\beta$ if $x \in (x_l + h, x_l + 2h]$, $f_l^{(1)}(x) = L(x - x_l - 2h)^\beta$ if $x \in (x_l + 2h, x_l + 3h]$, $f_l^{(1)}(x) = L(x_l + 4h - x)^\beta$ if $x \in (x_l + 3h, x_l + 4h]$ and $f_l^{(1)}(x) = 0$ otherwise. In addition, let $f_0(x) = 0$ for all $x \in [s_l, s_r]$. Finally, let $\{\varepsilon_i\}$ be a sequence of independent $N(0, 1)$ random variables.

For $l = \overline{0, L}$, consider a model $M_l = M_{n,l}$ with the regression function f_l , X distributed uniformly over $[0, 1]$, and ε distributed as $N(0, 1)$ random variable independently of X . Note that M_0 belongs to \mathcal{M}_2 and satisfies \mathcal{H}_0 . In addition, for $l \geq 1$, M_l belongs to \mathcal{M}_2 , does not satisfy \mathcal{H}_0 , and, moreover, has $\inf_{x \in [s_l, s_r]} f_l^{(1)}(x) < -C(\log n/n)^{\beta/(2\beta+3)}$ for some small $C > 0$.

Consider any test $\psi = \psi(\{X_i, Y_i\})$ such that $\mathbb{E}_{M_0}[\psi] \leq \alpha + o(1)$. Then following the argument from Dumbgen and Spokoiny (2001) gives

$$\begin{aligned} \inf_{M \in \mathcal{M}_2} \mathbb{E}_M[\psi] - \alpha &\leq \min_{1 \leq l \leq L} \mathbb{E}_{M_l}[\psi] - \mathbb{E}_{M_0}[\psi] + o(1) \leq \sum_{1 \leq l \leq L} \mathbb{E}_{M_l}[\psi]/L - \mathbb{E}_{M_0}[\psi] + o(1) \\ &= \sum_{1 \leq l \leq L} \mathbb{E}_{M_0}[\psi \rho_l]/L - \mathbb{E}_{M_0}[\psi] + o(1) = \sum_{1 \leq l \leq L} \mathbb{E}_{M_0}[\psi(\rho_l - 1)]/L + o(1) \\ &\leq \mathbb{E}_{M_0} \left[\psi \left| \sum_{1 \leq l \leq L} \rho_l/L - 1 \right| \right] + o(1) \leq \mathbb{E}_{M_0} \left[\left| \sum_{1 \leq l \leq L} \rho_l/L - 1 \right| \right] + o(1) \end{aligned}$$

where ρ_l is the likelihood ratio of observing $\{X_i, Y_i\}$ under the models M_l and M_0 . Further,

$$\rho_l = \exp \left(\sum_{1 \leq i \leq n} Y_i f_l(X_i) - \sum_{1 \leq i \leq n} f_l(X_i)^2/2 \right) = \exp(\omega_{n,l} \xi_{n,l} - \omega_{n,l}^2/2)$$

where $\omega_{n,l} = (\sum_{1 \leq i \leq n} f_l(X_i)^2)^{1/2}$ and $\xi_{n,l} = \sum_{1 \leq i \leq n} Y_i f_l(X_i)/\omega_{n,l}$. Note that under the model M_0 , conditional on $\{X_i\}$, $\{\xi_{n,l}\}_{1 \leq l \leq L}$ is a sequence of independent $N(0, 1)$ random variables, so that

$$\mathbb{E}_{M_0} \left[\left| \sum_{1 \leq l \leq L} \rho_l/L - 1 \right| \middle| \{X_i\} \right] \leq 2.$$

In addition, by the construction of the functions f_l , Lemma H.1 shows that $\omega_{n,l} \leq C(nh)^{1/2}h^{\beta+1}$ w.p.a.1 for some $C > 0$, and so $\omega_{n,l} \leq (C \log n)^{1/2}$ w.p.a.1 for some $C > 0$ where C can be made arbitrarily small by selecting sufficiently small c in the definition of h_n . Let A_n denote the event $\omega_{n,l} \leq (C \log n)^{1/2}$, so that $\mathbb{P}(A_n) \rightarrow 1$. Moreover,

$$\mathbb{E}_{M_0} \left[\left| \sum_{1 \leq l \leq L} \rho_l/L - 1 \right| \right] \leq \mathbb{E}_{M_0} \left[\left| \sum_{1 \leq l \leq L} \rho_l/L - 1 \right| | A_n \right] + 2(1 - \mathbb{P}(A_n)). \quad (42)$$

Further, on A_n ,

$$\begin{aligned}
\mathbb{E}_{M_0} \left[\left| \sum_{1 \leq l \leq L} \rho_l/L - 1 \right| \middle| \{X_i\} \right]^2 &\leq \mathbb{E}_{M_0} \left[\left(\sum_{1 \leq l \leq L} \rho_l/L - 1 \right)^2 \middle| \{X_i\} \right] \\
&\leq \sum_{1 \leq l \leq L} \mathbb{E}_{M_0} [\rho_l^2/L^2 | \{X_i\}] \\
&\leq \sum_{1 \leq l \leq L} \mathbb{E}_{M_0} [\exp(2\omega_{n,l}\xi_{n,l} - \omega_{n,l}^2)/L^2 | \{X_i\}] \\
&\leq \sum_{1 \leq l \leq L} \exp(\omega_{n,l}^2)/L^2 \leq \max_{1 \leq l \leq L} \exp(\omega_{n,l}^2)/L \\
&\leq \exp(C \log n - \log L) = o(1)
\end{aligned}$$

because C in the last line is arbitrarily small and $\log n \lesssim \log L$. Combining the last bound with (42) gives $\inf_{M \in \mathcal{M}_2} \mathbb{E}_M[\psi] \leq \alpha + o(1)$, and so the result follows. \square

APPENDIX E. PROOFS FOR SECTION 5

Proof of Theorem 5.1. In the set-up of this theorem, Lemmas D.1, D.2, D.3, D.5, D.7, D.8, and D.9 hold with X_i replaced by X_i, Z_i and Assumptions A1, A2, and A5 replaced by Assumptions A1', A2', and A5', respectively. Further, Lemma D.4 now follows by applying Lemma H.4, case (i), and assumptions A1 and A5 replaced by A1' and A5'. The result of Lemma D.6 holds but the proof has to be modified. Recall that in the proof of Lemma D.6, it was derived that

$$\mathbb{E} [\Delta_\Sigma | \{X_i, Z_i\}] \leq C \left(A_n \sqrt{\log p} + A_n^2 \log p \mathbb{E} \left[\max_{1 \leq i \leq n} \varepsilon_i^4 \right]^{1/2} \right).$$

Now, $A_n(\log p)^{5/2} = o_p(1)$ by Assumption A5'. In addition, by a maximal inequality (Lemma 2.2.2 in Van der Vaart and Wellner (1996)),

$$\mathbb{E} \left[\max_{1 \leq i \leq n} \varepsilon_i^4 \right] \leq C(\log n)^4$$

for some $C > 0$ under Assumption A1'. Therefore,

$$A_n^2 (\log p)^3 \mathbb{E} \left[\max_{1 \leq i \leq n} \varepsilon_i^4 \right]^{1/2} \leq C A_n^2 (\log(pn))^5 = o_p(1)$$

for some $C > 0$ under Assumption A5'. Hence, $(\log p)^2 \Delta_\Sigma = o_p(1)$. The rest of the proof of Lemma D.6 is the same.

Now, the first claim of Theorem 5.1 follows from an argument similar to that used in the proof of Theorem 4.1 by noting that under \mathcal{H}_0 , $(\log p)^{1/2} \max_{s \in \mathcal{S}_n} \widehat{f}(s) \leq o_p(1)^9$, which holds

⁹Specifically, the only required change in the proof of the first claim of Theorem 4.1 is that starting from inequality (3) α should be replaced by $\alpha - o_p(1)$.

by Assumption A11. The second claim of the theorem again follows from an argument similar to that used in the proof of Theorem 4.1 by noting that when $f(\cdot)$ is identically constant, $(\log p)^{1/2} \max_{s \in \mathcal{S}_n} |\hat{f}(s)| \leq o_p(1)$, which holds by Assumption A11. \square

Proof of Theorem 5.2. Denote $Y_i^0 = f(X_i) + \varepsilon_i$. Then $Y_i = Y_i^0 + Z_i^T \beta$ and $\tilde{Y}_i = Y_i^0 - Z_i^T (\hat{\beta}_n - \beta)$. So, $|\tilde{Y}_i - Y_i^0| \leq \|Z_i\| \|\hat{\beta}_n - \beta\|$ uniformly over $i = 1, \dots, n$ and all models in \mathcal{M}_{PL} . Further, note that since $\sum_{1 \leq i, j \leq n} Q(X_i, X_j, s)/V(s)^{1/2} = o_p(\sqrt{n/\log p})$ uniformly over $s \in \mathcal{S}_n$, there exists a sequence $\{\omega_n\}$ such that $\omega_n \rightarrow 0$ and $\sum_{1 \leq i \leq n} |a_i(s)| = o_p(\omega_n \sqrt{n/\log p})$ uniformly over $s \in \mathcal{S}_n$. Let \mathcal{A}_n denote the event that $\sum_{1 \leq i \leq n} |a_i(s)| > \omega_n \sqrt{n/\log p}$, so that $P(\mathcal{A}_n) = o(1)$. In addition, note that for any $C > 0$,

$$\begin{aligned} P\left(\max_{s \in \mathcal{S}_n} \sum_{1 \leq i \leq n} |\hat{a}_i(s)(\tilde{Y}_i - Y_i^0)| > C/\sqrt{\log p}\right) &\leq P\left(\max_{s \in \mathcal{S}_n} \sum_{1 \leq i \leq n} |a_i(s)(\tilde{Y}_i - Y_i^0)| > C/\sqrt{2 \log p}\right) + o(1) \\ &\leq P(C_9 \|\hat{\beta} - \beta\| \max_{s \in \mathcal{S}_n} \sum_{1 \leq i \leq n} |a_i(s)| > C/\sqrt{2 \log p}) + o(1) \\ &\leq P(\|\hat{\beta} - \beta\| > C/(C_9 \omega_n \sqrt{2n})) + P(\mathcal{A}_n) + o(1) = o(1) \end{aligned}$$

where the last conclusion follows from Assumption A12 and from $P(\mathcal{A}_n) = o(1)$. Therefore, for any $C > 0$,

$$P\left(|T - \max_{s \in \mathcal{S}_n} \sum_{1 \leq i \leq n} \hat{a}_i(s) Y_i^0| > C/\sqrt{\log p}\right) \leq P\left(\max_{s \in \mathcal{S}_n} \sum_{1 \leq i \leq n} |\hat{a}_i(s)(\tilde{Y}_i - Y_i^0)| > C/\sqrt{\log p}\right) = o(1).$$

From this point, in view of Lemma D.2, the result follows by the argument similar to that used in the proof of Theorem 4.1. \square

APPENDIX F. PROOF FOR SECTION 6

Proof of Theorem 6.1. The proof follows from the same argument as in the proof of Theorem 5.2 and Comment 5.2. \square

APPENDIX G. PROOFS FOR APPENDIX B

Proof of Proposition B.1. Since $\sigma_i \geq c_1$ by Assumption A1, it follows that

$$|\hat{\sigma}_i - \sigma_i| = \left| \frac{\hat{\sigma}_i^2 - \sigma_i^2}{\hat{\sigma}_i + \sigma_i} \right| \leq |\hat{\sigma}_i^2 - \sigma_i^2|/c_1,$$

so that it suffices to prove that $\hat{\sigma}_i^2 - \sigma_i^2 = O_p(b_n^2 + \log n/(b_n n^{1/2}))$ uniformly over $i = \overline{1, n}$. Now, note that

$$\hat{\sigma}_i^2 - \sigma_i^2 = \sum_{j \in J(i)'} (Y_{j+1} - Y_j)^2 / (2|J(i)|) - \sigma_i^2$$

where $J(i)' = \{j \in J(i) : j+1 \in J(i)\}$. Therefore,

$$\hat{\sigma}_i^2 - \sigma_i^2 = p_{i,1} + p_{i,2} + p_{i,3} - p_{i,4} + O_p(b_n^2)$$

where

$$\begin{aligned}
p_{i,1} &= \sum_{j \in J(i)'} (\varepsilon_{j+1}^2 - \sigma_j^2) / (2|J(i)|), \\
p_{i,2} &= \sum_{j \in J(i)'} (\varepsilon_j^2 - \sigma_j^2) / (2|J(i)|), \\
p_{i,3} &= \sum_{j \in J(i)'} (f(X_{j+1}) - f(X_j))(\varepsilon_{j+1} - \varepsilon_j) / |J(i)|, \\
p_{i,4} &= \sum_{j \in J(i)'} \varepsilon_j \varepsilon_{j+1} / |J(i)|.
\end{aligned}$$

Since $b_n \geq (\log n)^2 / (C_6 n)$, Lemma H.1 implies that $cb_n n \leq |J(i)| \leq Cb_n n$ for all $i = \overline{1, n}$ w.p.a.1 for some $c, C > 0$. Note also that $J(i)$'s depend only on $\{X_i\}$. Therefore, applying Lemma H.2 conditional on $\{X_i\}$ shows that $p_{i,1}$ and $p_{i,2}$ are both $O_p(\log n / (b_n n^{1/2}))$ uniformly over $i = \overline{1, n}$ since $E[\max_{1 \leq i \leq n} \varepsilon_i^4 | \{X_i\}] \leq C_1 n$ by Assumption A1. Further, applying Lemma H.2 conditional on $\{X_i\}$ separately to $\sum_{j \in J(i)': j \text{ odd}} \varepsilon_j \varepsilon_{j+1} / |J(i)|$, $\sum_{j \in J(i)': j \text{ even}} \varepsilon_j \varepsilon_{j+1} / |J(i)|$, $\sum_{j \in J(i)'} (f(X_{j+1}) - f(X_j)) \varepsilon_{j+1} / |J(i)|$, and $\sum_{j \in J(i)'} (f(X_{j+1}) - f(X_j)) \varepsilon_j / |J(i)|$ shows that $p_{i,3}$ and $p_{i,4}$ are also $O_p(\log n / (b_n n^{1/2}))$ uniformly over $i = \overline{1, n}$. Combining these bounds gives the claim of the proposition and completes the proof. \square

Proof of Proposition B.2. Let $s = (x, h) \in \mathcal{S}_n$. Since $h \leq (s_r - s_l) / 2$, either $s_l + h \leq x$ or $x + h \leq s_r$ holds. Let $\mathcal{S}_{n,1}$ and $\mathcal{S}_{n,2}$ denote the subsets of those elements of \mathcal{S}_n that satisfy the former and latter cases, respectively, so that $\mathcal{S}_n = \mathcal{S}_{n,1} \cup \mathcal{S}_{n,2}$. Consider $\mathcal{S}_{n,1}$. Let $C_K \in (0, 1)$. Since the kernel $K(\cdot)$ is continuous and strictly positive on the interior of its support, $\min_{t \in [-C_K, 0]} K(t) > 0$. In addition, since $K(\cdot)$ is bounded, it is possible to find a constant $c_K \in (0, 1)$ such that and $c_K + C_K \leq 1$ and

$$6c_K^{k+1} C_3 \max_{t \in [-1, -1+c_K]} K(t) \leq c_3 (1 - c_K)^k C_K \min_{t \in [-C_K, 0]} K(t) \quad (43)$$

where the constant k appears in the definition of kernel weighting functions.

Denote

$$\begin{aligned}
M_{n,1}(x, h) &= \{i = \overline{1, n} : X_i \in [x - C_K h, x]\}, \\
M_{n,2}(x, h) &= \{i = \overline{1, n} : X_i \in [x - h, x - (1 - c_K)h]\}, \\
M_{n,3}(x, h) &= \{i = \overline{1, n} : X_i \in [x - (1 - c_K/2)h, x - (1 - c_K)h]\}, \\
M_{n,4}(x, h) &= \{i = \overline{1, n} : X_i \in [x - h, x + h]\}.
\end{aligned}$$

Since $h_{\min} \geq (\log n)^2 / (C_6 n)$ w.p.a.1 by assumption, Lemma H.1 and Assumption 8 give

$$\begin{aligned}
(1/2)c_3 C_K n h &\leq |M_{n,1}(x, h)| \leq (3/2)C_3 C_K n h, \\
(1/2)c_3 c_K n h &\leq |M_{n,2}(x, h)| \leq (3/2)C_3 c_K n h, \\
(1/2)c_3 (c_K/2) n h &\leq |M_{n,3}(x, h)| \leq (3/2)C_3 (c_K/2) n h, \\
(1/2)c_3 n h &\leq |M_{n,4}(x, h)| \leq (3/2)C_3 2 n h
\end{aligned} \tag{44}$$

w.p.a.1. uniformly over $s = (x, h) \in \mathcal{S}_{n,1}$. Note also that (44) holds w.p.a.1 uniformly over $s = (x, h) \in \mathcal{S}_{n,2}$ as well. Then

$$\begin{aligned}
\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) |X_j - X_i|^k K((X_j - x)/h) &\geq \sum_{j \in M_{n,1}(x, h)} ((1 - c_K)h)^k K((X_j - x)/h) \\
&\quad - \sum_{j \in M_{n,2}(x, h)} (c_K h)^k K((X_j - x)/h) \\
&\geq ((1 - c_K)h)^k (1/2)c_3 C_K n h \min_{t \in [-C_K, 0]} K(t) \\
&\quad - (c_K h)^k (3/2)C_3 c_K n h \max_{t \in [-1, -1+c_K]} K(t) \\
&\geq ((1 - c_K)h)^k c_3 C_K n h \min_{t \in [-C_K, 0]} K(t)/4 \\
&\geq C n h^{k+1}
\end{aligned}$$

w.p.a.1 uniformly over $X_i \in M_{n,3}(x, h)$ and $s = (x, h) \in \mathcal{S}_{n,1}$ for some $C > 0$ where the inequality preceding the last one follows from (43). Hence,

$$\begin{aligned}
V(s) &= \sum_{1 \leq i \leq n} \sigma_i^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \\
&= \sum_{1 \leq i \leq n} \sigma_i^2 K((X_i - x)/h)^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) |X_j - X_i|^k K((X_j - x)/h) \right)^2 \\
&\geq \sum_{i \in M_{n,3}(x, h)} \sigma_i^2 K((X_i - x)/h)^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) |X_j - X_i|^k K((X_j - x)/h) \right)^2,
\end{aligned}$$

and so $V(s) \geq C(nh)^3 h^{2k}$ w.p.a.1 uniformly over $s = (x, h) \in \mathcal{S}_{n,1}$ for some $C > 0$. Similar argument gives $V(s) \geq C(nh)^3 h^{2k}$ w.p.a.1 uniformly over $s = (x, h) \in \mathcal{S}_{n,2}$ for some $C > 0$. In addition,

$$\left| \sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right| \leq (2h)^k |M_{n,4}(x, h)| \left(\max_{t \in [-1, +1]} K(t) \right)^2 \leq C n h^{k+1} \tag{45}$$

w.p.a.1 uniformly over $i = \overline{1, n}$ and $s = (x, h) \in \mathcal{S}_n$ for some $C > 0$. Combining (45) with the bound on $V(s)$ above gives $A_n \leq C / (n h_{\min})$ w.p.a.1 for some $C > 0$. Now, for the basic

set of weighting functions, $\log p \leq C \log n$ and $h_{\min} \geq Cn^{-1/3}$ w.p.a.1 for some $C > 0$, and so Assumption A5 holds, which gives the claim (a).

To prove the claim (b), note that

$$\sum_{1 \leq i \leq n} \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \leq (2h)^{2k} |M_{n,4}(x, h)|^3 \left(\max_{t \in [-1, +1]} K(t) \right)^4 \quad (46)$$

$$\leq C(nh)^3 h^{2k} \quad (47)$$

w.p.a.1 uniformly over $s = (x, h) \in \mathcal{S}_n$ for some $C > 0$ by (44). Therefore, under Assumption A3,

$$\begin{aligned} |\widehat{V}(s) - V(s)| &\leq \max_{1 \leq i \leq n} |\widehat{\sigma}_i^2 - \sigma_i^2| \sum_{1 \leq i \leq n} \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \\ &\leq (nh)^3 h^{2k} o_p(n^{-\kappa_2}) \end{aligned}$$

uniformly over $s = (x, h) \in \mathcal{S}_n$. Combining this bound with the lower bound for $V(s)$ established above shows that under Assumption A3, $|\widehat{V}(s)/V(s) - 1| = o_p(n^{-\kappa_2})$, and so

$$\begin{aligned} |(\widehat{V}(s)/V(s))^{1/2} - 1| &= o_p(n^{-\kappa_2}), \\ |(V(s)/\widehat{V}(s))^{1/2} - 1| &= o_p(n^{-\kappa_2}) \end{aligned}$$

uniformly over \mathcal{S}_n , which is the asserted claim (b).

To prove the last claim, note that for $s = (x, h) \in \mathcal{S}_n$,

$$|\widehat{V}(s) - V(s)| \leq I_1(s) + I_2(s)$$

where

$$\begin{aligned} I_1(s) &= \left| \sum_{1 \leq i \leq n} (\varepsilon_i^2 - \sigma_i^2) \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \right|, \\ I_2(s) &= \left| \sum_{1 \leq i \leq n} (\widehat{\sigma}_i^2 - \varepsilon_i^2) \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \right|. \end{aligned}$$

Consider $I_1(s)$. Combining (45) and Lemma H.1 applied conditional on $\{X_i\}$ gives

$$I_1(s) = (nh)^2 h^{2k} \log p o_p(n^{1/2}) \quad (48)$$

uniformly over $s = (x, h) \in \mathcal{S}_n$.

Consider $I_2(s)$. Note that

$$I_2(s) \leq I_{2,1}(s) + I_{2,2}(s)$$

where

$$I_{2,1}(s) = \sum_{1 \leq i \leq n} (\hat{\sigma}_i - \varepsilon_i)^2 \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2,$$

$$I_{2,2}(s) = 2 \sum_{1 \leq i \leq n} |\varepsilon_i (\hat{\sigma}_i - \varepsilon_i)| \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2.$$

Now, Assumption A2 combined with (46) and (47) gives

$$I_{2,1}(s) \leq \max_{1 \leq i \leq n} (\hat{\sigma}_i - \varepsilon_i)^2 \sum_{1 \leq i \leq n} \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \leq (nh)^3 h^{2k} o_p(n^{-2\kappa_1}) \quad (49)$$

uniformly over $s = (x, h) \in \mathcal{S}_n$. In addition,

$$I_{2,2}(s) \leq 2 \max_{1 \leq i \leq n} |\hat{\sigma}_i - \varepsilon_i| \sum_{1 \leq i \leq n} |\varepsilon_i| \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2,$$

and since

$$\mathbb{E} \left[\sum_{1 \leq i \leq n} |\varepsilon_i| \left(\sum_{1 \leq j \leq n} \text{sign}(X_j - X_i) Q(X_i, X_j, s) \right)^2 \mid \{X_i\} \right] \leq C(nh)^3 h^{2k}$$

w.p.a.1 uniformly over $s = (x, h) \in \mathcal{S}_n$ for some $C > 0$, it follows that

$$I_{2,2}(s) = (nh)^3 h^{2k} o_p(n^{-\kappa_1}) \quad (50)$$

uniformly over $s = (x, h) \in \mathcal{S}_n$. Combining (48)-(50) with the lower bound for $V(s)$ established above shows that under Assumption A2, $|\hat{V}(s)/V(s) - 1| = o_p(n^{-\kappa_1}) + O_p(\log p / (hn^{1/2}))$ uniformly over $s \in \mathcal{S}_n$. This gives the asserted claim (c) and completes the proof of the theorem. \square

Proof of Proposition B.3. Recall that $h_n = (\log p/n)^{1/(2\beta+3)}$. Therefore, for the basic set of weighting functions, there exists $c \in (0, 1)$ such that for all n , there is $\tilde{h} \in H_n$ satisfying $\tilde{h} \in (ch_n/3, h_n/3)$. By Lemma H.1 and Assumption A8, w.p.a.1, for all $[x_1, x_2] \subset [s_l, s_r]$ with $x_2 - x_1 = h_n$, there exists $i = \overline{1, n}$ such that $X_i \in [x_1 + h_n/3, x_2 - h_n/3]$. Then the weighting function with $(x, h) = (X_i, \tilde{h})$ satisfies conditions of Assumption A9. \square

APPENDIX H. USEFUL LEMMAS

Lemma H.1. *Let W_1, \dots, W_n be an i.i.d. sequence of random variables with the support $[s_l, s_r]$ such that $c_1(x_2 - x_1) \leq \mathbb{P}(W_1 \in [x_1, x_2]) \leq C_1(x_2 - x_1)$ for some $c_1, C_1 > 0$ and all $[x_1, x_2] \subset [s_l, s_r]$. Then for any $c_2 > 0$,*

$$(1/2)c_1 n(x_2 - x_1) \leq |\{i = \overline{1, n} : W_i \in [x_1, x_2]\}| \leq (3C_1/2)n(x_2 - x_1) \quad (51)$$

simultaneously for all intervals $[x_1, x_2] \subset [s_l, s_r]$ satisfying $x_2 - x_1 \geq c_2(\log n)^2/n$ w.p.a.1.

Proof. Let $L = [(s_r - s_l)/(c_2(\log n)^2/n)]$ where $[x]$ denotes the largest integer smaller than or equal to x . Denote $\Delta = (s_r - s_l)/L$. For $l = 0, \dots, L$, denote $y_l = s_l + l\Delta/L$. It suffices to show that (51) holds simultaneously for all intervals $[x_1, x_2]$ of the form $[y_{l-1}, y_l]$ for $l = \overline{1, L}$.

For $l = \overline{1, L}$, let $I_{i,l} = 1\{W_i \in [y_{l-1}, y_l]\}$. Then

$$\sum_{1 \leq i \leq n} I_{i,l} = |\{i = \overline{1, n} : W_i \in [y_{l-1}, y_l]\}|.$$

In addition, $E[I_{i,l}] = E[I_{i,l}^2] = P(W_i \in [y_{l-1}, y_l])$, so that $c_1 n \Delta \leq E[\sum_{1 \leq i \leq n} I_{i,l}] \leq C_1 n \Delta$ and $\text{Var}(\sum_{1 \leq i \leq n} I_{i,l}) \leq n E[I_{i,l}^2] \leq C_1 n \Delta$. Hence, Bernstein's inequality (see, for example, Lemma 2.2.9 in Van der Vaart and Wellner (1996)) gives

$$\begin{aligned} P\left(\sum_{1 \leq i \leq n} I_{i,l} > (3/2)C_1 n \Delta\right) &\leq \exp(-C(\log n)^2), \\ P\left(\sum_{1 \leq i \leq n} I_{i,l} < (1/2)c_1 n \Delta\right) &\leq \exp(-C(\log n)^2) \end{aligned}$$

for some $C > 0$. Therefore, by the union bound,

$$\begin{aligned} &P\left(\sum_{1 \leq i \leq n} I_{i,l} > (3/2)C_1 n \Delta \text{ or } \sum_{1 \leq i \leq n} I_{i,l} < (1/2)c_1 n \Delta \text{ for some } l = \overline{1, L}\right) \\ &\leq 2L \exp(-C(\log n)^2) = o(1) \end{aligned}$$

where the last conclusion follows from $L \leq Cn$ for some $C > 0$. This completes the proof of the lemma. \square

Lemma H.2. *Let W_1, \dots, W_n be independent random vectors in \mathbb{R}^p with $p \geq 2$. Let W_{ij} denote the j th component of W_i , that is $W_i = (W_{i1}, \dots, W_{ip})^T$. Define $M = \max_{1 \leq j \leq n} \max_{1 \leq j \leq p} |W_{ij}|$ and $\sigma^2 = \max_{1 \leq j \leq p} \sum_{1 \leq i \leq n} E[W_{ij}^2]$. Then*

$$E\left[\max_{1 \leq j \leq p} \left|\sum_{1 \leq i \leq n} (W_{ij} - E[W_{ij}])\right|\right] \leq C\left(\sigma\sqrt{\log p} + \sqrt{E[M^2] \log p}\right)$$

for some universal $C > 0$.

Proof. See Lemma 8 in Chernozhukov, Chetverikov, and Kato (2011). \square

Lemma H.3. *Let W^1 and W^2 be zero-mean Gaussian vectors in \mathbb{R}^p with covariances Σ^1 and Σ^2 , respectively. Let W_j^k denote the j th component of W^k , $k = 1$ or 2 , that is $W^k = (W_1^k, \dots, W_p^k)^T$. Then for any $g \in \mathbb{C}^2(\mathbb{R}, \mathbb{R})$,*

$$\left|E\left[g\left(\max_{1 \leq j \leq p} W_j^1\right) - g\left(\max_{1 \leq j \leq p} W_j^2\right)\right]\right| \leq \|g^{(2)}\|_\infty \Delta_\Sigma / 2 + 2\|g^{(1)}\|_\infty \sqrt{2\Delta_\Sigma \log p}$$

where $\Delta_\Sigma = \max_{1 \leq j, k \leq p} |\Sigma_{jk}^1 - \Sigma_{jk}^2|$.

Proof. See Theorem 1 and following comments in Chernozhukov, Chetverikov, and Kato (2011). \square

Lemma H.4. *Let x_1, \dots, x_n be a sequence of independent zero-mean vectors in \mathbb{R}^p with x_{ij} denoting the j th component of x_i , that is $x_i = (x_{i1}, \dots, x_{ip})^T$. Let y_1, \dots, y_n be a sequence of independent zero-mean Gaussian vectors in \mathbb{R}^p with y_{ij} denoting the j th component of y_i , that is $y_i = (y_{i1}, \dots, y_{ip})^T$. Assume that $E[x_i x_i^T] = E[y_i y_i^T]$ for all $i = \overline{1, n}$. Further, assume that for all i and j , $x_{ij} = z_{ij} u_i$ where z_{ij} 's are non-stochastic with $|z_{ij}| \leq B_n$ and $\sum_{1 \leq i \leq n} z_{ij}^2 / n = 1$ where $\{B_n\}$ is a sequence of positive constants. Finally, assume that for some constants $c_1, C_1, c_2, C_2 > 0$, one of the following conditions holds: (i) $E[u_i^2] \geq c_1$, $E[\exp(|u_i|/C_1)] \leq 2$, and $B_n^2 (\log(pn))^7 / n \leq C_2 n^{-c_2}$ or (ii) $E[u_i^2] \geq c_1$, $E[u_i^4] \leq C_1$, and $B_n^4 (\log(pn))^7 / n \leq C_2 n^{-c_2}$. Then there exist constants $c, C > 0$ depending only on c_1, C_1, c_2, C_2 such that*

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left(\max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{1 \leq i \leq n} x_{ij} \leq t \right) - \mathbb{P} \left(\max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{1 \leq i \leq n} y_{ij} \leq t \right) \right| \leq C n^{-c} \quad (52)$$

for all n . In addition, if the terms $C_2 n^{-c_2}$ above are replaced by η_n where $\{\eta_n\}$ is a sequence of positive numbers converging to zero, then there exists another sequence $\{\eta'_n\}$ of positive numbers converging to zero and depending only on $\{\eta_n\}$ such that

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left(\max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{1 \leq i \leq n} x_{ij} \leq t \right) - \mathbb{P} \left(\max_{1 \leq j \leq p} \frac{1}{\sqrt{n}} \sum_{1 \leq i \leq n} y_{ij} \leq t \right) \right| \leq \eta'_n \quad (53)$$

for all n .

Proof. The result in (52) is proven in Corollary 2.1 of Chernozhukov, Chetverikov, and Kato (2012). Further, inspecting the proof of Corollary 2.1 of Chernozhukov, Chetverikov, and Kato (2012) shows that the sequences $C_2 n^{-c_2}$ and $C n^{-c}$ in (52) can be replaced by general sequences $\{\eta_n\}$ and $\{\eta'_n\}$ of positive numbers converging to zero, and so the result in (53) holds as well. \square

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SUPPLEMENTARY APPENDIX

This supplementary Appendix contains additional simulation results. In particular, I consider the test developed in this paper with weighting functions of the form given in equation (6) with $k = 1$. The simulation design is the same as in Section 8. The results are presented in table 2. For ease of comparison, I also repeat the results for the tests of GSV, GHJK, and HH in this table. Overall, the simulation results in table 2 are similar to those in table 1, which confirms the robustness of the findings in this paper.

TABLE 3. Results of Monte Carlo Experiments

Noise	Case	Sample	Proportion of Rejections for								
			GSV	GHJK	HH	CS-PI	CS-OS	CS-SD	IS-PI	IS-OS	IS-SD
normal	1	100	.118	.078	.123	.129	.129	.129	.166	.166	.166
		200	.091	.051	.108	.120	.120	.120	.144	.144	.144
		500	.086	.078	.105	.121	.121	.121	.134	.134	.134
normal	2	100	0	.001	0	.002	.009	.009	.006	.024	.024
		200	0	.002	0	.001	.012	.012	.007	.016	.016
		500	0	.001	0	.002	.005	.005	.005	.016	.016
normal	3	100	0	.148	.033	.238	.423	.432	0	0	0
		200	.010	.284	.169	.639	.846	.851	.274	.615	.626
		500	.841	.654	.947	.977	.995	.996	.966	.994	.994
normal	4	100	.037	.084	.135	.159	.228	.231	.020	.040	.040
		200	.254	.133	.347	.384	.513	.515	.372	.507	.514
		500	.810	.290	.789	.785	.833	.833	.782	.835	.836
uniform	1	100	.109	.079	.121	.120	.120	.120	.200	.200	.200
		200	.097	.063	.109	.111	.111	.111	.154	.154	.154
		500	.077	.084	.107	.102	.102	.102	.125	.125	.125
uniform	2	100	.001	.001	0	0	.006	.006	.015	.031	.031
		200	0	0	0	.001	.009	.009	.013	.021	.024
		500	0	.003	0	.003	.012	.012	.011	.021	.021
uniform	3	100	0	.151	.038	.225	.423	.433	0	0	0
		200	.009	.233	.140	.606	.802	.823	.261	.575	.590
		500	.811	.582	.947	.976	.993	.994	.971	.990	.991
uniform	4	100	.034	.084	.137	.150	.216	.219	.020	.046	.046
		200	.197	.116	.326	.355	.483	.488	.328	.466	.472
		500	.803	.265	.789	.803	.852	.855	.796	.859	.861

Nominal Size is 0.1. GSV, GHJK, and HH stand for the tests of Ghosal, Sen, and van der Vaart (2000), Gijbels et. al. (2000), and Hall and Heckman (2000) respectively. CS-PI, CS-OS, and CS-SD refer to the test developed in this paper with σ_i estimated using Rice's formula and plug-in, one-step, and step-down critical values respectively. Finally, IS-PI, IS-OS, and IS-SD refer to the test developed in this paper with σ_i estimated by $\hat{\sigma}_i = \hat{\varepsilon}_i$ and plug-in, one-step, and step-down critical values respectively.