Bubbles and Self-Enforcing Debt*

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Abstract

We characterize equilibria with endogenous debt constraints for a general equilibrium economy with limited commitment in which the only consequence of default is losing the ability to borrow in future periods. First, we show that equilibrium debt limits must satisfy a simple condition that allows agents to exactly roll over existing debt period by period. Second, we provide an equivalence result, whereby the resulting set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations that are sustained with unbacked public debt or rational bubbles; for the latter, there exist well known existence and characterization results. In contrast to the classic result by Bulow and Rogoff (AER 1989), positive levels of debt are sustainable in our environment because the interest rate is sufficiently low to provide repayment incentives.

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1 Introduction

In a seminal paper, Bulow and Rogoff (1989a — henceforth BR) question the sustainability of debt purely by reputational considerations. Consider a small open economy that borrows at a given positive world interest rate, and suppose that the only consequence of default is that the country is denied credit in all future periods. BR show that if the country ever borrows a positive amount, there will eventually come a time at which it is better off defaulting and financing all future consumption with positive asset positions. This result has widely been interpreted as stating that the denial of future credit alone is insufficient to provide repayment incentives. A large literature has then considered alternative explanations for positive levels of debt and international capital flows, for example reductions in trade flows, loss of trade credit, explicit non-financial sanctions, collusion among non-competitive lenders, loss of reputation in other dimensions, reduced access to state-contingent securities, outright market exclusion, or time inconsistency in the borrower's preferences that prevent efficient savings schemes.[1]

An important ingredient in BR's argument is that the borrower is able to save at market interest rates after a default.[2] When the borrower decides to default he can enter a "cash in advance" contract with some other agent (some other country or international financial institution), by paying upfront in exchange for future state-contingent payments. This other agent therefore basically turns into a borrower, since he accepts a payment today in exchange for future payments. But what guarantees that this agent will fulfill his future obligations? In BR, this question does not arise, since the latter agent is assumed to have commitment power for exogenous reasons. The issue, however, seems especially relevant in the context of international capital markets, in which all participating countries have the option to default on their outstanding debt, if it is in their interest to do so. In such an environment, one country's ability to save after a default, and therefore its default incentives, also rests on the other countries' repayment incentives.

In this paper, we reconsider the incentives for debt repayment in an environment with *multilateral* lack of commitment, in which no agent can credibly commit to repay their obligations. As

[1] A non-exhaustive list of contributions includes Amador (2003), Bulow and Rogoff (1989b), Cole and Kehoe (1995, 1998), Eaton and Gersovitz (1981), Gul and Pesendorfer (2004), and Kletzer and Wright (2000).

[2] Similar results apply to government debt in a model with distortive taxes and lack of government commitment (Chari and Kehoe, 1993) and to competitive insurance markets with one-sided commitment by insurers, but not households (Kruger and Uhlig, 2006). These papers share with BR the assumption of one-sided commitment, and the loss of access to credit (but not savings) after a default. Krueger and Uhlig (2005) further discuss how the latter naturally emerges from competition by insurers with one-sided commitment. As a consequence, the only implementable contracts require the uncommitted agents to smooth consumption by lending to the committed agents.

in BR, we suppose that the only consequence of default is the denial of credit in all future periods. Following Kehoe and Levine (1993) and Alvarez and Jermann (2000), we formulate our model as a competitive equilibrium with endogenous debt limits, in which interest rates and debt limits adjust endogenously to regulate the incentives to repay outstanding debt.

In contrast to what happens in BR, we show that under some conditions positive levels of debt are sustainable in an environment with multilateral lack of commitment.[3] In the process, we identify an unexpected connection between the sustainability of debt by reputation and the sustainability of rational asset pricing bubbles (Tirole 1982).

The key to our result is that equilibrium interest rates adjust to ensure that agents repay their debt. We first show that the incentive to default disappears if interest rates are sufficient low. Reputational incentives for debt repayment thus rely not only on the amount of credit to which agents have access in future periods, but, perhaps more importantly, on the interest rate at which this credit is made available. We then go on to show that interest rates low enough to be consistent with repayment can emerge in equilibrium in an economy where no agent can commit to repay.

To illustrate these results, we first present a simple example where, in equilibrium, positive borrowing is sustained and the interest rate is equal to zero. More generally, debt is self-enforcing, as long as the real interest rate is less than or equal to the growth rate of debt limits, which equals the growth rate of aggregate endowments in steady-state.

In the rest of the paper, we give a full characterization of the conditions under which private debt is sustainable. We consider a stochastic endowment economy with sequential trade in complete contingent securities markets. Agents may issue securities up to a state-contingent limit. If they default, they are denied credit in all future periods. The equilibrium debt limits are determined endogenously as the largest possible limits such that repayment is always individually rational. Our first general result (Theorem 1) states that debt limits are self-enforcing if and only if they allow all individuals to exactly re-finance outstanding obligations by issuing new claims.

We then establish conditions for the existence of an equilibrium with self-enforcing debt and give a characterization of sustainable equilibrium allocations by means of an equivalence result. Consider an alternative environment with no private debt, but where a government issues state-contingent debt that is not backed by any fiscal revenue, i.e., where the government must finance

[3] The idea that intertemporal trade is sustainable when *all* agents have limited commitment also appears in Cole and Kehoe (1995), Kocherlakota (1995), and in Kletzer and Wright (2000), among others. However, these papers suppose that all financial transactions are observable, and debt is sustained using a Folk Theorem logic, with strategies that implement autarky as an optimal punishment after a default. Kehoe and Levine (1993) and Alvarez and Jermann (2000) show that the corresponding equilibrium allocations can be decentralized through competitive debt markets.

all existing claims by issuing new debt. This unbacked public debt has the feature of a rational bubble; in a deterministic environment, it can be reinterpreted as fiat money. Theorem 2 shows that any equilibrium allocation of the economy with self-enforcing private debt can also be sustained as an equilibrium allocation of the economy with unbacked public debt, and vice versa. Since there exist well known conditions for the sustainability of positive levels of unbacked public debt, or, more generally, for the existence of rational bubbles (see Santos and Woodford 1997 for a general analysis), these conditions also characterize the sustainability of positive levels of private debt in a general equilibrium BR economy. In particular, the same condition of "high vs. low equilibrium interest rates" determines the sustainability of both self-enforcing private debt and rational asset pricing bubbles. Positive levels of debt require low equilibrium interest rates, or more precisely, equilibrium state prices such that endowments are infinite-valued. In contrast, BR assume that the net present value of a borrower's life-time endowments is finite. This assumption exactly rules out the debt contracts and state-prices that emerge in general equilibrium.

The possibility of rational bubbles in models with borrowing constraints *à la* Bewley (1980) has been recognized in Scheinkman and Weiss (1986), Kocherlakota (1992) and Santos and Woodford (1997). Our equivalence result shows that self-enforcing private debt plays the same role as a rational asset bubble, and that it can take its place in facilitating intertemporal exchange.

In Section 2, we describe our general model and define competitive equilibria with self-enforcing private debt and unbacked public debt. In Section 3, we illustrate our main results in a simple example. In Section 4, we study repayment incentives for individual agents, and characterize self-enforcing debt limits (Theorem 1). In section 5, we study the resulting general equilibrium implications (Theorem 2). Proofs omitted from the text are in the appendix; some of the more technical proofs, and additional results can be found in an online appendix.

2 The Model

Uncertainty, preferences and endowments: Consider an infinite-horizon endowment economy with a single non-storable consumption good at each date $t \in \{0, 1, 2, ...\}$. For each t, there is a positive finite set S^t of date-t events s^t. Each s^t has a unique predecessor $\sigma\left(s^t\right) \in S^{t-1}$, and a positive, finite number of successors $s^{t+1} \in S^{t+1}$, for which $\sigma\left(s^{t+1}\right) = s^t$. There exists a unique initial date-0 event s^0. Event $s^{t+\tau}$ is said to follow event s^t (denoted $s^{t+\tau} \succ s^t$) if $\sigma^{(\tau)}\left(s^{t+\tau}\right) = s^t$. The set $\mathcal{S}\left(s^t\right) = \left\{s^{t+\tau} : s^{t+\tau} \succ s^t\right\} \cup \left\{s^t\right\}$ denotes the subtree of all events starting from s^t, and $\mathcal{S} \equiv \mathcal{S}\left(s^0\right)$ the complete event tree.

At date 0, nature draws a sequence $\{s^0, s^1, ...\}$, such that $s^{t-1} = \sigma(s^t)$ for all t. At date t, s^t is then publicly revealed. The unconditional probability that s^t is observed is denoted by $\pi(s^t)$, where $\pi(s^t) > 0$ for all $s^t \in \mathcal{S}$. For $s^{t+\tau} \in \mathcal{S}(s^t)$, $\pi(s^{t+\tau}|s^t) = \pi(s^{t+\tau})/\pi(s^t)$ denotes the conditional probability of $s^{t+\tau}$, given s^t.

There is a finite number J of consumer types, each represented by a unit measure of agents, and indexed by j. Each consumer type is characterized by a sequence of endowments of the consumption good, $Y^j \equiv \{y^j(s^t)\}_{s^t \in \mathcal{S}}$.[4] Preferences over consumption sequences $C \equiv \{c(s^t)\}_{s^t \in \mathcal{S}}$ are represented by the lifetime expected utility functional

$$U(C) = \sum_{s^t \in \mathcal{S}} \beta^t \pi(s^t) u(c(s^t)) \tag{1}$$

where $\beta \in (0,1)$, and $u(\cdot)$ is strictly increasing, concave, bounded, and twice differentiable.

Markets: At each date s^t, agents can issue and trade a complete set of contingent securities, which promise to pay one unit of period $t+1$ consumption, contingent on the realization of event $s^{t+1} \succ s^t$, in exchange for current consumption. If no agent ever defaults (as will be the case in equilibrium), securities issued by different agents are perfect substitutes for each other, and trade at a common price.

If agents had the ability to fully commit to their promises, they would be able to smooth all type-specific endowment fluctuations. In our model, agents cannot commit: at any date s^t, they can refuse to honor the securities they have issued and default. Any default becomes common knowledge and the defaulting agent loses the ability to issue claims in all future periods. Creditors can seize the financial assets he holds at the moment of default (i.e., his holdings of claims issued by other agents), but they are unable to seize any of his current or future endowments $Y^j(s^t)$, nor any of his future asset holdings. In sum, after a default, an agent loses the ability to issue debt, starts with a net financial position of 0, but he retains the ability to purchase assets.[5]

This form of punishment follows the assumptions of BR. It captures the idea that it is much easier for market participants to coordinate on not accepting the claims issued by a given borrower, than to enforce an outright ban from financial markets. As the future denial of credit eliminates the incentive to repay, a potential lender will assign zero value to the claims issued by a borrower

[4] Throughout the paper, for any variable x, $x(s^t)$ denotes the realization of x at event s^t, X denotes the sequence $\{x(s^t)\}_{s^t \in \mathcal{S}}$, and $X(s^t)$ denotes the subsequence $\{x(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$.

[5] The assumption that any positive holdings of other agents' claims are confiscated in case of default implies that agents can default only on their *net* financial position. This assumption is made only for analytic and expositional purposes, and we will discuss later how it can be relaxed without changing our results. Therefore, the only disciplining element that may prevent agents from defaulting is losing the privilege to borrow in future periods.

who has defaulted in the past. Enforcing an outright ban from financial markets, on the other hand, requires that potential borrowers are dissuaded from accepting loans at market prices from agents who have defaulted in the past. The denial of future credit thus only requires the issuer of each security to be known, while a ban from financial markets requires that the identity of buyers and sellers in all financial transactions are observable, so that agents can be punished for dealing with others who have defaulted in the past.

Let $q\left(s^t\right)$ denote the price of a s^t-contingent bond at the preceding event $\sigma\left(s^t\right)$. The date-0 price of consumption at s^t, $p\left(s^t\right)$, is defined recursively by $p\left(s^t\right) = q\left(s^t\right) \cdot p\left(\sigma\left(s^t\right)\right)$ for all $s^t \in \mathcal{S}$. Let $a^j\left(s^t\right)$ denote the agent's net financial position at s^t, that is, the amount of s^t-contingent securities he holds net of the amount of s^t-contingent securities he has issued. An agent chooses a profile of consumption and asset holdings $C^j \equiv \left\{c^j\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ and $A^j \equiv \left\{a^j\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ subject to the sequence of flow budget constraints

$$c^j\left(s^t\right) \leq y^j\left(s^t\right) + a^j\left(s^t\right) - \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) a^j\left(s^{t+1}\right), \text{ for each } s^t \in \mathcal{S}. \tag{2}$$

The amount of securities an agent issues is observable, and subject to a state-contingent upper bound $-\phi^j\left(s^{t+1}\right)$, which then determines a lower bound on his net financial position at s^{t+1}:

$$a^j\left(s^{t+1}\right) \geq \phi^j\left(s^{t+1}\right), \text{ for all } s^{t+1} \succ s^t, s^t \in \mathcal{S}. \tag{3}$$

Given the initial asset position $a^j\left(s^0\right)$, the optimal consumption and asset profile for an agent who never defaults maximizes (1), subject to the constraints (2) and (3).

Self-enforcing private debt: Since several arguments in the paper require the manipulation of budget sets, it is convenient to denote by $\mathcal{C}^j(a, \Phi^j\left(s^t\right); s^t)$ the set of feasible consumption profiles $C^j\left(s^t\right)$ for a type-j agent starting at event s^t with an asset position $a \in \mathbb{R}$, and facing future debt limits $\Phi^j\left(s^t\right)$, that is,

$$\mathcal{C}^j(a, \Phi^j\left(s^t\right); s^t) \equiv \left\{C\left(s^t\right) : C\left(s^t\right) \text{ satisfies (2), for some } A^j\left(s^t\right) \geq \Phi^j\left(s^t\right), \text{ with } a^j\left(s^t\right) = a\right\}. \tag{4}$$

If the agent chooses never to default, his life-time expected utility is given by:

$$V^j(a, \Phi^j\left(s^t\right); s^t) \equiv \max_{C(s^t) \in \mathcal{C}^j(a, \Phi^j(s^t); s^t)} \sum_{s^{t+\tau} \in \mathcal{S}(s^t)} \beta^\tau \pi\left(s^{t+\tau} | s^t\right) u(c\left(s^{t+\tau}\right)). \tag{5}$$

The life-time utility of a consumer who has defaulted in the past is $D^j\left(a; s^t\right) \equiv V^j(a, O\left(s^t\right); s^t)$, where $O\left(s^t\right)$ stands for the sequence of borrowing constraints equal to zero at every $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. Since $V^j(a, \Phi^j\left(s^t\right); s^t)$ is increasing in a, if $\phi^j\left(s^t\right)$ is such that $V^j(\phi^j\left(s^t\right), \Phi^j\left(s^t\right); s^t) = D^j\left(0; s^t\right)$,

then for all $a > \phi^j\left(s^t\right)$, not defaulting is strictly preferred to default, whereas for all $a < \phi^j\left(s^t\right)$, default is strictly preferred. This leads to the following definition.

Definition 1 *The debt limits* $\Phi^j \equiv \left\{\phi^j\left(s^t\right)\right\}_{s^t \in \mathcal{S}}$ *are self-enforcing, if and only if*

$$V^j(\phi^j\left(s^t\right), \Phi^j\left(s^t\right); s^t) = D^j\left(0; s^t\right) \text{ for all } s^t \in \mathcal{S}. \tag{6}$$

These debt limits imply that at each s^t, an agent is exactly indifferent between default and no default if his net financial position equals $\phi^j\left(s^t\right)$. This is akin to Alvarez and Jermann's (2000) notion of debt limits being "not too tight," and implies that debt limits adjust to allow for the maximum amount of credit that is compatible with repayment incentives. In principle, any set of debt limits for which $V^j(\phi^j\left(s^t\right), \Phi^j\left(s^t\right); s^t) \geq D^j\left(0; s^t\right)$ for all $s^t \in \mathcal{S}$ eliminates default incentives. However, if it were the case that $V^j(\phi^j\left(s^t\right), \Phi^j\left(s^t\right); s^t) > D^j\left(0; s^t\right)$, an agent facing a binding debt limit at $\phi^j\left(s^t\right)$ would be willing to borrow at a rate slightly higher than the market interest rate and market participants would not be willing to refuse him credit. Our debt limits are thus set so that (i) no borrower has an incentive to default, and (ii) no lender has an incentive to extend credit beyond a borrower's debt limit.

A competitive equilibrium with self-enforcing private debt is then defined as follows:

Definition 2 *For given* $\left\{a^j\left(s^0\right)\right\}_{j=1,\ldots,J}$ *with* $\sum_{j=1}^J a^j\left(s^0\right) = 0$*, a competitive equilibrium with self-enforcing private debt consists of consumption and asset profiles and debt limits* $\{C^j, A^j, \Phi^j\}_{j=1,\ldots,J}$*, and state-contingent bond prices* Q*, such that* (i) *for each* j*,* $\{C^j, A^j\}$ *maximize (1), subject to (2) and (3), for given* $a^j\left(s^0\right)$*,* (ii) *for each* j*,* Φ^j *satisfies(6), and* (iii) *markets clear:* $\sum_{j=1}^J c^j\left(s^t\right) = \sum_{j=1}^J y^j\left(s^t\right)$ *and* $\sum_{j=1}^J a^j\left(s^t\right) = 0$ *for all* $s^t \in \mathcal{S}$*.*

Our equilibrium definition exactly follows Alvarez and Jermann (2000), with the exception of the default consequence, which allows only for denial of future credit, instead of complete autarky. Conceptually, the debt limits are similar to prices in Walrasian markets, in that individuals optimize taking prices and debt limits as given, but both are endogenously determined by the market equilibrium to satisfy the market-clearing and self-enforcement conditions.

Unbacked public debt: For our equivalence result, we consider an alternative economy with unbacked public securities. As before, there are sequential markets with complete contingent securities. However, unlike before, agents can no longer issue these claims. Claims are only supplied by a government, which rolls over a fixed initial stock of claims $d\left(s^0\right)$ period by period by issuing new securities. The government must satisfy its budget constraint $d\left(s^t\right) \leq \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) d\left(s^{t+1}\right)$,

for all $s^t \in \mathcal{S}$, i.e. at s^t the amount of resources raised by issuing new claims for all $s^{t+1} \succ s^t$ must be sufficient to honor the previous period's commitments. This budget constraint captures the notion that these securities are not backed by any tax revenues or other government income. We assume that the government's budget constraint is satisfied with equality each period:

$$d\left(s^t\right) = \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) d\left(s^{t+1}\right) \text{ for all } s^t \in \mathcal{S}. \tag{7}$$

Given initial asset holdings $a^j\left(s^0\right) \geq 0$, optimal consumption allocations and asset holdings maximize (1), subject to (2) and the non-negativity constraint $a^j\left(s^t\right) \geq 0$ for all $s^t \in \mathcal{S}$. A competitive equilibrium with unbacked public debt is then defined as follows:

Definition 3 *For given initial asset positions $a^j\left(s^0\right) \geq 0$ for all j, and debt supply $d\left(s^0\right) = \sum_{j=1}^{J} a^j\left(s^0\right)$, a competitive equilibrium with unbacked public debt consists of consumption and asset profiles $\{C^j, A^j\}_{j=1,\ldots,J}$, a debt supply profile D, and bond prices Q, such that (i) for each j, $\{C^j, A^j\}$ are optimal given $a^j\left(s^0\right)$, (ii) D satisfies (7), and (iii) markets clear: $\sum_{j=1}^{J} c^j\left(s^t\right) = \sum_{j=1}^{J} y^j\left(s^t\right)$ and $\sum_{j=1}^{J} a^j\left(s^t\right) = d\left(s^t\right)$ for all $s^t \in \mathcal{S}$.*

3 Example

In this section, we illustrate the main results of our paper by means of a simple example with two types of consumers. In each period, one type receives the high endowment $\overline{e}$ and the other type receives the low endowment $\underline{e}$, with $\overline{e}+\underline{e} = 1$. The types switch endowment with probability α from one period to the next. Formally, uncertainty is captured by the Markov process s_t, with state space $S = \{\mathsf{s}_1, \mathsf{s}_2\}$ and symmetric transition probabilities $\Pr\left[s_{t+1} = \mathsf{s}_1 | s_t = \mathsf{s}_2\right] = \Pr\left[s_{t+1} = \mathsf{s}_2 | s_t = \mathsf{s}_1\right] = \alpha$. The event s^t corresponds here to the sequence $\{s_0, ..., s_t\}$ and the endowments $y^j\left(s^t\right)$ only depend on the current realization of s_t, with $y^j\left(s^t\right) = \overline{e}$, if $s_t = \mathsf{s}_j$ and $y^j\left(s^t\right) = \underline{e}$ if $s_t \neq \mathsf{s}_j$.

We construct a symmetric Markov equilibrium, in which consumption allocations, asset holdings, debt limits, and the prices of state-contingent bonds depend only on the current state s_t, consumption allocations and asset holdings are symmetric across types and states, and the debt limit is binding in the high-endowment state for each agent. To focus on a stationary equilibrium, assume that the economy begins in state $s_0 = \mathsf{s}_1$ and that the initial asset positions are $a^1\left(s_0\right) = -\omega$ and $a^2\left(s_0\right) = \omega$, where $-\omega$ is the debt limit for both agents. Proposition 1 shows that equilibria with positive debt levels can exist.

Proposition 1 *Let $\overline{c}$ be the solution to $1-\beta(1-\alpha)=\beta\alpha u'(1-\overline{c})/u'(\overline{c})$.*[6] *If $\overline{c}<\overline{e}$, there exists a stationary equilibrium with self-enforcing private debt in which:*

(i) State-contingent bond prices are $q\left(s^{t+1}\right)=q_c\equiv 1-\beta(1-\alpha)$ if $s_{t+1}\neq s_t$, and $q\left(s^{t+1}\right)=q_{nc}\equiv\beta(1-\alpha)$ if $s_{t+1}=s_t$;

(ii) Consumption allocations are $c^j\left(s^t\right)=\overline{c}$ if $s_t=\mathsf{s}_j$ and $c^j\left(s^t\right)=\underline{c}$ if $s_t\neq\mathsf{s}_j$, where $\underline{c}\equiv 1-\overline{c}$;

(iii) Asset holdings are $a^j\left(s^t\right)=-\omega$ if $s_t=\mathsf{s}_j$ and $a^j\left(s^t\right)=\omega$ if $s_t\neq\mathsf{s}_j$, where $\omega\equiv(\overline{e}-\overline{c})/(2q_c)$;

(iv) Debt limits are $\phi^j\left(s^t\right)=-\omega$ for all $s^t\in\mathcal{S}$.

Conditional on no agent ever defaulting, it is straightforward to check that the proposed consumption allocations and asset holdings are optimal, and asset markets clear. Therefore, we only need to check the no-default condition (6). This condition follows as an immediate consequence of our first general result, Theorem 1 below, which states that a sequence of debt limits Φ is self-enforcing if and only if $\phi\left(s^t\right)=\sum_{s^{t+1}\succ s^t}q\left(s^{t+1}\right)\phi\left(s^{t+1}\right)$ for all $s^t\in\mathcal{S}$. In our example, this condition reduces to $1=q_c+q_{nc}$, which is satisfied by our equilibrium prices.

Where does this characterization of self-enforcing debt limits come from, and why does it relate the incentives for repayment to the bond prices? To clarify this point consider an agent who contemplates the option of defaulting in a period where he receives the high endowment and is expected to repay a debt of ω. If, instead, this agent defaults, he can choose an asset profile of $\hat{a}\left(s^t\right)=2\omega$ if $s_t\neq\mathsf{s}_j$, and $\hat{a}\left(s^t\right)=0$ if $s_t=\mathsf{s}_j$, carrying a higher asset position into low-endowment states and a zero asset position into high-endowment states. The resulting consumption profile is $\hat{c}^j\left(s^t\right)=\overline{e}-2\omega q_c$ if $s_t=\mathsf{s}_j$, and $\hat{c}^j\left(s^t\right)=\underline{e}+2\omega-2\omega q_{nc}$ if $s_t\neq\mathsf{s}_j$. In comparison, the consumption profile without default is, by construction, $c^j\left(s^t\right)=\overline{c}=\overline{e}-2\omega q_c$ if $s_t=\mathsf{s}_j$, and $c^j\left(s^t\right)=1-\overline{c}=\underline{e}+2\omega q_c$ if $s_t\neq\mathsf{s}_j$. Therefore, whenever $1-q_{nc}>q_c$, a default leads to strictly higher consumption in low-endowment periods and to the same consumption in high-endowment periods hence default is strictly preferred to no-default.

A symmetric argument shows that if $1-q_{nc}<q_c$, the agent strictly prefers no-default to default. If the agent were to default, the resulting optimal asset profile takes the form $\hat{a}\left(s^t\right)=\hat{a}>0$ if $s_t\neq\mathsf{s}_j$, and $\hat{a}\left(s^t\right)=0$ if $s_t=\mathsf{s}_j$, where $\hat{a}$ is determined from the agent's first-order condition in high endowment periods. The resulting consumption profile is $\hat{c}^j\left(s^t\right)=\overline{e}-\hat{a}q_c$ if $s_t=\mathsf{s}_j$ and $\hat{c}^j\left(s^t\right)=\underline{e}+\hat{a}(1-q_{nc})$ if $s_t\neq\mathsf{s}_j$. If instead the agent does not default, he can set $a^j\left(s^t\right)=-\omega$ if $s_t=\mathsf{s}_j$ and $a^j\left(s^t\right)=\hat{a}-\omega(1-q_{nc})/q_c>-\omega$ if $s_t\neq\mathsf{s}_j$, where the last inequality follows from

[6] Inada conditions on $u(.)$ are sufficient to ensure that there is a unique $\overline{c}\in(0,1)$ that solves this equation.

$\hat{a} > 0$ and $1 - q_{nc} < q_c$. The resulting consumption allocations are $\hat{c}^j\left(s^t\right) = \overline{e} - \hat{a}q_c$ if $s_t = \mathsf{s}_j$ and $\hat{c}^j\left(s^t\right) = \underline{e} + \hat{a}\left(1 - q_{nc}\right) + \omega\left(q_{nc} + q_c - 1\right)/q_c$ if $s_t \neq \mathsf{s}_j$. Thus, whenever $1 - q_{nc} < q_c$, the agent strictly prefers not to default. The condition $q_c + q_{nc} = 1$ is therefore required to ensure that agents are exactly indifferent between default and no-default.

Proposition 1 illustrates our first general result: positive levels of debt are sustainable in equilibrium if interest rates are sufficiently low. Repayment incentives depend not only on how much the agent is allowed to borrow in the future, but also on the interest rate at which borrowing and lending will take place. The higher the interest rate, the less appealing is the opportunity to borrow in the future, and the more appealing the option to be a net lender after default. In general equilibrium the interest rate and the debt limits will jointly adjust to ensure that debt is self-enforcing and markets clear. Notice that the resulting interest rate is so low that aggregate endowments are infinite-valued.

A stationary equilibrium with positive borrowing only exists when $\overline{e} > \overline{c}$.[7] In addition, there always exists an autarkic equilibrium with zero borrowing and prices equal to $q_c^{aut} = \beta\alpha u'\left(\underline{e}\right)/u'\left(\overline{e}\right)$ and $q_{nc}^{aut} = \beta\left(1 - \alpha\right)$. Whether the inequality $\overline{e} > \overline{c}$ holds is equivalent to whether the risk-free interest rate in the autarkic equilibrium, $1/(q_c^{aut} + q_{nc}^{aut})$, is smaller than 1. If it is larger than 1, aggregate endowments are finite-valued at the autarkic equilibrium, i.e., the autarky allocation displays "high implied interest rates," in the language of Alvarez and Jermann (2000). In this case, autarky is the unique equilibrium. If instead $1/(q_c^{aut} + q_{nc}^{aut})$ is less than 1, aggregate endowments are infinite-valued in the autarkic equilibrium and there exist additional equilibria with positive debt circulation.[8]

The condition that $\overline{e} > \overline{c}$ also ensures that this economy admits a stationary equilibrium with valued unbacked public debt, with consumption allocations and bond prices that are identical to the ones in the equilibrium with self-enforcing private debt.

Proposition 2 *If $\overline{e} > \overline{c}$, there exists an equilibrium with unbacked public debt, in which consumption allocations and prices are the same as in Proposition 1, asset holdings are $a^j\left(s^t\right) = 0$ if $s_t = \mathsf{s}_j$ and $a^j\left(s^t\right) = 2\omega$ if $s_t \neq \mathsf{s}_j$, and the government's supply of debt is $d\left(s^t\right) = 2\omega$ for all $s^t \in \mathcal{S}$.*

[7] Whether or not $\bar{e} > \bar{c}$ depends on on the model parameters, and in particular on the volatility and persistence of the endowment process and on the agents' risk aversion.

[8] These results lead to an interesting comparison with the autarky punishment considered in Alvarez and Jermann (2000): in both cases, a non-autarkic equilibrium exists if and only if the autarky allocation displays "low implied interest rates" (Proposition 4.8 in Alvarez and Jermann (2000) proves the necessity part in general), but when there exist non-autarkic equilibria, the maximum sustainable level of debt in Alvarez and Jermann leads to high implied interest rate, and is hence strictly higher than in our model with exclusion from credit.

Proposition 2 illustrates our second general result: equilibrium allocations in an economy with self-enforcing private debt are equivalent to equilibrium allocations in an economy with unbacked public debt. This is proved in full generality in Theorem 2 and allows us to use equilibrium characterizations that apply in known environments with unbacked public debt to establish the existence and characterization of equilibria with positive levels of self-enforcing private debt.

In Appendix C, we extend the analysis of this example in several dimensions. First, we augment the example to include aggregate endowment growth, showing that the equilibrium interest rate must equal the economy's growth rate, which must equal the growth rate of the aggregate debt supply and the individual debt limits in steady-state. We also extend our results to non-stationary equilibria, in which there is an expectations-driven, self-fulfilling collapse in the real value of debt (equivalent to the hyper-inflations of the public debt/fiat money economy). Finally, we show how the transition dynamics depend on type-specific debt limits and initial asset holdings.

4 Characterizing Repayment Incentives

In this section, we characterize the repayment incentives of an individual borrower. The main result in this section (Theorem 1) is that debt limits are self-enforcing if and only if they "allow for exact roll-over," i.e., if and only if at each history, the agent is able to exactly repay his maximum outstanding debt $-\phi\left(s^t\right)$ by issuing new debt, up to the limit $-\phi\left(s^{t+1}\right)$, for each $s^{t+1} \succ s^t$. Since we are exclusively concerned with the single-agent problem, we simplify notation throughout this section by dropping the superscript j.

Theorem 1 *The debt limits Φ are self-enforcing if and only if they allow for exact roll-over:*

$$\phi\left(s^t\right) = \sum_{s^{t+1} \succ s^t} q\left(s^{t+1}\right) \phi\left(s^{t+1}\right) \text{ for all } s^t \in \mathcal{S}. \tag{8}$$

Moreover, if Φ is self-enforcing, $\mathcal{C}(a, \Phi\left(s^t\right); s^t) = \mathcal{C}(a - \phi\left(s^t\right), O\left(s^t\right); s^t)$ for all $s^t \in \mathcal{S}$.

This theorem also shows that the budget set of an agent who faces self-enforcing debt limits Φ is identical to that of an agent facing zero debt limits, but starting from a higher initial asset position. Hence, optimal consumption and asset profiles are the same for the two agents. This simplifies the equilibrium characterization, since, rather than computing the fixed point between the consumer's optimization problem and the self-enforcement condition (20), we only need to compute optimal consumption allocations for agents with zero debt limits, and these are identical to the optimal

consumption allocations without default. Equilibrium debt limits are then constructed in such a way that they satisfy (20) and market clearing in the asset market.

Condition (20) states that debt can only be sustained, if, instead of repaying, the borrower is able to infinitely roll over outstanding debt. To see what this characterization of debt limits entails, we can compare it to BR's no lending result.

Proposition 3 *(Bulow and Rogoff) Suppose that* $\hat{Y}\left(s^t\right) \equiv \sum_{s^{t+\tau}\in\mathcal{S}(s^t)} y\left(s^{t+\tau}\right) p\left(s^{t+\tau}\right)/p\left(s^t\right) < \infty$ *and consider any asset profile* $\left\{a\left(s^t\right)\right\}_{s^t\in\mathcal{S}}$ *such that* $a\left(s^t\right) \geq -\hat{Y}\left(s^t\right)$*. If* $a\left(s^t\right) < 0$ *for some event* $s^t \in \mathcal{S}$*, then the agent strictly prefers to default at some subsequent event* $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$*.*

BR show that lending is not sustainable if two conditions hold: (i) endowments are finite-valued at the prevailing state prices, i.e. interest rates are high, and (ii) the agent's debt is bounded by the present value of his future endowments, his "natural debt limit." Condition (i) is imposed as an exogenous restriction on state-prices. Condition (ii) on the other hand is a standard restriction which is usually imposed to rule out Ponzi games.[9] With high interest rates, Theorem 1 implies that any self-enforcing debt limits have to eventually be inconsistent with the natural debt limits. With low interest rates, however, natural debt limits are infinite, and condition (ii) imposes no restriction on debt levels.

To sustain positive debt levels, we must abandon one of these conditions. From a partial equilibrium point of view, relaxing either one can lead to self-enforcing debt. However, a general equilibrium argument shows that the interesting case arises when we dispose of condition (i). If we relax condition (ii), but maintain high interest rates, Theorem 1 implies that, if there are positive levels of debt, the aggregate stock of debt, and thus the savings of some lender, will eventually exceed the value of aggregate endowments. This clearly cannot happen in general equilibrium. On the other hand, it is possible to construct economies where, in general equilibrium, condition (i) fails to hold, as shown in the example in Section 3 and, more generally, in Section 5 below.

Exact roll-over implies self-enforcement: The sufficiency part of Theorem 1 is established by the following proposition.

Proposition 4 *Suppose that the debt limits* Φ *allow for exact roll-over. Then*

$$V(a, \Phi\left(s^t\right); s^t) = D\left(a - \phi\left(s^t\right); s^t\right) \text{ for all } s^t \in \mathcal{S}, \text{ and any } a \geq \phi\left(s^t\right). \tag{9}$$

[9] In the working-paper version, Bulow and Rogoff (1988, p. 5) hint at the idea that relaxing this condition may lead to positive debt, when they remark that this assumption rules out "'Ponzi'-type reputational contracts."

The self-enforcement condition (6) follows from setting $a = \phi\left(s^t\right)$ in (9). Condition (9) further implies that an agent who defaults on his maximum gross amount of debt $-\phi\left(s^t\right)$, but keeps his own asset holdings $a - \phi\left(s^t\right)$ after a default is always exactly indifferent between defaulting and not defaulting. The assumption that agents start with a net financial position of zero after default can therefore be relaxed without weakening repayment incentives.

We can illustrate the characterization of self-enforcing debt limits and the relation to the roll-over condition with a series of figures. For this, we assume that endowment fluctuations are deterministic, and agents trade a single uncontingent bond.[10] The agents' budget constraint can then be rewritten as $c_t = y_t + \left(p_t a_t - p_{t+1} a_{t+1}\right)/p_t$. For a given sequence of prices $\{p_t\}$, we can thus compare the consumption profiles resulting from different asset plans simply by comparing the period-by-period changes in the present value of asset holdings, $p_t a_t - p_{t+1} a_{t+1}$. In the following figures, we plot the time paths $\{p_t a_t\}$ of the present values of asset profiles with and without defaults to evaluate repayment incentives.

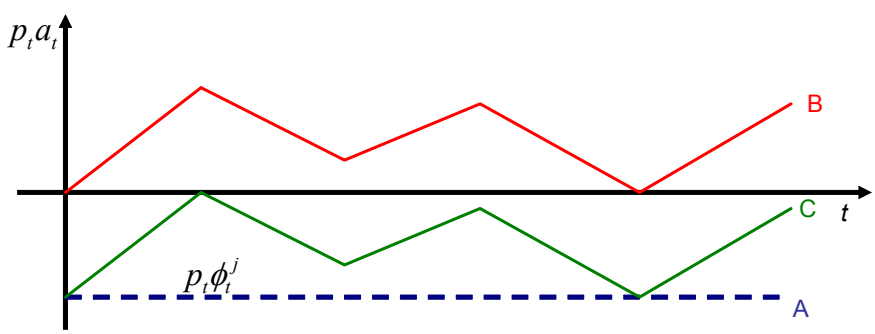

Figure 1: Debt limits satisfying exact roll-over

Figure 1 considers repayment incentives when debt limits allow for exact roll-over. In a deterministic environment, this requires that $p_t \phi_t$ is constant over time; such debt limits are plotted by the dotted line A. Line B represents an arbitrary asset profile that is feasible for an agent who defaults at date t. Line C represents a parallel downwards shift of the asset profile B, to an initial asset position of ϕ_t. Notice that C generates the same consumption sequence as B. Moreover, since profile B remains non-negative, C always remains above A, and is therefore feasible for an agent who starts with an asset position of ϕ_t and does not default. Hence, this agent must be weakly better off not defaulting at date t. On the other hand, for any asset profile C that is feasible without default starting from an asset position of ϕ_t, there is some asset profile B that is feasible starting

[10] We thus replace the dependence on s^t by a time subscript to simplify notation.

from a default at date t, and gives the agent the same consumption profile as C. If debt limits allow for exact roll-over, the agent must therefore be exactly indifferent between defaulting on an asset position of ϕ_t, and not defaulting, i.e. the self-enforcement condition is satisfied with equality.

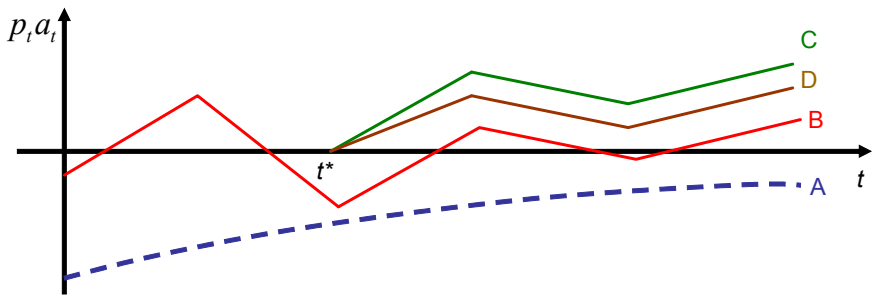

Figure 2: Shrinking debt limits

Along similar lines, we can also illustrate how repayment incentives are violated, when the present value of debt limits is shrinking over time. Figure 2 plots the case of BR, in which the natural debt limits (line A) are finite and hence tightening over time, in present value, and they act as a lower bound on the agent's asset profile. Then, for any asset profile B that is consistent with these debt limits and that admits positive debt at some date t, there exists a date $t^* \geq t$ at which the present value of the debt reaches a maximum. At that point, the agent can replicate the same consumption profile as B after a default, just using positive asset holdings (line C), or even improve upon the no-default profile by strictly increasing consumption at date t^* (line D).

Likewise, if the present value of debt limits is expanding over time (Figure 3), for every profile B that is feasible after a default at some date t, there exists a profile C that is feasible without default

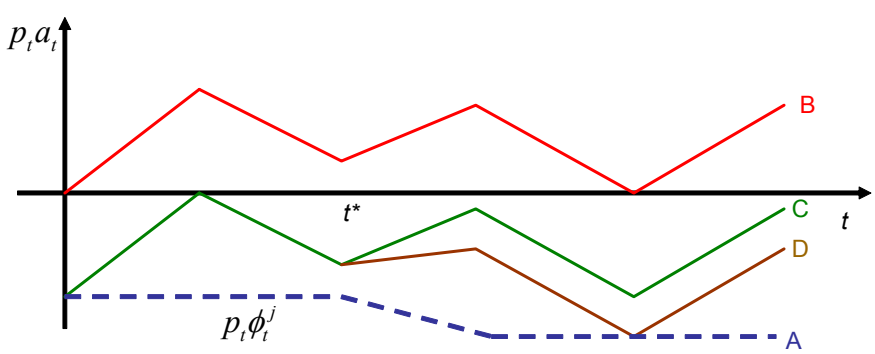

Figure 3: Expanding debt limits

starting from asset position ϕ_t and implements the same consumption. Moreover, profile D remains feasible without default starting from asset position ϕ_t, but delivers strictly higher consumption at date t^*, where the debt limits are expanding. Hence, agents strictly prefer not to default, when the present value of debt limits is expanding over time.

Self-enforcement implies exact roll-over: Proposition 5 completes the proof of Theorem 1 by showing that debt limits are self-enforcing only if they allow for exact roll-over. This was already suggested by the graphical arguments in Figure 2 and 3. However, this graphical intuition is incomplete, since it only applies to sequences of debt limits whose present values are monotone increasing or decreasing. Ruling out the possibility that arbitrary non-monotone sequences of debt limits may be self-enforcing turns out to be considerably more involved.

Proposition 5 *Any sequence of self-enforcing debt limits Φ allows for exact roll-over.*

The proof of this proposition is in Appendix B, here we sketch the key steps. Consider a sequence of self-enforcing debt limits Φ. Starting from some arbitrary event s^t, we first construct a sequence of auxiliary debt limits $\tilde{\Phi}\left(s^t\right)$, as follows:

$$\tilde{\phi}\left(s^{t+\tau}\right) = \begin{cases} \phi\left(s^{t+\tau}\right) & \text{if } a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) \\ \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \min\left\{\phi\left(s^{t+\tau+1}\right), \tilde{\phi}\left(s^{t+\tau+1}\right)\right\} & \text{otherwise} \end{cases}, \tag{10}$$

where $A^*\left(s^t\right)$ denotes the optimal no-default asset profile of an agent starting from s^t with asset position $a^*\left(s^t\right) = \phi\left(s^t\right)$. The first step of the proof (and the major technical hurdle) consists in showing that this sequence is well-defined and finite-valued. This is complicated by the fact that present discounted values need not be well-defined in our environment since we cannot rely on an assumption of high interest rates or finite-valued endowments. The characterization in turn makes use of the time-separability, concavity and boundedness of $u(\cdot)$.

Next, using arbitrage arguments, we show that (i) $\tilde{\phi}\left(s^{t+\tau}\right) \leq \phi\left(s^{t+\tau}\right)$, and $\tilde{\phi}\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right)$ whenever $a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right)$, and (ii) $\tilde{\phi}\left(s^{t+\tau}\right) \geq \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \tilde{\phi}\left(s^{t+\tau+1}\right)$, for every $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. The first property states that $\tilde{\Phi}$ is a lower bound of Φ, and is equal to Φ whenever the actual debt limit is binding. The second property states that $\tilde{\Phi}$ satisfies (ER) with a weak inequality, so that under $\tilde{\Phi}$, at any event, the maximum outstanding debt obligations are weakly less than the funds that can be raised by exhausting debt limits on the continuation events.

Property (i), together with the concavity of $u(\cdot)$ then implies that the value of the no-default problem starting from asset holdings of $a\left(s^t\right) = \phi\left(s^t\right)$ at s^t is the same under the original

debt limits, $\Phi\left(s^t\right)$, as under the auxiliary debt limits, $\tilde{\Phi}\left(s^t\right)$ (since the latter only relaxes non-binding debt limits). Since the original debt limits are self-enforcing, and $\phi\left(s^t\right) \geq \tilde{\phi}\left(s^t\right)$, we obtain $D\left(0; s^t\right) = V\left(\phi\left(s^t\right); \Phi\left(s^t\right), s^t\right) = V\left(\phi\left(s^t\right); \tilde{\Phi}\left(s^t\right), s^t\right) \geq V\left(\tilde{\phi}\left(s^t\right); \tilde{\Phi}\left(s^t\right), s^t\right)$. On the other hand, using the same arbitrage argument as Proposition 4, property (ii) implies that $\mathcal{C}(0, O\left(s^t\right); s^t) \subseteq \mathcal{C}\left(\tilde{\phi}\left(s^t\right), \tilde{\Phi}\left(s^t\right), s^t\right)$ and $V\left(\tilde{\phi}\left(s^t\right); \tilde{\Phi}\left(s^t\right), s^t\right) \geq D\left(0; s^t\right)$, with strict inequality if $\tilde{\phi}\left(s^{t+\tau}\right) > \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \tilde{\phi}\left(s^{t+\tau+1}\right)$ for some $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. Therefore, both inequalities must hold with equality, which requires $\phi\left(s^t\right) = \tilde{\phi}\left(s^t\right)$, and that $\tilde{\Phi}\left(s^t\right)$ satisfies the exact roll-over condition as an equality, for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$.

To complete the proof, we show that $\tilde{\phi}\left(s^{t+1}\right) = \phi\left(s^{t+1}\right)$, for all $s^{t+1} \succ s^t$. Repeating the same steps as above, we construct additional sequences of auxiliary debt limits $\hat{\Phi}\left(s^{t+1}\right)$, together with optimal asset holdings $\hat{A}\left(s^{t+1}\right)$, for each $s^{t+1} \succ s^t$. Clearly, $\hat{\Phi}\left(s^{t+1}\right)$ satisfies (ER), and $\phi\left(s^{t+1}\right) = \hat{\phi}\left(s^{t+1}\right)$. Moreover, due to concavity and additive separability of U, optimal asset profiles are monotone in initial asset holdings, so that $\hat{a}\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) = \hat{\phi}\left(s^{t+\tau}\right)$, whenever $a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) = \tilde{\phi}\left(s^{t+\tau}\right)$. Together with (10), this implies that $\tilde{\phi}\left(s^{t+\tau}\right) = \hat{\phi}\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \mathcal{S}\left(s^{t+1}\right)$, and hence $\tilde{\phi}\left(s^{t+1}\right) = \hat{\phi}\left(s^{t+1}\right) = \phi\left(s^{t+1}\right)$, which completes our proof.

Remark: Proposition 5 is the only result where we use the assumptions of additive time-separability, concavity and boundedness of $u(\cdot)$. All other results rely purely on arbitrage arguments and therefore require only strict monotonicity. The boundedness assumption is a strong restriction, but it is required only for a partial equilibrium characterization. If one restricts attention to debt limits Φ such that optimal consumption allocations are bounded above by aggregate endowments (a condition that must hold in general equilibrium), Proposition 5 holds under the following weaker restriction.

Assumption 1 *For all C, such that $U(C) \geq \min_j U\left(Y^j\right)$ and $c\left(s^t\right) \in \left[0, \sum_{j=1}^J y^j\left(s^t\right)\right]$ for all $s^t \in \mathcal{S}$, $\sum_{s^t \in \mathcal{S}} \beta^t \pi\left(s^t\right) c\left(s^t\right) u'(c\left(s^t\right)) < \infty$.*

This regularity condition bounds the rate at which individual and aggregate endowments can grow or decline, relative to the discount factor β. When relative risk aversion is bounded, this assumption holds whenever $U\left(\sum_{j=1}^J Y^j\right)$ and $\min_j U\left(Y^j\right)$ are both finite.

5 General Equilibrium Characterization

We now turn to the question whether there exist equilibria with positive levels of self-enforcing debt, and how they can be characterized. Theorem 2 shows that a given consumption allocation

and price vector constitute a competitive equilibrium with self-enforcing private debt, if and only if the same allocation and prices are an equilibrium of the corresponding economy with unbacked public debt. For the latter economy, there are known existence and characterization results (e.g. Santos and Woodford 1997), which then extend immediately to the economy with self-enforcing private debt.

Theorem 2 *An allocation $\{C^j\}_{j=1,\ldots,J}$ and prices Q are sustainable as a competitive equilibrium with self-enforcing private debt, if and only if $\{C^j\}_{j=1,\ldots,J}$ and Q are also sustainable as a competitive equilibrium with unbacked public debt.*

The proof of Theorem 2 relies on three observations: First, as a consequence of Theorem 1, if debt limits Φ^j are self-enforcing, then for suitably chosen initial asset positions, the feasible, and hence the optimal, consumption allocations in the default and no-default problems are identical. Second, if debt limits Φ^j are self-enforcing, then the sequence of public debt supply constructed by $d\left(s^t\right) = -\sum_{j=1}^{J} \phi^j\left(s^t\right)$ satisfies the government roll-over constraint (7). Likewise, for any debt supply sequence D that satisfies (7), it is possible to construct sequences of debt limits Φ^j for each type that allow for exact roll-over, are consistent with the condition on initial asset holdings, and satisfy $-\sum_{j=1}^{J} \phi^j\left(s^t\right) = d\left(s^t\right)$. Finally, market-clearing conditions in the two environments are equivalent. For goods markets, this is immediate; for asset markets this follows from the fact that the above mapping between private debt limits and government debt supply generates the same aggregate amount of debt circulation.

Theorem 2 formally establishes the connection between sustaining repayment incentives for private debt and sustaining rational bubbles. In the private debt economy, the agents' commitment and enforcement power is so limited that any contract that, at some date, requires a positive transfer of resources in net present value is not sustainable. Likewise, in the economy with unbacked public debt, the government does not have the power to use taxation to guarantee its debtholders a positive net transfer of resources. Existing debt must instead be rolled over indefinitely. In both cases, the result is that the only sustainable allocations roll over existing debt forever. The equivalence arises because neither side can credibly commit to future transfers, either via contract enforcement or via taxation.

6 Concluding Remarks

In this paper, we have studied a general equilibrium economy with self-enforcing private debt, in which, after a default, borrowers are excluded from future credit, but retain the ability to save

in the market. For a partial equilibrium version of this model, in which a small open economy borrows internationally at fixed, positive interest rates, BR show that debt cannot be sustainable by reputational mechanisms only: eventually, the country always has an incentive to default. The BR result can be interpreted as follows: if there are some agents who are able to commit to intertemporal transfers at "high interest rates," then the remaining agents, who are unable to commit, will accumulate and decumulate the securities issued by the committed agents, but will never become net borrowers. Kruger and Uhlig (2006) provide the analytical foundations for this interpretation.

In contrast, we show that positive levels of debt can be sustained when no party has commitment power. The key to our result is that interest rates adjust downwards to provide repayment incentives to all the potential borrowing parties. As a result, "low interest rates" emerge in equilibrium. The assumption of multilateral lack of commitment seems especially appropriate for international financial markets, where it is reasonable to assume that sovereign nations will act in a self-interested manner when evaluating whether to repay or default on their international obligations, and whether to enforce the international obligations of their residents.

Our theoretical results help to clarify the BR result in two directions. From a formal point of view, they highlight the role played by the interest rates in the BR argument. From a more substantive point of view, they clarify the role of unilateral vs. multilateral lack of commitment for the sustainability of debt in general equilibrium, and provide a characterization of the degree of insurance which can be achieved when the punishment for default is only credit exclusion, showing in particular that the sustainable levels of debt need not be zero. This analysis helps bridge the gap between the negative result of BR and the positive results obtained with stronger forms of punishment, in particular those in Kehoe and Levine (1993) and Alvarez and Jermann (2000).[11]

Our analysis leads to the question whether the debt dynamics of international borrowers display the self-financing features identified in this paper, that is, whether borrowing countries face conditions that enable them to roll over their debt indefinitely. In practice, this amounts to testing for rational bubbles in international debt, a quantitative issue whose resolution is outside the scope of this paper. In closed economies, the sustainability of rational bubbles has been questioned both on theoretical and on empirical grounds (e.g. Abel et al. 1989).[12] However, some authors have

[11] See footnote 8. Our analysis also relates to the model of private international capital flows of Jeske (2006) and Wright (2006). In their model, the ability of agents to participate in domestic capital markets after defaulting on external debt has the same effects as the ability to save in our model. See the discussion in Wright (2006) for formal details.

[12] In theory, rational bubbles are commonly ruled out by the presence of assets that pay an infinite stream of

recently argued that rational bubbles may be present in international financial flows (e.g. Ventura 2004, and Caballero and Krishnamurthy 2006). In this respect our theoretical results point out the connection between the sustainability of rational bubbles and the problem of providing incentives for repayment in a world with multilateral lack of commitment. Much work remains to be done to incorporate this insight in a realistic model of international capital flows and to test it empirically.

Finally, our paper also has implications for the literature on inside and outside money. In particular, in our setup unbacked public debt and self-enforcing private debt are analogous to outside (fiat) money and inside money. The existing monetary literature discusses the circulation of fiat money and inside money largely in separation from each other. The circulation of fiat money requires that an intrinsically useless asset (a rational bubble) is traded at a positive price. The circulation of inside money instead relies on having the proper reputational mechanisms in place to guarantee that outstanding claims are honored. Although on the surface these seem to be conceptually distinct problems, our analysis shows that they are closely related.[13]

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dividends, or are backed by physical collateral, such as claims to an economy's stock of capital - such assets eliminate both the need for and the possibility of bubbles for intertemporal trade. Whether a similar argument can be applied to international capital markets is less clear, since such assets are typically also tied to a physical location at which the collateral is located, or the dividend is generated. The country controlling that location may however choose to expropriate foreign nationals or prevent them from accessing the collateral. In international capital markets, the same lack of commitment that makes rational bubbles possible thus also limits the transferability of assets that would eliminate them.

[13] See Cavalcanti, Erosa and Temzelides (1999) and Berentsen, Camera and Waller (2007) for matching models of private debt circulation under limited commitment. In Cavalcanti et al., a fixed subset of agents is allowed to issue notes, which are sustained by the loss of a non-competitive note-issuing rent if outstanding notes are not redeemed on demand. Berentsen et al. study a matching model with money and competitive supply of bank credit, and show among other things that with lack of commitment, such credit is sustainable only if the inflation rate is non-negative.

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7 Appendix A: Proofs

Proof of Proposition 1. Given the prices q_c and q_{nc}, the first order conditions for consumer j can be rewritten as $q_c u'(\overline{c}) = \beta\alpha u'(\underline{c})$, $q_{nc} = \beta(1-\alpha)$, and $q_c u'(\underline{c}) \geq \beta\alpha u'(\overline{c})$. These conditions

hold given our definition of $\overline{c}$ (for the last inequality we make use of the fact that the definition of $\overline{c}$ implies $u'(\overline{c}) < u'(\underline{c})$). In addition, budget constraints and market-clearing conditions are satisfied by construction, given our definition of ω. As explained in the text, Theorem 1 below also implies that these debt limits are self-enforcing. ■

Proof of Proposition 2. It is immediate to check that the proposed consumption allocations and asset holdings are optimal for consumers, and they satisfy market-clearing in goods and asset markets by construction. Moreover, since $d(s^t)$ is constant for all $s^t \in \mathcal{S}$ and $q_c + q_{nc} = 1$, the government's roll-over condition is also satisfied. ■

Proof of Proposition 3. First, we show that for each asset profile $\{a(s^t)\}_{s^t \in \mathcal{S}}$, such that $a(s^t) \geq -\hat{Y}(s^t)$, for all $s^t \in \mathcal{S}$, there exists a sequence of 'auxiliary debt limits' $\tilde{\Phi}$, with $\tilde{\phi}(s^t) \in \left[-\hat{Y}(s^t), 0\right]$, for all s^t, which satisfies $\tilde{\phi}(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\left\{\tilde{\phi}(s^{t+1}), a(s^{t+1})\right\}$, for all $s^t \in \mathcal{S}$. Define the sequence $\{\Phi^{(K)}\}$ by $\phi^{(K)}(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\left\{\phi^{(K-1)}(s^{t+1}), a(s^{t+1})\right\}$, where $\Phi^{(0)}$ is given by $\phi^{(0)}(s^t) = -\hat{Y}(s^t)$. Since $a(s^{t+1}) \geq -\hat{Y}(s^{t+1}) = \phi^{(0)}(s^{t+1})$ for all s^{t+1}, $\phi^{(1)}(s^t) \geq -\sum_{s^{t+1} \succ s^t} q(s^{t+1}) \hat{Y}(s^{t+1}) = -\hat{Y}(s^t) + y(s^t) \geq \phi^{(0)}(s^t)$ for all $s^t \in \mathcal{S}$, and therefore $\{\Phi^{(K)}\}$ is monotone increasing in K. Moreover, since $\{\Phi^{(K)}\}$ is bounded above by 0, this sequence converges to a finite limit $\tilde{\Phi}$ which satisfies the above recursive construction.

Next, we show that if $a(s^t) < \tilde{\phi}(s^t)$ for some $s^t \in \mathcal{S}$, then the agent strictly prefers to default at s^t, and set $\hat{a}(s^{t+\tau}) = a(s^{t+\tau}) - \min\left\{\tilde{\phi}(s^{t+\tau}), a(s^{t+\tau})\right\} \geq 0$, for all $s^{t+\tau} \in \mathcal{S}(s^t)$. The difference in consumption between default and no default, $\{\Delta c(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$, is

$$\begin{aligned}
\Delta c(s^{t+\tau}) &= -\min\left\{a(s^{t+\tau}), \tilde{\phi}(s^{t+\tau})\right\} + \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q(s^{t+\tau+1}) \min\left\{a(s^{t+\tau+1}), \tilde{\phi}(s^{t+\tau+1})\right\} \\
&= -\min\left\{a(s^{t+\tau}) - \tilde{\phi}(s^{t+\tau}), 0\right\}
\end{aligned}$$

which is non-negative for all $s^{t+\tau} \in \mathcal{S}(s^t)$, and strictly positive at s^t.

Finally, we show that if $a(s^t) < 0$ for some $s^t \in \mathcal{S}$, the agent strictly prefers to default at some $s^{t+\tau} \in \mathcal{S}(s^t)$. Suppose to the contrary that $a(s^{t+\tau}) \geq \tilde{\phi}(s^{t+\tau})$ for all $s^{t+\tau} \in \mathcal{S}(s^t)$. Then, using the definition of $\tilde{\phi}$, we find, for $K = 1, 2, ...$

$$\tilde{\phi}(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \tilde{\phi}(s^{t+1}) = \sum_{s^{t+K} \succ s^t} \frac{p(s^{t+K})}{p(s^t)} \tilde{\phi}(s^{t+K}) \geq -\sum_{s^{t+K} \succ s^t} \frac{p(s^{t+K})}{p(s^t)} \hat{Y}(s^{t+K})$$

and therefore $\tilde{\phi}(s^t) \geq -\lim_{K \to \infty} \sum_{s^{t+K} \succ s^t} \hat{Y}(s^{t+K}) p(s^{t+K}) / p(s^t) = 0$, by the assumption that $\hat{Y}(s^t)$ is finite. But this contradicts $\tilde{\phi}(s^t) \leq a(s^t) < 0$. ■

Proof of Proposition 4. Take an arbitrary event s^t, and asset profiles $\{a(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$ and $\{\hat{a}(s^{t+\tau})\}_{s^{t+\tau} \in \mathcal{S}(s^t)}$, s.t. $\hat{a}(s^{t+\tau}) = a(s^{t+\tau}) - \phi(s^{t+\tau})$ for all $s^{t+\tau} \in \mathcal{S}(s^t)$, with $a(s^t) = a \geq \phi(s^t)$

and $\hat{a}\left(s^t\right) = a - \phi\left(s^t\right)$. Clearly, $\left\{a\left(s^{t+\tau}\right)\right\}_{s^{t+\tau}\in\mathcal{S}(s^t)}$ is feasible under no default, if and only if $\left\{\hat{a}\left(s^{t+\tau}\right)\right\}_{s^{t+\tau}\in\mathcal{S}(s^t)}$ is feasible after defaulting at s^t. Moreover, the exact roll-over condition (20) implies $a\left(s^{t+\tau}\right) - \sum_{s^{t+\tau+1}\succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) a\left(s^{t+\tau+1}\right) = \hat{a}\left(s^{t+\tau}\right) - \sum_{s^{t+\tau+1}\succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \hat{a}\left(s^{t+\tau+1}\right)$ for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$, and therefore, starting from $a\left(s^t\right) = a$, asset plan $\left\{a\left(s^{t+\tau}\right)\right\}_{s^{t+\tau}\in\mathcal{S}(s^t)}$ implements the same consumption allocation $\left\{c\left(s^{t+\tau}\right)\right\}_{s^{t+\tau}\in\mathcal{S}(s^t)}$ as asset plan $\left\{\hat{a}\left(s^{t+\tau}\right)\right\}_{s^{t+\tau}\in\mathcal{S}(s^t)}$ starting from $\hat{a}\left(s^t\right) = a - \phi^j\left(s^t\right)$. But then, $\mathcal{C}(a, \Phi\left(s^t\right); s^t) = \mathcal{C}(a - \phi\left(s^t\right), O\left(s^t\right); s^t)$, and $V(a, \Phi\left(s^t\right); s^t) = D\left(a - \phi\left(s^t\right); s^t\right)$. ■

Proof of Theorem 2. *Step 1:* Let $\{C^j, A^j, \Phi^j, Q\}_{j=1,\ldots,J}$ be a CE with self-enforcing private debt, starting from initial asset positions $\{a^j\left(s^0\right)\}_{j=1,\ldots,J}$. For each j, consider the modified asset profiles $\hat{A}^j = A^j - \Phi^j$, and government debt supply $D = -\sum_{j=1}^J \Phi^j$. By proposition 4, $\mathcal{C}^j(a^j\left(s^0\right), \Phi^j; s^0) = \mathcal{C}^j\left(\hat{a}^j\left(s^0\right); O\left(s^0\right), s^0\right)$, which implies that $\{C^j, \hat{A}^j\}$ are optimal, given initial asset position $\hat{a}^j\left(s^0\right) = a^j\left(s^0\right) - \phi^j\left(s^0\right)$, and zero debt limits. Since Φ^j is self-enforcing, D satisfies (7), and since $\sum_{j=1}^J c^j\left(s^t\right) = \sum_{j=1}^J y^j\left(s^t\right)$ and $\sum_{j=1}^J A^j = 0$, we have $\sum_{j=1}^J \hat{A}^j = -\sum_{j=1}^J \Phi^j = D$. Hence $\{C^j, \hat{A}^j, D, Q\}_{j=1,\ldots,J}$ is a CE with unbacked public debt, for initial asset positions $\{\hat{a}^j\left(s^0\right)\}_{j=1,\ldots,J}$.

Step 2: Let $\{C^j, \hat{A}^j, D, Q\}_{j=1,\ldots,J}$ be a CE with unbacked public debt, for initial asset positions $\{\hat{a}^j\left(s^0\right)\}_{j=1,\ldots,J}$. For each j, consider debt limits $\tilde{\Phi}^j = \hat{a}^j\left(s^0\right)/d\left(s^0\right) \cdot D$, and the asset profile $\tilde{A}^j = \hat{A}^j + \tilde{\Phi}^j$, with initial asset holdings of $\tilde{a}^j\left(s^0\right) = 0$. The debt limits $\tilde{\Phi}^j$ are self-enforcing by construction, and using proposition 4, $\{C^j, \tilde{A}^j\}$ are optimal, given inital asset position $\tilde{a}^j\left(s^0\right) = 0$ and debt limits $\tilde{\Phi}^j$. Moreover, since $\sum_{j=1}^J \hat{A}^j = D$, we have $\sum_{j=1}^J \tilde{A}^j = D + \sum_{j=1}^J \tilde{\Phi}^j = 0$, so that goods and asset markets clear. Therefore $\{C^j, \tilde{a}^j, \tilde{\Phi}^j, p\}_{j=1,\ldots,J}$ is a CE with self-enforcing private debt, for initial asset positions of zero. ■

8 Appendix B: Proof of Proposition 5

The proof of Proposition 5 proceeds in five steps, which are stated as separate lemmas. Lemma 1 establishes useful properties of the solution to the consumer problem (5). Starting from an arbitrary date $s^t \in \mathcal{S}$, Lemma 2 constructs a sequence of auxiliary debt limits $\tilde{\Phi}\left(s^t\right)$. Lemma 3 then establishes that $\tilde{\Phi}\left(s^t\right)$ satisfies the (ER) condition as a weak inequality, and bounds the actual debt limits from below. Lemma 4 establishes that $\tilde{\Phi}\left(s^t\right)$ satisfies the exact roll-over condition as an equality, and shows that the auxiliary and actual debt limits coincide at the initial date s^t. Finally, Lemma 5 shows that the auxiliary and actual debt limits also coincide for all $s^{t+1} \succ s^t$.

Suppose that $V\left(\phi\left(s^t\right), \Phi\left(s^t\right); s^t\right) = D\left(0; s^t\right)$ for all $s^t \in \mathcal{S}$, and that $\phi\left(s^t\right) < 0$ for some s^t

(otherwise, the proposition holds trivially). Let $A^*\left(s^t\right)$ and $C^*\left(s^t\right)$ be the optimal consumption and asset profiles, starting from asset holdings of $a^*\left(s^t\right)=\phi\left(s^t\right)$ at history s^t. Suppose either that $u\left(\cdot\right)$ is bounded, or that $c^*\left(s^{t+\tau}\right)\leq\sum_{j=1}^{J}y^j\left(s^{t+\tau}\right)$ for all $s^{t+\tau}\in\mathcal{S}\left(s^t\right)$, and that Assumption 1 holds. Moreover, let $\mathcal{N}\left(s^t\right)$ denote the subtree of events starting from s^t, for which the debt limits are non-binding: Starting from $\mathcal{N}_0\left(s^t\right)\equiv\left\{s^t\right\}$, define

$$\begin{aligned}
\mathcal{N}_\tau\left(s^t\right) &= \left\{s^{t+\tau}\succ s^t : a^*\left(s^{t+\tau}\right)>\phi\left(s^{t+\tau}\right)\text{ and }\sigma\left(s^{t+\tau}\right)\in\mathcal{N}_{\tau-1}\left(s^t\right)\right\},\\
\mathcal{B}_\tau\left(s^t\right) &= \left\{s^{t+\tau}\succ s^t : a^*\left(s^{t+\tau}\right)=\phi\left(s^{t+\tau}\right)\text{ and }\sigma\left(s^{t+\tau}\right)\in\mathcal{N}_{\tau-1}\left(s^t\right)\right\},\\
\mathcal{N}\left(s^t\right) &= \bigcup\nolimits_{\tau=0}^{\infty}\mathcal{N}_\tau\left(s^t\right),\ \mathcal{B}\left(s^t\right)=\bigcup\nolimits_{\tau=0}^{\infty}\mathcal{B}_\tau\left(s^t\right),\\
\mathcal{N}\left(s^{t+\tau};s^t\right) &= \mathcal{N}\left(s^t\right)\cap\mathcal{S}\left(s^{t+\tau}\right)\text{ and }\mathcal{B}\left(s^{t+\tau};s^t\right)=\mathcal{B}\left(s^t\right)\cap\mathcal{S}\left(s^{t+\tau}\right),\text{ for all }s^{t+\tau}\in\mathcal{N}\left(s^t\right).
\end{aligned}$$

$\mathcal{N}_\tau\left(s^t\right)$ denotes the set of histories $s^{t+\tau}$ along which the debt limit was never binding between event s^t and $s^{t+\tau}$, and $\mathcal{N}\left(s^t\right)$ the union of all such sets, including s^t. $\mathcal{B}_\tau\left(s^t\right)$ denotes the set of histories $s^{t+\tau}$ at which the debt limit is binding for the first time after s^t, and $\mathcal{B}\left(s^t\right)$ the union of all such sets. $\mathcal{N}\left(s^{t+\tau};s^t\right)$ defines the 'subtree' of $\mathcal{N}\left(s^t\right)$, which starts at $s^{t+\tau}$, and $\mathcal{B}\left(s^{t+\tau};s^t\right)$ denotes the set of events at which the debt limit is binding for the first time after $s^{t+\tau}$. Finally, define

$$\hat{\phi}\left(s^{t+\tau};s^t\right)=\sum_{s^{t+\tau+k}\in\mathcal{B}(s^{t+\tau};s^t)}\frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)}\phi\left(s^{t+\tau+k}\right)\text{ and }\hat{Y}\left(s^{t+\tau};s^t\right)\equiv\sum_{s^{t+\tau+k}\in\mathcal{N}(s^{t+\tau};s^t)}\frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)}y\left(s^{t+\tau+k}\right)$$

as the present value of the first binding debt limits and the endowments over $\mathcal{N}\left(s^t\right)$.

Lemma 1 establishes that optimal asset plans and consumption allocations are increasing in initial asset holdings, and that $\hat{Y}\left(s^{t+\tau};s^t\right)$ and $\hat{\phi}\left(s^{t+\tau};s^t\right)$ are both finite-valued.

Lemma 1 *(i) $V\left(a,\Phi\left(s^t\right);s^t\right)$ is strictly increasing and the sequences $C^*\left(s^t\right)$ and $A^*\left(s^t\right)$ of optimal consumption and asset holdings are non-decreasing in initial asset holdings a, for $a\geq -y\left(s^t\right)+\sum_{s^{t+1}\succ s^t}q\left(s^{t+1}\right)\phi\left(s^{t+1}\right)$.*

(ii) $\hat{Y}\left(s^{t+\tau};s^t\right)<\infty$ and $\hat{\phi}\left(s^{t+\tau};s^t\right)>-\infty$. Moreover, if $\Phi\left(s^t\right)$ is self-enforcing, $\phi\left(s^{t+\tau}\right)+\hat{Y}\left(s^{t+\tau};s^t\right)>\hat{\phi}\left(s^{t+\tau};s^t\right)$.

Proof. Part (i): That V is strictly increasing, concave and differentiable in a follows immediately from the properties of the consumer problem (5). Since $c\left(s^t\right)$ must be non-negative, and $a\left(s^{t+1}\right)\geq\phi\left(s^{t+1}\right)$, the budget set $\mathcal{C}\left(a,\Phi\left(s^t\right);s^t\right)$ is non-empty, and hence the problem is well-defined, only if $y\left(s^t\right)+a\left(s^t\right)-\sum_{s^{t+1}\succ s^t}q\left(s^{t+1}\right)\phi\left(s^{t+1}\right)\geq 0$.

For given a, the optimal $c^*\left(s^t\right)$ and $\left\{a^*\left(s^{t+1}\right)\right\}$ satisfy the first-order conditions $u'\left(c^*\left(s^t\right)\right)=\lambda\left(s^t\right)$ and $\beta\pi\left(s^{t+1}|s^t\right)V_a\left(a^*\left(s^{t+1}\right),\Phi\left(s^{t+1}\right),s^{t+1}\right)=\lambda\left(s^t\right)q\left(s^{t+1}\right)+\mu\left(s^{t+1}\right)$, where $\lambda\left(s^t\right)$ and

$\mu\left(s^{t+1}\right)$ are the Lagrange multipliers on, respectively, the budget constraint at s^t and the debt limit for s^{t+1}. Re-arranging, we have $c^*\left(s^t\right) = \gamma\left(\lambda\left(s^t\right)\right)$ and $a^*\left(s^{t+1}\right) = \max\left\{\psi\left(\lambda\left(s^t\right); s^{t+1}\right); \phi\left(s^{t+1}\right)\right\}$, , where $\gamma(\cdot) = (u')^{-1}(\cdot)$ and $\psi\left(\cdot; s^{t+1}\right) = (V_a)^{-1}\left(\lambda\left(s^t\right) q\left(s^{t+1}\right)/(\beta\pi\left(s^{t+1}|s^t\right))\right)$. Substituting into the budget constraint, we find $y\left(s^t\right)+a = \gamma\left(\lambda\left(s^t\right)\right)+\sum_{s^{t+1}\succ s^t} q\left(s^{t+1}\right)\max\left\{\psi\left(\lambda\left(s^t\right); s^{t+1}\right); \phi\left(s^{t+1}\right)\right\}$, which has a unique solution $\lambda\left(a; s^t\right)$ (since u is concave in c and V is concave in a, and hence $\gamma(\cdot)$ and $\psi\left(\cdot; s^{t+1}\right)$ are strictly decreasing in λ). Moreover $\lambda\left(\cdot; s^t\right)$ is strictly decreasing in a, and therefore, $c^*\left(s^t\right)$ and $\left\{a^*\left(s^{t+1}\right)\right\}$ are all non-decreasing in a. Applying the same argument recursively to all $s^{t+k} \in \mathcal{S}\left(s^t\right)$ completes the proof.

Part (ii): Summing the agent's budget constraint over $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, we get

$$\begin{aligned}\sum_{s^{t+\tau+k}\in\mathcal{N}(s^{t+\tau};s^t)} \frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)} c^*\left(s^{t+\tau+k}\right) &= a^*\left(s^{t+\tau}\right) + \hat{Y}\left(s^{t+\tau}; s^t\right) - \hat{\phi}\left(s^{t+\tau}; s^t\right) \\ &\quad - \lim_{K\to\infty} \sum_{s^{t+\tau+K}\in\mathcal{N}(s^{t+\tau};s^t)} \frac{p\left(s^{t+\tau+K}\right)}{p\left(s^{t+\tau}\right)} a^*\left(s^{t+\tau+K}\right)\end{aligned}$$

Since the first-order condition holds with equality for all $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, we have

$$\sum_{s^{t+\tau+K}\in\mathcal{N}(s^{t+\tau};s^t)} \frac{p\left(s^{t+\tau+K}\right)}{p\left(s^{t+\tau}\right)} a^*\left(s^{t+\tau+K}\right) = \sum_{s^{t+\tau+K}\in\mathcal{N}(s^{t+\tau};s^t)} \beta^K \frac{\pi\left(s^{t+\tau+K}\right)}{\pi\left(s^{t+\tau}\right)} \frac{u'\left(c^*\left(s^{t+\tau+K}\right)\right)}{u'\left(c^*\left(s^{t+\tau}\right)\right)} a^*\left(s^{t+\tau+K}\right).$$

Taking limits w.r.t. K, this equals zero, due to the agent's transversality condition. Likewise, we have

$$\sum_{s^{t+\tau+k}\in\mathcal{N}(s^{t+\tau};s^t)} \frac{p\left(s^{t+\tau+k}\right)}{p\left(s^{t+\tau}\right)} c^*\left(s^{t+\tau+k}\right) = \sum_{s^{t+\tau+k}\in\mathcal{N}(s^{t+\tau};s^t)} \beta^k \frac{\pi\left(s^{t+\tau+k}\right)}{\pi\left(s^{t+\tau}\right)} \frac{u'\left(c^*\left(s^{t+\tau+k}\right)\right)}{u'\left(c^*\left(s^{t+\tau}\right)\right)} c^*\left(s^{t+\tau+k}\right),$$

which is finite, either because $u'(c)\,c \leq u(c) - u(0) \leq \bar{U}$, if $u(\cdot)$ is bounded, or using the fact that $\sum_{s^{t+\tau}\in\mathcal{S}(s^t)} \beta^\tau \pi\left(s^{t+\tau}|s^t\right)\left(u\left(c^*\left(s^{t+\tau}\right)\right) - u\left(y\left(s^{t+\tau}\right)\right)\right) \geq 0$ and $c^*\left(s^{t+\tau}\right) \leq \sum_{j=1}^J y^j\left(s^{t+\tau}\right)$, along with Assumption 1. But then, it follows immediately that $\hat{Y}\left(s^{t+\tau}; s^t\right) < \infty$ and $\hat{\phi}\left(s^{t+\tau}; s^t\right) > -\infty$.

Finally, notice that if $a\left(s^t\right) = -y\left(s^t\right) + \sum_{s^{t+1}\succ s^t} q\left(s^{t+1}\right)\phi\left(s^{t+1}\right)$, the only feasible allocation without default yields $c\left(s^t\right) = 0$, and $a\left(s^{t+1}\right) = \phi\left(s^{t+1}\right)$ for all $s^{t+1} \succ s^t$. This yields a life-time expected utility of $u(0) + \beta\sum_{s^{t+1}\succ s^t} \pi\left(s^{t+1}|s^t\right) V\left(\phi\left(s^{t+1}\right), \Phi\left(s^{t+1}\right), s^{t+1}\right)$. If instead the agent defaults and sets $d\left(s^{t+1}\right) = 0$ for all $s^{t+1} \succ s^t$, his life-time expected utility is $u\left(y\left(s^t\right)\right) + \beta\sum_{s^{t+1}\succ s^t} \pi\left(s^{t+1}|s^t\right) D\left(0, s^{t+1}\right)$. If $D\left(0, s^{t+1}\right) = V\left(\phi\left(s^{t+1}\right), \Phi\left(s^{t+1}\right), s^{t+1}\right)$ for all s^{t+1}, it follows immediately that default is strictly preferred to no default, implying that $\phi\left(s^t\right) > -y\left(s^t\right) + \sum_{s^{t+1}\succ s^t} q\left(s^{t+1}\right)\phi\left(s^{t+1}\right)$, for all $s^t \in \mathcal{S}$. Summing this inequality over $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, we find $\hat{Y}\left(s^{t+\tau}; s^t\right) + \phi\left(s^{t+\tau}\right) - \hat{\phi}\left(s^{t+\tau}; s^t\right) > 0$. ■

Lemma 2 establishes the existence of a finite-valued sequence of 'auxiliary' debt limits $\tilde{\Phi}\left(s^t\right)$.

Lemma 2 *There exists a finite-valued sequence* $\tilde{\Phi}\left(s^t\right) : \mathcal{S}\left(s^t\right) \to \mathbb{R}$ *which satisfies*

$$\tilde{\phi}\left(s^{t+\tau}; s^t\right) = \begin{cases} \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q\left(s^{t+\tau+1}\right) \min\left\{\phi\left(s^{t+\tau+1}\right), \tilde{\phi}\left(s^{t+\tau+1}; s^t\right)\right\} & \text{if } a^*\left(s^{t+\tau}\right) > \phi\left(s^{t+\tau}\right) \\ \phi\left(s^{t+\tau}\right) \text{ if } a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) \end{cases} \tag{11}$$

for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. *Moreover,* $\lim_{K\to\infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^{t+\tau}; s^t)} p\left(s^{t+\tau+K}\right) \tilde{\phi}\left(s^{t+\tau+K}; s^t\right) = 0$.

Proof. Step 1 establishes the existence of such a solution for $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, together with the limit property. Step 2 extends the construction to $\mathcal{S}\left(s^t\right)$.

Step 1: Let $\Phi^{(0)}$ be defined by $\phi^{(0)}\left(s^{t+\tau}\right) = \hat{\phi}\left(s^{t+\tau}; s^t\right) - Y\left(s^{t+\tau}; s^t\right)$, and define $\Phi^{(K)}\left(s^t\right)$ recursively by $\Phi^{(K)} = \mathcal{T}\Phi^{(K-1)}$, where the operator $\mathcal{T}$ on sequences $B \in \mathbb{R}^{\mathcal{N}(s^t)}$ is defined by

$$(\mathcal{T}B)\left(s^{t+\tau}\right) = \sum_{s^{t+\tau+1} \in \mathcal{N}(s^{t+\tau}; s^t)} q\left(s^{t+\tau+1}\right) \min\left\{\phi\left(s^{t+\tau+1}\right), b\left(s^{t+\tau+1}\right)\right\} + \sum_{s^{t+\tau+1} \in \mathcal{B}(s^{t+\tau}; s^t)} q\left(s^{t+\tau+1}\right) \phi\left(s^{t+\tau+1}\right).$$

Since $\phi\left(s^{t+\tau}\right) \geq \hat{\phi}\left(s^{t+\tau}; s^t\right) - \hat{Y}\left(s^{t+\tau}; s^t\right) = \phi^{(0)}\left(s^{t+\tau}\right)$, we have $\phi^{(1)}\left(s^{t+\tau}\right) \geq \phi^{(0)}\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, and therefore $\Phi^{(1)} \geq \mathcal{T}\Phi^{(0)}$. But $\mathcal{T}$ is a monotone operator, and therefore, $\left\{\Phi^{(K)}\right\}$ is a non-decreasing sequence. Moreover, since $\phi^{(K)}\left(s^{t+\tau}\right) \leq 0$, for all K and $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, $\left\{\Phi^{(K)}\right\}$ must converge to a finite limit $\tilde{\Phi}\left(s^t\right)$ which satisfies (11). The limit property then follows immediately from $0 \geq \tilde{\phi}\left(s^{t+\tau+K}; s^t\right) \geq \phi^{(0)}\left(s^{t+\tau+K}\right) = \hat{\phi}\left(s^{t+\tau+K}; s^t\right) - \hat{Y}\left(s^{t+\tau+K}; s^t\right)$, and the fact that $\lim_{K\to\infty} \sum_{s^{t+\tau+K} \in \mathcal{N}(s^{t+\tau}; s^t)} p\left(s^{t+\tau+K}\right) \left[\hat{\phi}\left(s^{t+\tau+K}; s^t\right) - \hat{Y}\left(s^{t+\tau+K}; s^t\right)\right] = 0$.

Step 2: Define $\mathcal{B}^{(1)}\left(s^t\right) = \mathcal{B}\left(s^t\right)$ and let

$$\mathcal{B}^{(k)}\left(s^t\right) = \bigcup_{\tau=k}^{\infty} \left\{s^{t+\tau} \succ s^t : a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right) \text{ and } \sigma\left(s^{t+\tau}\right) \in \mathcal{N}\left(s^{t+\tau'}\right) \text{ for some } s^{t+\tau'} \in \mathcal{B}^{(k-1)}\left(s^t\right)\right\}.$$

denote the subset of histories in $\mathcal{S}\left(s^t\right)$, at which the debt limit is binding for the k th time after s^t. Since $A^*\left(s^t\right)$ solves the consumer problem starting from s^t with asset position $\phi\left(s^t\right)$, and $a^*\left(s^{t+\tau}\right) = \phi\left(s^{t+\tau}\right)$ for all $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}\left(s^t\right)$, $\left\{a^*\left(s^{t+\tau'}\right)\right\}_{s^{t+\tau'} \in \mathcal{S}(s^{t+\tau})}$ solves the consumer problem starting from any $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}\left(s^t\right)$ with asset position $\phi\left(s^{t+\tau}\right)$. Since s^t was chosen arbitrarily, we can replicate the same arguments as above for all $s^{t+\tau} \in \bigcup_{k=1}^{\infty} \mathcal{B}^{(k)}\left(s^t\right)$ to construct a solution $\tilde{\Phi}\left(s^t\right)$ to (11) for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$. ■

Lemma 3 establishes that (i) $\tilde{\Phi}\left(s^t\right)$ bounds $\Phi\left(s^t\right)$ from below, and (ii) $\tilde{\Phi}\left(s^t\right)$ satisfies the exact roll-over property as a weak inequality. We prove these properties for $\mathcal{N}\left(s^t\right)$ and $\mathcal{B}\left(s^t\right)$; they can immediately be extended to $\mathcal{S}\left(s^t\right)$ using the construction of the previous proof.

Lemma 3 (i) *For all* $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, $\phi\left(s^{t+\tau}\right) \geq \tilde{\phi}\left(s^{t+\tau}; s^t\right) = \hat{\phi}\left(s^{t+\tau}; s^t\right)$ *and* $\tilde{\phi}\left(s^{t+\tau}; s^t\right) = \sum_{s+\tau+1 \succ s+\tau} q\left(s^{t+\tau+1}\right) \tilde{\phi}\left(s^{t+\tau+1}; s^t\right)$.

(ii) *For all* $s^{t+\tau} \in \mathcal{B}\left(s^t\right)$, $\tilde{\phi}\left(s^{t+\tau}; s^t\right) \geq \sum_{s+\tau+1 \succ s+\tau} q\left(s^{t+\tau+1}\right) \tilde{\phi}\left(s^{t+\tau+1}; s^t\right)$.

Proof. Part (i): Suppose to the contrary that $\phi\left(s^{t+\tau}\right) < \tilde{\phi}\left(s^{t+\tau}; s^t\right)$ for some $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$, and let $\left\{\tilde{a}\left(s^{t+\tau+k}\right)\right\}_{s^{t+\tau+k} \in \mathcal{S}(s^t)\setminus\{s^t\}}$ denote the optimal asset profile without default starting from a position of $\phi\left(s^{t+\tau}\right)$ at $s^{t+\tau}$. Consider an agent who defaults and sets

$$\hat{a}\left(s^{t+\tau+k}\right) = \begin{cases} \tilde{a}\left(s^{t+\tau+k}\right) - \min\left\{\phi\left(s^{t+\tau+k}\right), \tilde{\phi}\left(s^{t+\tau+k}; s^t\right)\right\} & \text{for all } s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right) \\ 0 & \text{for all } s^{t+\tau+k} \in \mathcal{B}\left(s^{t+\tau}; s^t\right) \end{cases},$$

which is feasible after a default. Whenever $s^{t+\tau+k} \in \mathcal{B}\left(s^{t+\tau}; s^t\right)$, Lemma 1*(i)* implies that $\tilde{a}\left(s^{t+\tau+k}\right) = \phi\left(s^{t+\tau+k}\right)$, and because of (SE), $V\left(\phi\left(s^{t+\tau+k}\right), \Phi\left(s^{t+\tau+k}\right), s^{t+\tau+k}\right) = D\left(0, s^{t+\tau+k}\right)$, so that the default profile provides the same life-time utility going forward from any $s^{t+\tau+k} \in \mathcal{B}\left(s^{t+\tau}; s^t\right)$ as the non-default profile. Moreover, for $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, the difference in consumption between default and no default $\left\{\Delta c\left(s^{t+\tau+k}\right)\right\}$ is

$$\begin{aligned} \Delta c\left(s^{t+\tau+k}\right) &= -\min\left\{\phi\left(s^{t+\tau+k}\right), \tilde{\phi}\left(s^{t+\tau+k}; s^t\right)\right\} + \\ &\quad \sum_{s^{t+\tau+k+1} \succ s^{t+\tau+k}} q\left(s^{t+\tau+k+1}\right) \min\left\{\phi\left(s^{t+\tau+k+1}\right), \tilde{\phi}\left(s^{t+\tau+k+1}; s^t\right)\right\} \\ &= -\min\left\{\phi\left(s^{t+\tau+k}\right) - \tilde{\phi}\left(s^{t+\tau+k}; s^t\right), 0\right\} \geq 0, \end{aligned}$$

where the inequality is strict at $s^{t+\tau}$. Therefore, consumption is weakly higher after a default for all $s^{t+\tau+k} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$, and strictly higher at $s^{t+\tau}$, implying that default must be optimal. Now, using $\phi\left(s^{t+\tau+1}\right) \geq \tilde{\phi}\left(s^{t+\tau+1}; s^t\right)$ for all $s^{t+\tau+1} \in \mathcal{N}\left(s^{t+\tau}; s^t\right)$ and $\phi\left(s^{t+\tau+1}\right) = \tilde{\phi}\left(s^{t+\tau}; s^{t+\tau}\right)$ for all $s^{t+\tau+1} \in \mathcal{B}\left(s^{t+\tau}; s^t\right)$ in (11) then implies (ER) for all $s^{t+\tau} \in \mathcal{N}\left(s^t\right)$. Solving (11) forward and using the limit property from lemma 2, we find $\tilde{\phi}\left(s^{t+\tau}; s^t\right) = \hat{\phi}\left(s^{t+\tau}; s^t\right)$.

Part (ii): Applying the same arbitrage argument as in part (i) at $s^{t+\tau} \in \mathcal{B}\left(s^t\right)$, we have $\phi\left(s^{t+\tau}\right) \geq \tilde{\phi}\left(s^{t+\tau}; s^{t+\tau}\right)$, and $\tilde{\phi}\left(s^{t+\tau}; s^{t+\tau}\right) = \sum_{s+\tau+1 \succ s+\tau} q\left(s^{t+\tau+1}\right) \tilde{\phi}\left(s^{t+\tau+1}; s^{t+\tau}\right)$. Moreover, by construction, $\tilde{\phi}\left(s^{t+\tau}; s^t\right) = \phi\left(s^{t+\tau}\right)$ and $\tilde{\phi}\left(s^{t+\tau+1}; s^t\right) = \tilde{\phi}\left(s^{t+\tau+1}; s^{t+\tau}\right)$, for all $s^{t+\tau+1} \succ s^{t+\tau}$, from which the result follows immediately. ■

Lemma 4 uses these two properties to show that $\phi\left(s^t\right) = \tilde{\phi}\left(s^t; s^t\right)$, and that $\tilde{\Phi}\left(s^t\right)$ satisfies (ER) with equality, for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$.

Lemma 4 *For all $s^t \in \mathcal{S}$, $\phi\left(s^t\right) = \tilde{\phi}\left(s^t; s^t\right)$, and $\tilde{\Phi}\left(s^t\right)$ satisfies (ER) with equality.*

Proof. Consider the consumer problem with borrowing constraints equal to $\tilde{\Phi}(s^t)$. Since the objective is strictly concave, $\phi(s^{t+\tau}) \geq \tilde{\phi}(s^{t+\tau}; s^t)$ for all $s^{t+\tau} \in \mathcal{S}\left(s^t\right)$, and $\phi(s^{t+\tau}) > \tilde{\phi}(s^{t+\tau}; s^t)$ only if $a^*(s^{t+\tau}) > \phi(s^{t+\tau})$, $\tilde{\Phi}(s^t)$ relaxes only non-binding constraints, and hence $A^*(s^t)$ is also optimal for the relaxed problem with borrowing constraints $\tilde{\Phi}(s^t)$, implying $V(\phi(s^t), \tilde{\Phi}(s^t), s^t) =$

$V(\phi(s^t), \Phi(s^t), s^t)$. Using the self-enforcement hypothesis and the monotonicity of $V(a, \tilde{\Phi}(s^t), s^t)$ in a and $\phi(s^t) \geq \tilde{\phi}(s^t; s^t)$, this implies $D\left(0, s^t\right) = V(\phi(s^t), \Phi(s^t), s^t) = V(\phi(s^t), \tilde{\Phi}(s^t), s^t) \geq V(\tilde{\phi}(s^t; s^t), \tilde{\Phi}(s^t), s^t)$. On the other hand, since $\tilde{\Phi}(s^t)$ satisfies the exact roll-over property as a weak inequality, the same argument as proposition 4 implies that $V(\tilde{\phi}(s^t; s^t), \tilde{\Phi}(s^t), s^t) \geq D\left(0, s^t\right)$, where the inequality is strict whenever $\tilde{\phi}(s^{t+\tau}; s^t) > \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q(s^{t+\tau+1}) \tilde{\phi}(s^{t+\tau+1}; s^t)$ for some $s^{t+\tau} \in \mathcal{S}(s^t)$. Together these inequalities can hold only as equalities, which requires that $\phi(s^t) = \tilde{\phi}(s^t; s^t)$, and $\tilde{\phi}(s^{t+\tau}; s^t) = \sum_{s^{t+\tau+1} \succ s^{t+\tau}} q(s^{t+\tau+1}) \tilde{\phi}(s^{t+\tau+1}; s^t)$ for all $s^{t+\tau} \in \mathcal{S}(s^t)$. ■

To complete the proof that $\Phi\left(s^t\right)$ satisfies (ER), we thus need to show that that $\phi\left(s^{t+1}\right) = \tilde{\phi}\left(s^{t+1}; s^t\right)$ for all $s^{t+1} \in \mathcal{S}\left(s^t\right)$. Whenever $s^{t+1} \in \mathcal{B}_1\left(s^t\right)$, i.e. whenever the debt limit is binding at s^{t+1}, this is true by construction. Our final lemma shows that this is also true whenever the debt limit is not binding.

Lemma 5 *For all $s^{t+1} \in \mathcal{N}_1\left(s^t\right)$, $\phi\left(s^{t+1}\right) = \tilde{\phi}\left(s^{t+1}; s^t\right)$.*

Proof. Since s^t was chosen arbitrarily, applying all the preceding arguments to $s^{t+1} \in \mathcal{N}_1\left(s^t\right)$ implies that $\phi\left(s^{t+1}\right) = \tilde{\phi}\left(s^{t+1}; s^{t+1}\right)$, so it suffices to show that $\tilde{\phi}\left(s^{t+1}; s^t\right) = \tilde{\phi}\left(s^{t+1}; s^{t+1}\right)$. Now, from the monotonicity of $\left\{a^*\left(s^{t+\tau}\right)\right\}$ w.r.t. the initial asset holdings (Lemma 1(i)), it follows that $\mathcal{N}\left(s^{t+1}\right) \subseteq \mathcal{N}\left(s^{t+1}; s^t\right)$ and $\mathcal{B}\left(s^{t+1}\right) \subseteq \mathcal{B}\left(s^{t+1}; s^t\right) \cup \mathcal{N}\left(s^{t+1}; s^t\right)$, i.e. debt limits must be binding for an agent starting from s^{t+1} with assets $\phi\left(s^{t+1}\right)$, whenever they are binding for an agent starting from s^{t+1} with assets $a^*\left(s^{t+1}\right) > \phi\left(s^{t+1}\right)$. But then, by the definition of (11), $\tilde{\phi}\left(s^{t+\tau}; s^{t+1}\right) = \tilde{\phi}\left(s^{t+\tau}; s^t\right)$ for all $s^{t+\tau} \in \mathcal{B}\left(s^{t+1}; s^t\right)$, and both $\tilde{\Phi}\left(s^t\right)$ and $\tilde{\Phi}\left(s^{t+1}\right)$ satisfy (ER) for all $s^{t+\tau} \in \mathcal{N}\left(s^{t+1}; s^t\right)$, from which it follows immediately that $\tilde{\phi}\left(s^{t+\tau}; s^{t+1}\right) = \tilde{\phi}\left(s^{t+\tau}; s^t\right)$ for all $s^{t+\tau} \in \mathcal{N}\left(s^{t+1}; s^t\right)$, and hence $\tilde{\phi}\left(s^{t+1}; s^t\right) = \tilde{\phi}\left(s^{t+1}; s^{t+1}\right)$. ■

9 Appendix C: Extensions of the example of Section 3

In this appendix, we examine three extensions of the example in Section 3. First, we discuss how initial debt limits and initial asset holdings determine the transition to a steady-state equilibrium. Second, we show the existence of non-stationary equilibria, in which the real value of debt collapses over time. These equilibria are the counter-part to the "hyper-inflation" equilibria that exist in the environment with unbacked public debt. Finally, we illustrate how our characterization of self-enforcing debt extends to environments with growth, showing that, with CRRA utility, a stationary equilibrium of our model is characterized by a real interest rate that is equal to the aggregate growth rate.

Transitional dynamics. In the environment with unbacked public debt, it is well known that the transition to steady-state is complete the first time the state switches. Before then, the consumption allocations of each type are determined by initial asset holdings. Here, we show that the same result applies to the economy with self-enforcing private debt, except that consumption allocations in the initial phase are determined by both the debt limits and the initial asset holdings of each type.

We begin by showing that the steady-state allocation $(\overline{c}, \underline{c})$ of Proposition 1 in the main text does not require debt limits to be identical for both types - instead, the same allocations and prices are sustained by any debt limits $\left(-\omega^1, -\omega^2\right)$, such that $\omega^1 + \omega^2 = 2\omega$. To see that the consumption allocations $(\overline{c}, \underline{c})$ and steady-state state prices (q_c, q_{nc}) continue to characterize an equilibrium with self-enforcing debt even when debt limits are asymmetric, consider asset holdings of $a^j\left(s^t\right) = -\omega^j$, if $s_t = \mathsf{s}_j$, and $a^j\left(s^t\right) = \omega^{-j}$, if $s_t \neq \mathsf{s}_j$. These asset holdings clear the market, and yield $c^j\left(s^t\right) = \overline{e} - \omega^j\left(1 - q_{nc}\right) - \omega^{-j} q_c = \overline{e} - q_c\left(\omega^j + \omega^{-j}\right) = \overline{c}$ if $s_t = \mathsf{s}_j$, and $c^j\left(s^t\right) = \underline{e} + \omega^{-j}\left(1 - q_{nc}\right) + \omega^j q_c = \underline{c}$, if $s_t \neq \mathsf{s}_j$. Therefore, the steady-state allocations can be supported by a continuum of different debt limits. In the extreme case, where $\omega^1 = 0$, type 1 never borrows, and type 2 never lends.

Now, suppose that the economy begins at date 0 in state $s_0 = \mathsf{s}_1$, and initial asset holdings are $a^1\left(s_0\right) = a = -a^2\left(s_0\right)$, for some value of a, and steady-state debt limits are $\left(-\omega^1, -\omega^2\right)$, with $\omega^1 + \omega^2 = 2\omega$. We construct equilibria where the consumption allocation is constant and equal to $(\overline{c}^o, \underline{c}^o)$, asset holdings are $a^1\left(s^t\right) = a = -a^2\left(s^t\right)$ as long as $s_t = \mathsf{s}_1$, and it switches to the steady state allocation $(\overline{c}, \underline{c})$ the first time $s_t = \mathsf{s}_2$. Let (q_c^o, q_{nc}^o) and $\left(-\omega^{1o}, -\omega^{2o}\right)$, and denote, respectively, the state-contingent prices and the debt limits of types 1 and 2 in the transitional phase (also assumed to be constant during this phase). $(\overline{c}^o, \underline{c}^o)$ and (q_c^o, q_{nc}^o) satisfy the consumer's budget constraints and first-order conditions:

$$\overline{c}^o = \overline{e} + a\left(1 - q_{nc}^o\right) - q_c^o \omega^2 \text{ and } \underline{c}^o = \underline{e} - a\left(1 - q_{nc}^o\right) + q_c^o \omega^2 \tag{12}$$

$$q_c^o = \beta\alpha \frac{u'\left(\underline{c}\right)}{u'\left(\overline{c}^o\right)} \text{ and } q_{nc}^o = \beta\left(1 - \alpha\right). \tag{13}$$

The initial debt limits $\left(\omega^{1o}, \omega^{2o}\right)$ then satisfy the exact roll-over condition, which requires that $\omega^{jo} = q_{nc}^o \omega^{jo} + q_c^o \omega^j$, or

$$\omega^{jo} = \frac{\beta\alpha}{1 - \beta\left(1 - \alpha\right)} \frac{u'\left(\underline{c}\right)}{u'\left(\overline{c}^o\right)} \omega^j. \tag{14}$$

Substituting the condition for $\overline{c}^o$ into the one for q_c^o and rearranging, we find

$$\overline{c}^o + \beta\alpha \frac{u'\left(\underline{c}\right)}{u'\left(\overline{c}^o\right)} \omega^2 = \overline{e} + a\left(1 - \beta\left(1 - \alpha\right)\right) \tag{15}$$

Since the LHS of (15) is strictly increasing in $\overline{c}^o$, (15) admits a unique solution, from which one can solve for the other variables. We thus have the following characterization of equilibrium transition paths.

Proposition 6 *For given initial asset holdings $a^1(s_0) = a = -a^2(s_0)$ and steady-state debt limits $\left(-\omega^1, -\omega^2\right)$, consider state-contingent prices, debt limits and consumption allocations (q_c^o, q_{nc}^o), $\left(-\omega^{1o}, -\omega^{2o}\right)$, and $(\overline{c}^o, \underline{c}^o)$ for the transitional phase prior to the first time the state switches from s_1 to s_2. This characterizes a competitive equilibrium, if and only if (12), (13), and (14) are satisfied, and initial asset holdings satisfy $a \in \left[-\omega^{1o}, \omega^{2o}\right]$.*

Proof. Conditions (12), (13), and (14), together with $a \in \left[-\omega^{1o}, \omega^{2o}\right]$ and $\overline{c}^o \geq \underline{c}$ are necessary and sufficient conditions for the characterization of a competitive equilibrium of the form that we construct here; the requirement that $a \in \left[-\omega^{1o}, \omega^{2o}\right]$ implies that debt limits for both types are satisfied during the transition phase, while $\overline{c}^o \geq \underline{c}$ implies that the type 2 agent's first-order condition holds as an inequality, when the state changes.

To prove our result, we thus need to show that this last condition is redundant, i.e. that it is always implied by the former. Using (15), one finds that $\overline{c}^o$ is an increasing function of initial asset holdings a. Moreover, rearranging (15) in terms of q_c^o, we have $q_c^o u'\left(\overline{e} + a\left(1 - q_{nc}^o\right) - q_c^o \omega^2\right) = \beta\alpha u'\left(\underline{c}\right)$. When $a = -\omega^{1o}$, the LHS of this expression reduces to $q_c^o u'\left(\overline{e} - q_c^o\left(\omega^1 + \omega^2\right)\right) = \beta\alpha u'\left(\underline{c}\right)$, from which it follows that $q_c^o = q_c$ and $\overline{c}^o = \overline{c}$ when $a = -\omega^{1o}$. It follows that for any $a \geq -\omega^{1o}$, $\overline{c}^o \geq \overline{c} > \underline{c}$, so that the type 2 agent's first-order condition is satisfied. Finally, notice that when $a = \omega^{2o}$, $\overline{c}^o = \overline{e}$ - for any higher initial asset position of type 1 (and lower asset position of type 2), type 2 would have a strict incentive to default, consume his autarky allocation for one period, and then reenter the market purely as a lender. ■

Thus, the amount of consumption smoothing that is feasible during the transition phase is a function of the debt limits allocated to each type, and the initial asset positions. In the special case where type 1's initial asset position is exactly at his debt limit, the economy starts out directly in the steady-state equilibrium. On the other hand, if type 2's initial asset position is at his debt limit, only the autarky allocation is feasible during the transition phase, and risk sharing starts only once there is a switch in states. For any intermediate configuration, the extent of consumption-smoothing in the transition depends on how far each type is from his debt limit - the further type 2 is from his limit, and the closer type 1 is to his, the more consumption smoothing is feasible.

Non-stationary equilibria. We begin by considering non-stationary equilibrium paths in the example of Section 3 of the paper. Let $\mathcal{K}\left(s^t\right)$ denote the number of times the state has switched

from s_1 to s_2 or from s_2 to s_1 along history s^t. We construct equilibria that are characterized by a sequence $\left\{q_{nc}^k, q_c^{k+1}, \overline{c}^k, \underline{c}^k, \omega^k\right\}_{k=0}^{\infty}$, where asset prices are $q\left(s^{t+1}\right) = q_c^k$ if $s_{t+1} \neq s_t$ and $k = \mathcal{K}\left(s^t\right)$ and $q\left(s^{t+1}\right) = q_{nc}^k$ if $s_{t+1} = s_t$ and $k = \mathcal{K}\left(s^t\right)$, consumption allocations are $c^j\left(s^t\right) = \overline{c}^k$ if $s_t = \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$ and $c^j\left(s^t\right) = \underline{c}^k$ if $s_t \neq \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$, asset holdings are $a^j\left(s^t\right) = -\omega^k$ if $s_t = \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$ and $a^j\left(s^t\right) = \omega^k$ if $s_t \neq \mathsf{s}_j$ and $k = \mathcal{K}\left(s^t\right)$, and debt limits are $\phi^j\left(s^t\right) = \omega^k$ if $k = \mathcal{K}\left(s^t\right)$. That is, as in the stationary equilibrium, agents are constrained at low-endowment histories, but the tightness of the constraint changes each time the state switches between s_1 and s_2.

To construct the sequence $\left\{q_{nc}^k, q_c^{k+1}, \overline{c}^k, \underline{c}^k, \omega^k\right\}_{k=0}^{\infty}$, notice that the consumption allocations and prices must satisfy the agents' budget constraint and first-order conditions at high-endowment histories:

$$\overline{c}^k = \overline{e} - \omega^k + q_{nc}^k \omega^k - q_c^{k+1}\omega^{k+1} \tag{16}$$

$$\underline{c}^k = \underline{e} + \omega^k - q_{nc}^k \omega^k + q_c^{k+1}\omega^{k+1} \tag{17}$$

$$q_c^{k+1} u'\left(\overline{c}^k\right) = \beta\alpha u'\left(\underline{c}^{k+1}\right) \tag{18}$$

$$q_{nc}^k = \beta\left(1-\alpha\right) \tag{19}$$

In addition, the sequence of debt limits must satisfy the exact roll-over condition:

$$\omega^k = q_c^{k+1}\omega^{k+1} + q_{nc}^k \omega^k. \tag{20}$$

Substituting (16)-(19) into (20), and then using (20), the dynamics of ω^k are then characterized by the following difference equation:

$$\omega^{k+1}\beta\alpha u'\left(\underline{e} + 2\left(1-\beta\left(1-\alpha\right)\right)\omega^{k+1}\right) - \left(1-\beta\left(1-\alpha\right)\right)\omega^k u'\left(\overline{e} - 2\left(1-\beta\left(1-\alpha\right)\right)\omega^k\right) = 0 \tag{21}$$

This difference equation has two stationary points at ω (the steady state value derived in Proposition 1 in the paper), and the other at zero. Moreover, we can rearrange this difference equation in the form $\omega^k = F\left(\omega^{k+1}\right)$, where the function F is continuous and has the property that if $\omega^{k+1} > \omega$, then $F\left(\omega^{k+1}\right) > \omega^{k+1}$, and if $\omega^{k+1} < \omega$, then $F\left(\omega^{k+1}\right) < \omega^{k+1}$. This in turn implies that for each $\omega^k < \omega$, there exists $\omega^{k+1} < \omega^k$ for which $\omega^k = F\left(\omega^{k+1}\right)$.[14] We thus have the following characterization of non-stationary equilibria:

[14] If F is invertible, then this is the unique equilibrium path starting from any equilibrium value of $\omega^0 \leq \omega$. If F is not invertible, there may be other solutions to (21), some of which satisfy $\omega^{k+1} \geq \omega^k$. A sufficient condition for invertibility is $-u''\left(c\right)c/u'\left(c\right) \leq 1$, for $c \in (0,1)$.

Proposition 7 *For given $\omega^0 \in (0, \omega)$, there exists a decreasing sequence $\{\omega^k\}_{k=0}^{\infty}$ that is recursively defined by (21), and a non-stationary equilibrium of the economy in Section 3, where prices and allocations are given by (16)-(19), for $k = 0, 1, 2, ...$*

Proof. To complete the above argument, we just need to check the agents' first-order conditions for low-endowment periods, which require $q_c^{k+1} u'\left(\underline{c}^k\right) \geq \beta\alpha u'\left(\overline{c}^{k+1}\right)$, or equivalently $u'\left(\overline{c}^k\right)/u'\left(\underline{c}^k\right) \leq u'\left(\underline{c}^{k+1}\right)/u'\left(\overline{c}^{k+1}\right)$. Using the fact that $\omega^k < \omega$ for all k, we have $\overline{c}^k > \overline{c}$ and $\underline{c}^k < \underline{c}^k$ for all k, and therefore $u'\left(\overline{c}^k\right)/u'\left(\underline{c}^k\right) \leq u'\left(\overline{c}\right)/u'\left(\underline{c}\right) < 1$ and $u'\left(\underline{c}^{k+1}\right)/u'\left(\overline{c}^{k+1}\right) \geq u'\left(\underline{c}\right)/u'\left(\overline{c}\right) > 1$, from which the result follows immediately. ■

These non-stationary equilibria are characterized by a self-fulfilling collapse of the value of debt: agents anticipate that debt limits will tighten in the future, which limits the incentives for repayment, and hence tightens current debt limits. These equilibria correspond to the 'hyper-inflationary' equilibria of the economy with unbacked public debt, in which the real value of public debt gradually collapses.

Growing endowments. Consider a variation on the economy of Section 3, where the two types still receive randomly alternating endowments but the aggregate endowments are stochastically growing over time. Uncertainty is represented by the Markov process $h_t = s_t \times z_t \in S \times Z$, where $S = \{\mathsf{s}_1, \mathsf{s}_2\}$ determines the share of aggregate endowments going to each type, and $Z = \{z_1, ..., z_N\}$ determines the growth rate of aggregate endowments. Endowments $y^j\left(h^t\right)$ are thus given by

$$\begin{aligned} y^j\left(h^t\right) &= \overline{e} f\left(z^t\right) \text{ if } s_t = \mathsf{s}_j, \\ y^j\left(h^t\right) &= \underline{e} f\left(z^t\right) \text{ if } s_t \neq \mathsf{s}_j, \end{aligned}$$

with $\overline{e} + \underline{e} = 1$, and aggregate endowments characterized recursively by $f\left(z^t\right) = g\left(z_t\right) \cdot f\left(z^{t-1}\right)$. Transition probabilities are defined by $\pi\left(h_{t+1}|h_t\right) = \Pr\left[s_{t+1}|s_t\right] \cdot \Pr\left[z_t\right]$; that is, aggregate and distributional shocks are independent of each other, and the growth rate of aggregate endowments is i.i.d. over time. The distributional shocks are characterized as before by symmetric transition probabilities $\Pr\left[s_{t+1} = \mathsf{s}_1 | s_t = \mathsf{s}_2\right] = \Pr\left[s_{t+1} = \mathsf{s}_2 | s_t = \mathsf{s}_1\right] = \alpha$. Agents have CRRA utility, $u\left(c\right) = c^{1-\sigma}/\left(1-\sigma\right)$. We further assume that $\beta \sum_{z'} \Pr\left[z'\right] g\left(z'\right)^{1-\sigma} < 1$, so that life-time expected utilities are finite.

We solve this extension of our model for a stationary equilibrium, in which state-prices, and consumption allocations, asset holdings and debt limits (normalized by aggregate endowments) are functions only of the current state h_t. Following the same steps as Alvarez and Jermann, we can re-cast this extension as an economy with constant endowments. In particular, consider an economy

with aggregate endowments normalized to 1 for all z^t, the probability of state z' given by $\hat{\pi}(z') = \Pr[z'] g(z')^{1-\sigma} / \sum_{z'} \Pr[z'] g(z')^{1-\sigma}$, and discount rate given by $\hat{\beta} = \beta \sum_{z'} \Pr[z'] g(z')^{1-\sigma}$.[15] Consider $\left\{\hat{C}, \hat{Y}, \hat{A}, \hat{\Phi}, \hat{Q}\right\}$ such that

$$\hat{c}^j\left(h^t\right) = \frac{c^j\left(h^t\right)}{f\left(z^t\right)}, \quad \hat{y}^j\left(h^t\right) = \frac{y^j\left(h^t\right)}{f\left(z^t\right)}, \quad \hat{a}^j\left(h^t\right) = \frac{a^j\left(h^t\right)}{f\left(z^t\right)}, \quad \hat{\phi}^j\left(h^t\right) = \frac{\phi^j\left(h^t\right)}{f\left(z^t\right)},$$

$$\hat{q}\left(h'|h\right) = q\left(h'|h\right) g\left(z'\right).$$

Then, it is straight-forward to check that $(\hat{C}, \hat{A})$ solve the consumer's problem with the modified allocations and probabilities, if and only if (C, A) solved the original consumer problem. Moreover, these allocations clear the markets if and only if the original allocations do, and given $\hat{Q}$, $\hat{\Phi}$ satisfies (ER) if and only if Φ satisfies (ER) given state-prices Q. The version of our model with i.i.d. shocks to aggregate endowment growth thus maps exactly into the example considered in the paper. The characterization of Proposition 1 (in the paper) then applies to the normalized quantities and prices of the economy with growth.

For the growth version of our model, this implies that $c^j\left(h^t\right) = \overline{c} f\left(z^t\right)$ if $s_t = \mathsf{s}_j$, and $c^j\left(h^t\right) = \underline{c} f\left(z^t\right)$ if $s_t \neq \mathsf{s}_j$, $\phi^j\left(h^t\right) = -\omega f\left(z^t\right)$, and $q(h'|h) = q_c \hat{\pi}(z') / g(z')$ if $s' \neq s$, and $q(h'|h) = q_{nc} \hat{\pi}(z') / g(z')$ if $s' = s$, where $\overline{c}$, $\underline{c}$, ω, q_c, and q_{nc} are defined as in the paper, for a discount rate $\hat{\beta}$. In particular, this implies that the state-prices divided by endowment growth must add up to 1, or that $\left(\sum_{z'} \hat{\pi}(z') / g(z')\right)^{-1} = 1$, i.e. that the risk-free real interest rate is given by the harmonic mean of the real growth rate. Thus, in equilibrium (as in the paper), the requirement that debt limits be self-enforcing ties down the risk-free real interest rate at a level that is close to the expected level of the aggregate growth rate.

[15] In this economy, z^t has no effects on aggregate endowments, so it will not affect real allocations - however, agents still trade in securities that are contingent on z^t. If one extends the analysis to arbitrary one-stage Markov processes for aggregate endowment growth. the same type of normalization leads to state-dependent discount rates (unless $\sigma = 1$), but this has little effect on the economic implications of the model.