The Limits of Ex-Post Implementation

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Abstract: The sensitivity of Bayesian implementation to agents' beliefs about others suggests the use of more robust notions of implementation such as ex-post implementation, which requires that each agent's strategy be optimal for every possible realization of the types of other agents. We show that the only deterministic social choice functions that are ex-post implementable in generic mechanism design frameworks with multi-dimensional signals, interdependent valuations and transferable utilities, are constant functions. In other words, deterministic ex-post implementation requires that the same alternative must be chosen irrespective of agents' signals. The proof shows that ex-post implementability of a non-trivial deterministic social choice function implies that certain rates of information substitution coincide for all agents. This condition amounts to a system of differential equations that are not satisfied by generic valuation functions.

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1 Introduction

Bayesian implementation is frequently criticized because it can be sensitive to the precise information that agents (and the designer) have about the characteristics of other agents. This seems especially important in practice because it is not clear how agents form beliefs about others. Dominant-strategy implementation responds to this criticism by requiring that each agent's strategy be optimal, not only against the actual strategies of other agents, but against all possible strategies of other agents. In particular, dominant strategy implementation requires that each agent's strategy be independent of the *actual* type of other agents, and in this sense is robust to informational errors.

Unfortunately, as Gibbard (1973) and Satterthwaite (1975) have shown, if there are at least three social alternatives and preferences are unrestricted, then only dictatorial choice rules are dominant-strategy implementable. On the other hand, for environments in which preferences are quasi-linear in money and agents preferences are independent of the information held by others, the celebrated Vickrey-Clarke-Groves mechanisms provide dominant-strategy implementation of the efficient choice function.

The assumption of private values is very restrictive: in many interesting situations, each agents' valuation of alternatives depends on information known only by other agents. The literature on implementation with interdependent values has typically maintained quasi-linear utilities as an assumption that is necessary (in view of the Gibbard-Satterthwaite results) and reasonable (when financial stakes are moderate). In such environments, insisting on robustness to the information of others is formalized as *ex-post implementation*, which requires the strategy of each agent to be optimal against the strategies of other agents for *every possible* realization of types (as opposed to Bayesian implementation, which requires that the strategy of each agent be optimal against the strategies of other agents for *the given* distribution of types).[2] Ex-post implementation is weaker than dominant-strategy implementation since it assumes that other agents follow their equilibrium

[2] The notion of ex-post equilibrium corresponds to *uniform equilibrium*, as defined by d'Aspremont and Gerard Varet (1979), and to *uniform incentive compatibility* as defined by Holmstrom and Myerson (1983). The term *ex-post equilibrium* is due to Cremer and McLean (1985).

strategy — but it shares the appealing property that agents need not know the distribution of others' signals in order to find it optimal to follow their equilibrium strategies.

Our main result is a generic impossibility theorem for ex-post implementation of deterministic social choice functions: restricting to environments in which utilities are quasi-linear but interdependent and types (or signals) are multi-dimensional, we show that, for generic valuation functions, the only deterministic social choice rules that are ex-post implementable are *constant* . Our assertion is uniform over deterministic social choice rules, and hence is much stronger than the assertion that, for each given deterministic social choice rule, the set of valuations for which the given rule is not ex-post implementable is generic.

The environments we consider include many familiar and practical social choice problems. For instance, consider the decision about whether to improve a roadway, and how to assign costs. Construction will typically affect firms along the roadway in a number of ways, such as lack of customer access during construction and increased customer access after completion. In particular, signals are multi-dimensional. Moreover, valuations are interdependent, because the estimates of each firm are imperfect (and would be improved by knowing the estimates of each other firm), because of competition between the firms, and because of positive spillovers across firms.

Our analysis proceeds in two steps. The first step shows that if any non-constant determinstic choice function is ex-post implementable then a certain geometric condition on utility functions must be satisfied; the second step shows that this geometric condition is not satisfied for generic utility functions. This is done both for a topological and for a measure-theoretic notion of genericity.

The geometric condition connects the agents' rates of information substitution, which measure how marginal variations in the several dimensions of one agent's signal affect the agents' payoffs. The condition is derived from the *taxation principle* which implies that, in an ex-post incentive compatible mechanism, all agents have the same indifference sets (the sets of states at which the agent is indifferent between two given alternatives). We show that, on these common indifference sets, marginal variations in signals must affect all agents' valuations in the same way. For multi-dimensional signals, the existence of transfers that equate the implied rates of information substitu-

tion amounts to the assertion that valuations satisfy a system of differential equations of a particular kind. We then show that generic valuations do not satisfy any such system of differential equations.

One way to put the present work in perspective is to recall the literature on efficient ex-post implementation. A number of authors have shown that efficient ex-post implementation is possible when signals are one-dimensional and satisfy a single-crossing property (see Cremer and McLean, 1985; Maskin, 1992; Ausubel, 1997; Dasgupta and Maskin, 2000; Jehiel and Moldovanu, 2001; Bergemann and Välimäki, 2002; and Perry and Reny, 2002). Maskin (2003) offers an excellent survey[3]. Postlewaite and McLean (2004) allow for multidimensional signals, but require that agents are "informationally small".

The restriction to one-dimensional signals is essential. It is not a-priori obvious what the analog of the single-crossing property is for settings with multi-dimensional signals, nor whether it would imply efficient implementability. When at least one agent's signal is two-dimensional (and the distribution of signals is independent across agents), Jehiel and Moldovanu (2001) have shown that, for generic valuations, the efficient social choice rule is not Bayesian implementable, and hence *a fortiori* not ex-post implementable. But, the impossibility of implementing the efficient social choice rule does *not* imply the impossibility of implementing other social choice rules. The present paper shows that, no matter what definition of single-crossing one uses, the set of valuations for which non-trivial implementation is possible is non-generic. Thus, the impossibility result of the present paper is much stronger than the impossibility result of Jehiel and Moldovanu. The proof of the present impossibility theorem is much more difficult as well. Jehiel and Moldovanu (2001) show that efficient implementation implies that the preferences of one agent must be aligned with the social preferences; we show that non-constant implementation implies that the preferences of two agents must be aligned with each other. The important difference is that the social preferences are fixed by the valuation functions, whereas the preferences of any pair of agents can be altered by an endogenous transfer.

A second way to put our work in perspective is to recall the literature

[3]The single-crossing property is satisfied for open sets of preferences in the one-dimensional framework studied by these authors. Yet, it is by no means satisfied by all preferences, and there are open sets of valuation functions that do not satisfy the single-crossing property.

on *robust mechanism design*. Wilson (1987) has pointed out that the success of many of the schemes that rely on Bayes-Nash implementation depend on the beliefs of the agents or of the mechanism designer in a sensitive way: if the agents or the designer are mistaken in their beliefs, the actual outcome of a supposedly optimal mechanism may be far from the intended one. To address this problem, it seems natural to require that the designer wants to implement a social choice function that depends only on the payoff-relevant types (the marginal distribution of which is more likely to be known to the designer) but not on the belief-types of the agents. Bergemann and Morris (2005) show that if a social choice function is Bayes-Nash implementable for *every* system of beliefs and higher order beliefs that can be associated with the given payoff-types, then it must be ex-post implementable[4]. Combining their result with ours implies in the present context that the designer can only implement constant choice rules. In particular, our impossibility result draws attention to a potential disadvantage of the "belief free" approach: in a simple example, we show that the designer may prefer a belief-dependent choice function over any belief-independent choice function (which would be trivial by our main result), even if she adopts the worst case scenario about agents' beliefs.

The rest of the paper is organized as follows: In Section 2 we describe the mechanism design problem, we define the ex-post equilibrium concept, and we derive a helpful "taxation principle." In Section 3 we provide a geometric condition on valuations that must hold in order for a non-trivial ex-post implementable and deterministic choice function to exist, and we apply the geometric intuition to a specific example, yielding generic impossibility in that case. In Section 4 we present the various employed notions of genericity, and we derive the impossibility result by showing that the above geometric conditions induce a system of differential equations that has no solution generically. In Section 5 we describe connections to related work, and we discuss our main assumptions and result. In particular, we review several interesting, but non-generic settings where non-trivial implementation is possible. Section 6 gathers several concluding remarks. Proofs are collected in Section 7.

[4]See Dekel et al. (2004) for a critique of the use of Nash equilibria in models without common priors. Their critique is attenuated in a mechanism design setting where the designer can recommend a plan of actions to the agents.

2 The Model

For ease of exposition, we consider a setting with two agents $i \in \mathcal{N} = \{1, 2\}$, who will be affected by a decision between two alternatives $k \in \mathcal{K}$. (Because this "2 by 2" model is embedded in every model with more agents and alternatives, the impossibility result for this special setting immediately extends to the general setting of N agents and K alternatives.)

Agent i's utility $u^i = v^i_k - t^i$ is determined by a quasi-linear utility function, taking into account the chosen alternative k and a monetary payment $t^i \in \mathbb{R}$. Her valuation $v^i_k = v^i_k(s)$ for alternative k depends on the state of the world $s \in S$.

Each agent holds private information $s^i \in S^i$ on the state of the world $s \in S$. The signal s^i results from an exogenous draw. There is no loss of generality in assuming that the agents' joint information $(s^i)_{i\in\mathcal{N}}$ completely determines the state of the world s. We thus identify states of the world with signal combinations: $S = \prod_{i\in\mathcal{N}} S^i$. When we focus on one agent i, we denote the other agent by $-i$ with signal $s^{-i} \in S^{-i}$. We assume $S^i = [0,1]^{d^i}$, and assume v to be a smooth function on S. (Assuming S^i to be the closure of any open connected subset of $\mathbb{R}^{d^i}$ would suffice as well.) We denote by ∇_{s^i} the d^i-dimensional vector of derivatives with respect to s^i, and by ∂_ρ the directional derivative in direction $\rho \in \mathbb{R}^{d^i}$. Two vectors $x, y \in R^d$ are *co-directional* if $x = \lambda y$ for $\lambda \geq 0$.

We consider deterministic choice functions $\psi : S \to \mathcal{K}$, with the property that there are transfers functions $t^i : S \to \mathbb{R}$, such that truth-telling is an **ex-post equilibrium** in the incomplete information game that is induced by the direct revelation mechanism $\left(\psi, (t^i)_{i\in N}\right)$, i.e.

$$v^i_{\psi(s)}(s) - t^i(s) \geq v^i_{\psi(\widetilde{s}^i, s^{-i})}(s) - t^i\left(\widetilde{s}^i, s^{-i}\right) \tag{1}$$

for all $s^i, \widetilde{s}^i \in S^i$ and $s^{-i} \in S^{-i}$, where $s := (s^i, s^{-i})$ [5]. We shall call such ψ **implementable**. We call a choice function ψ **trivial,** if it is constant on the interior $\operatorname{int} S$ of the type space.[6]

[5] Since we exclude random choice rules, a "social choice rule" implicitly stands henceforth for a "deterministic social choice rule".

[6] Restricting attention to the interior of the type space is justified since the interior has full measure. This assumption is necessary since the main geometric argument in the

By requiring optimality of i's truth-telling for every realization of other agents information s^{-i}, equation (1) treats s^{-i} as if it was known to agent i. Her incentive constraint is thus equivalent to a monopolistic screening problem for every s^{-i}. Thus, the central authority can post personalized prices $t_k^i(s^{-i})$ for the various alternatives, and let the individuals choose among them. In equilibrium all agents must agree on a most favorable alternative, yielding:

Lemma 2.1 (Ex-Post Taxation Principle) *(see Chung and Ely, 2003) The choice function ψ is implementable if and only if for all $i \in \mathcal{N}$, $k \in \mathcal{K}$ and $s^{-i} \in S^{-i}$, there are transfers $(t_k^i(s^{-i}))_k \in (\mathbb{R} \cup \{\infty\})^2$ such that:*

$$\psi(s) \in \arg\max_{k \in \mathcal{K}} \left\{ v_k^i(s) - t_k^i\left(s^{-i}\right) \right\}. \tag{2}$$

The proof of our main result, Theorem 4.2 consists of two major steps: Proposition 3.3 in the next Section shows that the existence of a non-trivial ex-post implementable choice function implies a geometric condition on the gradients of the relative valuation functions; Proposition 4.3 in Section 4 shows that this geometric condition cannot be satisfied generically.

3 The Geometry of Ex-Post Implementation

Because agents' incentives are only responsive to differences in payoffs, it is convenient to focus on *relative* valuations μ^i and *relative* transfers τ^i :

$$\mu^i(s) = v_k^i(s) - v_l^i(s); \ \tau^i\left(s^{-i}\right) = t_k^i\left(s^{-i}\right) - t_l^i\left(s^{-i}\right)$$

For technical simplicity, we assume that relative valuations satisfy the mild requirement $\nabla_{s^i}\mu^i(s) \neq 0$ for all $s \in S$.[7]

proof fails on the boundary of the type space. Alternatively, we could have assumed open type spaces to start with.

[7] That is, agent i's relative valuation is everywhere responsive to i's own signal. Theorem 4.2 can be adapted to allow for relative valuations that are not everywhere responsive to own signals — and in particular to allow for interior maxima — but the additional complication makes the argument less transparent without seeming to add any useful insights.

The geometric condition derived in Proposition 3.3 below relies on an argument on the intersection of the closures of the areas in the signal space S where alternatives k and l, respectively, are chosen (in other words, this intersection is the boundary that separates the two areas.)

Definition 3.1 *The indifference set I of a choice function ψ is defined by:*

$$I := \overline{\psi^{-1}\{k\}} \cap \overline{\psi^{-1}\{l\}} \cap \operatorname{int} S$$

For an indifference signal $\widehat{s} \in I$, we define the indifference set with fixed $\widehat{s}^i$ to be

$$I^i(\widehat{s}) := \left\{ s \in I : s^i = \widehat{s}^i \right\}$$

The taxation principle states that, in an incentive compatible mechanism, all agents agree that the chosen alternative is the most favorable one. If relative transfers τ are continuous, this implies that the indifference set of the choice function and the indifference sets of all agents must coincide. The following lemma formalizes this assertion.

Lemma 3.2 *Let (ψ, t) be a non-trivial ex-post incentive compatible mechanism with continuous relative transfers τ^i.*

1. *The indifference set of the choice function ψ coincides with the indifference set of each of the agents, i.e., for every $\widehat{s} \in \operatorname{int} S$ and $i \in \{1, 2\}$, we have*[8]
$$\mu^i(\widehat{s}) - \tau^i\left(\widehat{s}^{-i}\right) = 0 \qquad \Leftrightarrow \qquad \widehat{s} \in I \tag{3}$$

2. *For all $\widehat{s} \in I$, $I^i(\widehat{s})$ coincides with $\{s \in \operatorname{int} S : s^i = \widehat{s}^i, \mu^{-i}(s) = \mu^{-i}(\widehat{s})\}$. $I^i(\widehat{s})$ is a $(d^{-i} - 1)$-dimensional sub-manifold of $\operatorname{int} S$.*

If relative transfers are differentiable, the gradient of an agent's payoff function is perpendicular to her indifference set. Thus, the coincidence of

[8] Continuity of μ^i and τ^i as well as $\nabla_{s^i}\mu^i(s) \neq 0$ are necessary for this result. Whereas the assumptions on μ^i are standard, the assumption on the endogenous function τ^i is only used for this intermediate result. The case of discontinuous τ^i is covered by point two of Proposition 3.3 which does not depend on this result.

the agents' indifference sets as expressed in (3) implies that the gradients of agents' payoff functions must be co-directional on the indifference set:

$$\begin{pmatrix} \nabla_{s^i}\mu^i(s) \\ \nabla_{s^{-i}}\mu^i(s) - \nabla_{s^{-i}}\tau^i(s^{-i}) \end{pmatrix} \text{ and } \begin{pmatrix} \nabla_{s^i}\mu^{-i}(s) - \nabla_{s^i}\tau^{-i}(s^i) \\ \nabla_{s^{-i}}\mu^{-i}(s) \end{pmatrix}$$

$$\text{are co-directional on } I \tag{4}$$

If condition (4) were to fail, there would be a perturbation s_ε of s that makes alternative k favorable for agent i and l more favorable to j, contradicting the taxation principle.

Condition (4) says that the payoff functions of agent i and $-i$ have the same rate of information substitution: the relative effect on payoffs of changing any two dimensions of the signal must coincide for all agents. While condition (4) carries the main geometric intuition, one might not immediately see the considerable restrictions it implies, as the transfer functions τ^i and τ^{-i} are chosen endogenously. The following proposition, which will serve as the basis for the genericity argument in Section 4, shows a condition that follows from (4) and that does not rely on the transfer functions.

Proposition 3.3 *Let (ψ, t) be a non-trivial ex-post incentive compatible mechanism.*

1. *If the relative transfers τ^i are continuous on $\operatorname{int} S^{-i}$ for all $i \in \{1,2\}$ then there are an indifference signal $\widehat{s} \in I$, and a vector $y \in \mathbb{R}^{d^i}$ such that*

$$\nabla_{s^i}\mu^i(s) \text{ and } (\nabla_{s^i}\mu^{-i}(s) - y)$$
$$\text{are co-directional for every } s \in I^i(\widehat{s}). \tag{5}$$

2. *If relative transfers τ^{-i} are discontinuous at a signal profile $\widehat{s}^i \in \operatorname{int} S^i$ for some $i \in \{1,2\}$ then agent i's incentives are locally independent of s^{-i}. That is, there are a vector $y \in \mathbb{R}^{d^i}$ and a non-empty open set $Q \subset S^{-i}$ such that*

$$\nabla_{s^i}\mu^i(\widehat{s}^i, q) \text{ and } y$$
$$\text{are co-directional for every } q \in Q. \tag{6}$$

For differentiable relative transfer functions, a proof for Proposition 3.3 is simple: Condition (5) is the upper half of Condition (4) after setting $y = \nabla_{s^i}\tau^{-i}(s^i)$. The full proof is slightly more complicated because the relative transfer functions are not known to be differentiable, or even continuous.

As an illustration, we apply Proposition 3.3 to a setting with bi-linear valuations and 2-dimensional signals $s^i = (s^i_k, s^i_l) \in [0,1]^2$. In this case, non-trivial implementation implies a simple algebraic condition (easily seen not to hold generically) on the coefficients of the valuation functions. Proposition 7.3 will generalize this example to the class of all polynomials of degree less than a sufficiently large integer.

Example 3.4 *Define valuations v by:*

$$\begin{aligned} v^i_k(s) &= a^i_k s^i_k + b^i_k s^i_k s^{-i}_k = s^i_k\left(a^i_k + b^i_k s^{-i}_k\right) \\ v^i_l(s) &= a^i_l s^i_l + b^i_l s^i_l s^{-i}_l = s^i_l\left(a^i_l + b^i_l s^{-i}_l\right) \end{aligned}$$

where $a^i_k, b^i_k, a^i_l, b^i_l \neq 0$. Thus,

$$\mu^i(s) = a^i_k s^i_k - a^i_l s^i_l + b^i_k s^i_k s^{-i}_k - b^i_l s^i_l s^{-i}_l.$$

For a vector $y = \begin{pmatrix} y_k \\ y_l \end{pmatrix}$, we have

$$\begin{aligned} \nabla_{s^i}\mu^i(s) &= \begin{pmatrix} a^i_k + b^i_k s^{-i}_k \\ -a^i_l - b^i_l s^{-i}_l \end{pmatrix} \\ \left(\nabla_{s^i}\mu^{-i}(s) - y\right) &= \begin{pmatrix} b^{-i}_k s^{-i}_k - y_k \\ -b^{-i}_l s^{-i}_l - y_l \end{pmatrix} \end{aligned}$$

It is readily verified that $b^i_l b^{-i}_k - b^i_k b^{-i}_l = 0$ is necessary for such vectors to remain co-directional when we vary s^{-i}_k and s^{-i}_l (see Appendix for details). It follows from Proposition 3.3 that a non-trivial choice function ψ is implementable only if

$$b^i_l b^{-i}_k - b^i_k b^{-i}_l = 0. \tag{7}$$

The above condition is obviously non-generic: the set of parameters where it is satisfied has zero Lebesgue-measure in the 8-dimensional space of coefficients that parameterize the bi-linear valuations in this example.

4 Generic Impossibility

We now show that the geometric conditions 5 and 6 derived in Proposition 3.3 cannot be generically satisfied.

We use two notions of genericity. The first is topological. If E is a complete metric space, recall that every open subset $U \subset E$ also admits a complete metric. A subset $A \subset U$ is *residual in* U if A contains the countable intersection $\bigcap_{\nu \in \mathbb{N}} A_\nu$ of open and dense sets $A_\nu \subset U$. Residual sets are generally viewed as (topologically) large, and their complements as small. In particular, the Baire Category Theorem guarantees that residual sets of complete metric spaces are dense.

The second notion of genericity is measure-theoretic. Let E be a complete metric topological vector space, U an open subset of E and A a Borel subset of U. We say that A is *finitely shy* in U if there is a finite dimensional subspace $F \subset E$ such that A meets every translate of F in a set of Lebesgue measure 0 (equivalently, if every translate of A meets F in a set of Lebesgue measure 0).[9]. A Borel set $A \subset U$ is *finitely prevalent* in U if the relative complement $U \setminus A$ is finitely shy in U. Hunt et al. (1992) and Anderson and Zame (2001) have argued that finite prevalence, and prevalence, which is a generalization, provide a sensible measure-theoretic notion of "largeness" for infinite-dimensional spaces of parameters. In particular, if $E = \mathbb{R}^n$ then $B = U \setminus A$ is finitely prevalent in U if and only if the Lebesgue measure of A is 0.

In general, these two notions of genericity are different — even in finite dimensional spaces. However, aside from a technical issue on the degree of differentiability required of the relative valuation function under consideration, we show that ex-post implementation is generically impossible in **both** the topological and the measure-theoretic sense.

Definition 4.1 *For each $m \geq 1$, Let $C^m(S, \mathbb{R}^2)$ be the (Banach) space of maps $S \to \mathbb{R}^2$ that admit an m-times continuously differentiable extension*

[9] If F has dimension n, say, any linear isomorphism between F and $\mathbb{R}^n$ induces a measure on F. All such measures are mutually absolutely continuous, and have the same null sets. Hence, it is consistent to abuse terminology by saying that a subset of F — or any translate of F — has Lebesgue measure 0 if it has measure 0 for one — hence all — of these induced measures.

to an open neighborhood of S, equipped with the topology of uniform convergence of maps and m derivatives. Let $\mathcal{H}^m \subset C^m(S, \mathbb{R}^2)$ be the open subset consisting of those pairs of relative valuation functions $(\mu^1, \mu^2) \in C^m(S, \mathbb{R}^2)$ for which the partial gradients $\nabla_{s^i}\mu^i$ do not vanish anywhere on S.

Theorem 4.2 *Assume that the individual signal spaces have dimensions $d^1 \geq 2$ and $d^2 \geq 2$, respectively. Fix an integer $r > \frac{2d^1+1}{d^1-1}$; set $d = d^1 + d^2$ and $p = dr + 2d^1 + 1 - 2d^1 r$.*

1. *There is a residual subset $\mathcal{G}^1 \subset \mathcal{H}^1$ such that for every $(\mu^1, \mu^2) \in \mathcal{G}^1$, only trivial social choice functions are ex-post implementable.*
2. *There is a residual **and** finitely prevalent subset $\mathcal{G}^{p+1} \subset \mathcal{H}^{p+1}$ such that for every $(\mu^1, \mu^2) \in \mathcal{G}^{p+1}$, only trivial choice functions are ex-post implementable.*

To prove the Theorem, fix valuation functions μ^1, μ^2. For each $\widehat{s} \in \operatorname{int} S$ define

$$\tilde{I}^i(\widehat{s}) = \{s \in \operatorname{int} S : s^i = \widehat{s}^i, \mu^{-i}(s) = \mu^{-i}(\widehat{s})\}$$

For mechanisms with continuous relative transfers, we know by assumption and by Lemma 3.2 that $\tilde{I}^i(\widehat{s})$ is a non-trivial manifold of dimension $d^{-i} - 1 \geq 1$. Moreover, for each such mechanism and for each $\widehat{s} \in I$, Lemma 3.2 guarantees that $\tilde{I}^i(\widehat{s}) = I^i(\widehat{s})$. The following Proposition (who also takes care of mechanisms where relative transfers are not necessarily continuous) is enough to complete the proof of the Impossibility Theorem:

Proposition 4.3 *There is a residual set $\mathcal{G}^1 \subset \mathcal{H}^1$ and a residual **and** finitely prevalent subset $\mathcal{G}^{p+1} \subset \mathcal{H}^{p+1}$ such that if $(\mu^1, \mu^2) \in \mathcal{G}^1$ or $(\mu^1, \mu^2) \in \mathcal{G}^{p+1}$ then*

(1) there do not exist $\widehat{s} \in \operatorname{int} S$ and $y \in \mathbb{R}^{d^1}$ such that $\nabla_{s^1}\mu^2(s) - y$ and $\nabla_{s^1}\mu^1(s)$ are co-directional for every $s \in \tilde{I}^1(\widehat{s})$

(2) there do not exist $\widehat{s} \in \operatorname{int} S$ and $y \in \mathbb{R}^{d^2}$ such that $\nabla_{s^2}\mu^1(s) - y$ and $\nabla_{s^2}\mu^2(s)$ are co-directional for every $s \in \tilde{I}^2(\widehat{s})$

(3) *there do not exist* $\widehat{s} \in \operatorname{int} S$, $y \in \mathbb{R}^{d^1}$, *and a non-empty open set* $Q \subset S^2$ *such that* y *and* $\nabla_{s^1}\mu^1(\widehat{s}^1, q)$ *are co-directional for every* $q \in Q$

(4) *there do not exist* $\widehat{s} \in \operatorname{int} S$, $y \in \mathbb{R}^{d^2}$, *and a non-empty open set* $Q \subset S^1$ *such that* y *and* $\nabla_{s^2}\mu^2(\widehat{s}^2, q)$ *are co-directional for every* $q \in Q$

To give some flavor of the argument, fix an indifference signal $\widehat{s} \in I$ and a vector $y \in \mathbb{R}^{d^i}$. If $\nabla_{s^i}\mu^i(s)$ and $\nabla_{s^i}\mu^{-i}(s) - y$ are co-directional for every $s \in I^i(\widehat{s}) = \{s \in S : s^i = \widehat{s}^i, \mu^{-i}(s) = \mu^{-i}(\widehat{s})\}$, then the valuation functions μ^1, μ^2 satisfy a certain set of first-order differential equations. It is not hard to see that generic valuations functions do not satisfy these differential equations. However, this is not enough, because Proposition 4.3 does not say that generic valuation functions fail to satisfy these differential equations for **prescribed** $\widehat{s}$ and y, but rather that generic valuation functions do not satisfy these differential equations for **any** $\widehat{s}$ and y. But, varying $\widehat{s}$ and y does not offer enough degrees of freedom to guarantee that $\nabla_{s^i}\mu^i(s)$ and $\nabla_{s^i}\mu^{-i}(s) - y$ are co-directional at every point of the non-trivial manifold $I^i(\widehat{s})$.

5 Discussion

5.1 Dictatorship

In the private values setting, the Gibbard-Satterthwaite theorem asserts that only dictatorial social choice functions are dominant strategy implementable. It might seem that dictatorial rules should be ex-post implementable in our interdependent valuations setting as well.

Note that "dictatorship" is ambiguous, because the dictator's valuation v^i depends on $-i$'s information s^{-i}. If a social choice rule ψ always selects the alternative for which, given all signals, dictator i has the highest valuation, then $\psi(s)$ depends of course on all signals. Point 1 of Proposition 3.3 shows that this is impossible, because the agents' incentive constraints cannot be simultaneously satisfied.

Secondly, a rule ψ that is dictatorial in the sense that $\psi(s)$ depends only on the dictator's information s^i is generically not implementable either: The relative transfer to the other agent $\tau^{-i}(s^i)$ implied by the taxation principle

has to be discontinuous, and point 2 of Proposition 3.3 shows that, generically, i's incentive constraint cannot be satisfied for all s^{-i}.

Lastly, the mechanism that lets agent i choose the alternative (solely based on i's information) does not induce a choice function according to our terminology since i's choice will depend both on her belief type and on her payoff relevant type s^i. In Example 5.1 below, we show that a designer may prefer this belief-dependent dictatorial choice rule over any ex-post implementable choice rule.

5.2 Efficient Implementation

As we have noted, Jehiel and Moldovanu (2001) show that for generic valuations, efficient Bayes implementation is impossible; hence for generic valuations, efficient ex-post implementation is impossible as well. Our result is stronger because it applies to *all* non-constant social choice rules simultaneously, not just to the efficient rule.

To understand the mathematical relation between the results, assume for simplicity that only agent i holds private information; write $\mu^{\mathcal{N}} = \mu^1 + \mu^2$ and assume $\nabla_{s^i}\mu \neq 0$. Efficient ex-post implementation implies that there is a difference in transfers $\Delta = \tau^i$, such that society is indifferent between the alternatives if and only if this is the case for agent i. Mathematically, this means that the level set $(\mu^i)^{-1}(\Delta)$ must coincide with the indifference set of the efficient choice function $I^{eff} := \left(\mu^{\mathcal{N}}\right)^{-1}(0)$. Hence:

$$\nabla_{s^i}\mu^i(s) \text{ and } \nabla_{s^i}\mu^{\mathcal{N}}(s) \text{ are co-directional for all } s \in I^{eff}. \tag{8}$$

Thus, efficient implementation is only possible if there is a congruence between the private and social rates of information substitution. In contrast, the condition given here for non-trivial implementation requires a congruence of private rates for any two agents i and $-i$. Whereas the social preference is **exogenously** fixed by the agents' valuations, agent $-i$'s preferences depend on the **endogenous** transfer τ^{-i}.

5.3 Max-Min Beliefs and Ex-Post Implementation

Chung and Ely (2004) study a private-values auction where the distribution of payoff-relevant types is known to the designer. They show that a revenue-maximizing designer who adopts a worst-case scenario about the agents' beliefs prefers a dominant-strategy mechanism over any Bayes-Nash implementable scheme. In contrast, the example below shows in our interdependent values framework that the designer may prefer a belief-dependent choice function, even if she adopts the "worst-case" scenario about the agents' beliefs.

Example 5.1 *There are two agents competing for a single indivisible object. Agents have two-dimensional payoff-relevant signals* $s^i = (p^i, c^i) \in [0,1]^2$*, where* (p^i, c^i) *are uniformly and independently distributed on* $[0,1]^2$*. The distributions of* (p^i, c^i) *are known to the designer. The valuation of agent* i *is given by* $v^i(s^i, s^{-i}) = p^i + \alpha c^i c^{-i}$*, where* α *is a small positive number. The good must be allocated to either agent* 1 *or* 2*, and the designer is happy (gets* 1*) whenever the good is allocated to an agent who values the good no less than* 0.5 *and not happy (gets* 0*) otherwise. Proposition 3.3 implies (see also Example 5.2 below) that only trivial choice rules are ex-post implementable. It is readily verified that, as* $\alpha \to 0$*, the designer's expected payoff associated with a trivial choice rule is* 0.5*. Consider now a non-trivial mechanism: the designer lets agent* 1 *decide first whether or not to buy the object at price* 0.5*; if agent* 1 *decides not to buy, the good is allocated to agent* 2*. Because agent* 1*'s choice depends on her belief about* c^2*, this mechanism is not ex-post implementable for any* $\alpha > 0$*. Assuming that the support of agent 1's belief remains bounded, even in the worst scenario about* 1*'s belief on* c^2*, the designer's payoff converges to* $\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = .75$ *as* α *converges to* 0*.*

5.4 The Limits of the Impossibility Result

In this Subsection we show how weakening the assumptions in our impossibility result opens the door to ex-post implementation in a number of interesting cases. We explain the mechanics in terms of our previous structural results (Propositions 3.3 and 4.3).

5.4.1 One strategic agent

Suppose that only agent i has private information, while the designer knows the information of all agents other than i. In this case, Proposition 3.3 is void of content. Let $t_k^i = t_k^i(s^{-i})$ be any transfer to agent i in alternative k (this may be constantly zero). Then the non-trivial social choice function that implements any outcome $\psi(s) \in \arg\max_k \{v_k^i(s) - t_k^i(s^{-i})\}$ for every signal profile $s = (s^i, s^{-i})$ is ex-post implementable.

Even though this seems a trivial point, we note, by contrast, that the *efficient* social choice rule is *not* ex-post implementable in this setting (see Jehiel and Moldovanu, 2001).

5.4.2 Separable Valuations

Suppose valuation functions are separable; i.e., there are functions $f_k^i : S^i \to \mathbb{R}$, $h_k^i : S^{-i} \to \mathbb{R}$, with

$$v_k^i(s) = f_k^i\left(s^i\right) + h_k^i\left(s^{-i}\right).$$

Of course, separable valuation functions are non-generic. Condition 4 requires that

$$\begin{pmatrix} \nabla_{s^i}\left(f_k^i - f_l^i\right)(s^i) \\ \nabla_{s^{-i}}\left(h_k^i - h_l^i\right)(s^{-i}) - \nabla_{s^{-i}}\tau^i\left(s^{-i}\right) \end{pmatrix}$$

is co-directional on I with

$$\begin{pmatrix} \nabla_{s^i}\left(h_k^{-i} - h_l^{-i}\right)(s^i) - \nabla_{s^i}\tau^{-i}\left(s^i\right) \\ \nabla_{s^{-i}}\left(f_k^{-i} - f_l^{-i}\right)(s^{-i}) \end{pmatrix}$$

Note that the upper half of the above expressions is independent of s^{-i}. Hence, the two gradients can be equalized everywhere by setting, for example, $\tau^{-i}(s^i) := \left(h_k^{-i} - h_l^{-i}\right)(s^i) - \left(f_k^i - f_l^i\right)(s^i)$ (Analogously for $\tau^i(s^{-i})$). These transfers implement the choice function $\psi(s) \in \arg\max_k \{\sum_i f_k^i(s^i)\}$. Under several technical conditions, Jehiel *et al.* (2004) have shown (using Roberts' (1979) result about dominant strategy implementation in private values settings) that a choice rule ψ is ex-post implementable only if it is an

affine maximizer , i.e., only if it is of the form

$$\psi(s) \in \arg\max_{k \in \mathcal{K}} \left\{ \sum_{j=1}^{N} \alpha^j f_k^j \left(s^j\right) + \lambda_k \right\} \tag{9}$$

for agent-specific weights $\alpha^j \geq 0$ and alternative-specific weights $\lambda_k \in \mathbb{R}$.[10]

5.4.3 One-object auctions without allocative externalities

Bikhchandani (2004) studies an one-object auction model where agents care only about their *own* allocation. Because agents are indifferent between all alternatives at which they are not winning, valuations are non-generic if there are three agents or if there are two agents and the seller may keep the object. Bikchandani shows that non-trivial ex-post implementation is possible in this important framework.

Example 5.2 *As in Example 5.1, consider two bidders $i \in \{1, 2\}$ competing for one object with valuations $v^i(s^i, s^{-i}) = p^i + c^i c^{-i}$ where $s^i = (p^i, c^i) \in [0, 1]^2$.*

Consider first the setting in which the seller is not allowed to keep the object. The relative valuations are $\mu^i = p^i + c^i c^{-i}$ and $\mu^{-i} = -p^{-i} - c^i c^{-i}$. Assume that (ψ, t) is a non-trivial ex-post incentive compatible mechanism with continuous relative transfers. Condition 5 of Proposition 3.3 requires the existence of an indifference signal $\widehat{s} \in (0, 1)^4$, of a vector $(y_a, y_b)^T$, and of a function $\lambda(c^{-i}) \in \mathbb{R}^+$ such that:

$$\lambda\left(c^{-i}\right) \begin{pmatrix} 1 \\ c^{-i} \end{pmatrix} = \begin{pmatrix} 0 - y_a \\ -c^{-i} - y_b \end{pmatrix}$$

for all c^{-i} in a neighborhood of $\widehat{c}^{-i}$. By the first equation, $\lambda(c^{-i})$ is independent of c^{-i} and equal to $-y_a$. But the second equation, $\lambda(c^{-i})\, c^{-i} = -c^{-i} - y_b$, can be satisfied for a continuum of c^{-i} only if $\lambda(c^{-i}) \equiv -1$. This contradicts

[10] But not every affine maximizer is implementable. Problems arise if the weight α^i of some agent i is zero; see Jehiel *et al.* (2004) for a way around this problem.

the fact that $\lambda(c^{-i}) \in \mathbb{R}^{+}$.[11,12]

Now suppose that the seller may keep the object. Bikchandani shows how to ex-post implement the allocation where buyer $i = 1, 2$ *gets the object if* $p^i - p^{-i} > c^{-i} - c^i c^{-i}$*, and where the good is not sold otherwise. Note that the boundary between the areas where either buyer wins consists of the two lines with* $\{p^i = p^{-i}, c^i = c^{-i} = 0\}$ *and* $\{p^i = p^{-i}, c^i = c^{-i} = 1\}$*. Thus, the indifference set is a manifold of dimension one contained in the boundary of the signal space, and the construction used to prove Proposition 4.3 does not work. Note too that the object is not sold if the buyers have valuations that are close to each other (eg., at* $p^i = p^{-i} = c^i = c^{-i} = 1$*). This must happen precisely in order to avoid a higher-dimensional boundary between the alternatives where the object is sold. It can be shown that Bikchandani's mechanism is ex-post incentive-efficient.*

The transfers used in Bikhchandani's subtle construction closely follow the logic of efficient implementation with interdependent values and one-dimensional signals. This works since in one-object auctions without allocative externalities agent i's multidimensional signal affects her utility in the unique alternative where i wins.

5.4.4 One-dimensional signals

As mentioned in the Introduction, efficient, ex-post implementation is possible if all agents have one-dimensional signals, and if a single-crossing property holds. The single-crossing property is determined by strict inequalities, and it is satisfied for an open set of valuations. The gradients of utility functions are now scalars, and the parallelism condition has no bite. The impossibility result requires that at least two agents have multi-dimensional signals. When only one agent has a multi-dimensional signal, the boundary between areas where different allocations are chosen may have dimension zero, so the door is open to possibility results, as we now illustrate:

[11] Alternatively, a consideration of the cross product $-c^{-i} - y_b + y_a c^{-i} = 0$ yields $y_b = 0$ and $y_a = 1$. This shows that $\nabla_{s^i}\mu^i(s)$ and $(\nabla_{s^i}\mu^{-i}(s) - (1,0)^T)$ are co-linear, but point in opposite directions.

[12] Condition (2) of that Proposition isn't satisfied either. To see this, note that the direction of $\nabla_{s^i}\mu^i(s) = \left(1, c^{-i}\right)^T$ cannot be locally independent of s^{-i}. Thus, non-trivial implementation fails also with discontinuous transfers.

Example 5.3 *There are two agents $i = 1, 2$ and two alternatives k, l. Agent 1 has a one-dimensional signal $s^1 \in [0, 1]$. Agent 2 has a two-dimensional signal, $s^2 = (s^2_k, s^2_l) \in [0, 1]^2$. Assume that the relative valuation μ^1 satisfies the condition: :*

$$\frac{\partial}{\partial s^1}\mu^1(s) > 0 \tag{10}$$

(Note that the set of valuations satisfying this condition is open.) We show how to implement a social choice function ψ that chooses alternative k for high values of s^1 and chooses alternative l for low values of s^1. Set first transfers $t^2(s^1)$ such that

$$\frac{\partial}{\partial s^1}\left(\mu^2(s) - \tau^2(s^1)\right) > 0 \tag{11}$$

and

$$\mu^2(s) - \tau^2(s^1) \text{ takes on values above and below zero on } S \tag{12}$$

Consider now the choice function

$$\psi(s) = \begin{cases} k & \text{if } \mu^2(s) - \tau^2(s^1) \geq 0 \\ l & \text{if } \mu^2(s) - \tau^2(s^1) < 0 \end{cases} \tag{13}$$

By condition (11), for a fixed s^2 there is $\overline{s}^1(s^2)$ such that

$$\psi(s) = \begin{cases} k & \text{if } s^1 \geq \overline{s}^1(s^2) \\ l & \text{if } s^1 < \overline{s}^1(s^2) \end{cases} \tag{14}$$

For agent 1 we apply the standard technique from the literature with one-dimensional signals, and we set transfer $\tau^1(s^2) = \mu^1(\overline{s}^1(s^2))$. Using the monotonicity assumption in equation (10) we get that

$$\psi(s) = \begin{cases} k & \text{if } \mu^1(s) - \tau^1(s^2) \geq 0 \\ l & \text{if } \mu^1(s) - \tau^1(s^2) < 0 \end{cases}. \tag{15}$$

By equations (15) and (13) (ψ, t) is incentive compatible. It is non-trivial by equation (12). Note that for generic μ, the choice function ψ is non-dictatorial.

6 Conclusion

Ex-post implementation (as opposed to Bayesian implementation) is important if we wish to allow for the possibility that the agents or the designer may have insufficient or erroneous information about relevant features of the environment. We have shown that non-trivial ex-post implementation is impossible in generic quasi-linear environments with interdependent preferences and multidimensional signal spaces.

We see a number of directions for future research:

1) Extend the impossibility result to stochastic social choice functions. This is technically demanding, but we expect that a similar impossibility result holds.

2) Identify additional important (non-generic) classes of valuations for which ex-post implementation is possible.

3) In every setting for which Bayesian implementation of some social choice function is possible with respect to some priors but ex-post implementation fails, there will be some "maximal information mechanism" that allows for posterior implementation *a la* Green and Laffont (1987). What are the properties of these mechanisms ?

4) Identify and characterize those situations where a designer who adopts "worst-case" beliefs would choose an ex-post implementable mechanism, and those where he would not. (This exercise will shed some light on the price that one has to pay for employing belief-free mechanisms.)

7 Proofs

7.1 The Geometric Characterization

Proof of Lemma 3.2. **1)** $\mu^i(\widehat{s}) - \tau^i(\widehat{s}^{-i}) = 0$ and $\nabla_{s^i}\mu^i(\widehat{s}) \neq 0$ imply that there are s'^i, s''^i arbitrarily close to $\widehat{s}^i$ such that $\mu^i(s''^i, \widehat{s}^{-i}) - \tau^i(\widehat{s}^{-i}) <$

$0 < \mu^i(s'^i, \widehat{s}^{-i}) - \tau^i(\widehat{s}^{-i})$. Applying the taxation principle to agent i yields $\psi(s'^i, \widehat{s}^{-i}) = k$ and $\psi^1(s''^i, \widehat{s}^{-i}) = l$. Hence $\widehat{s} \in I$.

For the converse, assume that $\mu^i(\widehat{s}) - \tau^i(\widehat{s}^{-i}) > 0$, say. By continuity, we have $\mu^i(s) - \tau^i(s) > 0$, and thus $\psi(s) = k$, for all s in a neighborhood of $\widehat{s}$. Thus, $\widehat{s} \notin I$.

2) $I^i(\widehat{s}) = \{s \in \operatorname{int} S : s^i = \widehat{s}^i, \mu^{-i}(s) = \mu^{-i}(\widehat{s})\}$ is immediate from the above. Since we assumed that $\nabla_{s^{-i}}\mu^{-i}$ is non-vanishing, we can apply the implicit function theorem to conclude that $I^i(\widehat{s})$ is a $d^{-i} - 1$ dimensional manifold ■

To prove Proposition 3.3, we first state a simple Lemma.

Lemma 7.1 *Let ϕ and ξ be smooth functions on an open set $X \subset \mathbb{R}^N$. Assume that there exists $x \in X$ such that $\phi(x) = \xi(x) = 0$, but $\nabla\phi(x)$ and $\nabla\xi(x)$ are not co-directional. Then there exists x' arbitrarily close to x such that $\phi(x') < 0 < \xi(x')$.*

Proof. As $\nabla\phi(x)$ and $\nabla\xi(x)$ are not co-directional, there exists a direction $\rho \in \mathbb{R}^N$ with $\rho \cdot \nabla\phi(x) < 0 < \rho \cdot \nabla\xi(x)$. For $x' = x + \varepsilon\rho$, with $\varepsilon > 0$, we get $\phi(x') < 0 < \xi(x')$, as desired. This argument is illustrated in Figure 1. ■

Proof of Proposition 3.3. Consider an ex-post incentive compatible mechanism (ψ, t) and the associated relative valuations and transfers.

1) If τ is differentiable, the discussion preceding the Proposition together with Lemma 7.1 completes the proof. More generally, we need to deal with two sub-cases:

1.a) The direction of the gradient $\nabla_{s^i}\mu^i(s)$ does not depend on $s \in I^i(\widehat{s})$. Instead of showing that τ^{-i} is differentiable, we directly construct the vector y. Denote the orthogonal complement of $\nabla_{s^i}\mu^i(s)$ by $(\nabla\mu^i)^\perp \subset \mathbb{R}^{d^i}$ and let $\rho \in (\nabla\mu^i)^\perp$. Fix for a moment s^{-i} with $(\widehat{s}^i, s^{-i}) \in I^i(\widehat{s})$. By Lemma 3.2, $\mu^{-i}(\cdot, s^{-i}) - \tau^{-i}(\cdot)$ must equal zero on the sub-manifold $\{s^i : \mu^i(s^i, s^{-i}) = \mu^i(\widehat{s}^i, s^{-i})\}$. Thus, restricted to that manifold, τ^{-i} is differentiable and we have $\partial_\rho\mu^{-i}(\widehat{s}^i, s^{-i}) = \partial_\rho\tau^{-i}(\widehat{s}^i)$. Therefore, $\rho\cdot\nabla_{s^i}\mu^{-i}(\widehat{s}^i, s^{-i}) = \partial_\rho\mu^{-i}(\widehat{s}^i, s^{-i})$ is independent of $(\widehat{s}^i, s^{-i}) \in I^i(\widehat{s})$.Set now $y := \nabla_{s^i}\mu^{-i}(\widehat{s}) + \lambda\nabla_{s^i}\mu^i(\widehat{s})$. By construction, we have $\rho \cdot (\nabla_{s^i}\mu^{-i}(\widehat{s}^i, s^{-i}) - y) = 0$ for $\rho \in$

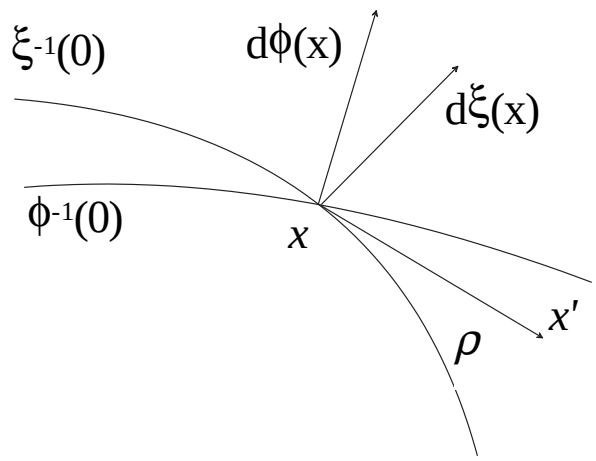

Figure 1: If the gradients of ϕ and ξ are not co-directional at x, the functions disagree at some x', i.e. $\phi(x') < 0 < \xi(x')$.

$(\nabla\mu^i)^\perp$. By choosing λ sufficiently large, $\nabla_{s^i}\mu^i(s)$ and $(\nabla_{s^i}\mu^{-i}(\widehat{s}^i, s^{-i}) - y)$ must be co-directional, and condition (5) is satisfied.

1.b) The direction of the gradient $\nabla_{s^i}\mu^i(s)$ varies in $s \in I^i(\widehat{s})$. In this case we will show that τ^{-i} is differentiable at some $\widetilde{s}^i$ close to $\widehat{s}^i$. As a first step, we show that the directional derivatives $\partial_\rho\tau^{-i}(\widehat{s}^i)$ in directions $\rho \in \nabla_{s^i}\mu^i(\widehat{s}^i, s^{-i})^\perp$ exist. Fix $s \in I^i(\widehat{s})$ and $\rho \in \nabla_{s^i}\mu^i(s)^\perp$ such that there are $\underline{s}, \overline{s} \in I^i(\widehat{s})$ close to s with $\rho\cdot\nabla_{s^i}\mu^i(\overline{s}) > 0 > \rho\cdot\nabla_{s^i}\mu^i(\underline{s})$. By agent i's incentive constraint, we have $\psi\left(\widehat{s}^i + \varepsilon\rho, \overline{s}^{-i}\right) = k$ and $\psi(\widehat{s}^i + \varepsilon\rho, \underline{s}^{-i}) = l$ for small enough $\varepsilon > 0$ (compare this argument to the one for Lemma 7.1). In turn, agent $(-i)$'s incentive constraint implies $\partial_\rho\mu^{-i}(\overline{s}) \geq -\frac{\tau^{-i}\left(\widehat{s}^i+\varepsilon\rho\right)-\tau^{-i}\left(\widehat{s}^i\right)}{\varepsilon} \geq \partial_\rho\mu^{-i}(\underline{s})$. As $\underline{s}^{-i}$ and $\overline{s}^{-i}$ approach s^{-i},and ε approaches zero, this entails $\partial_\rho\tau^{-i}(\widehat{s}^i) = \partial_\rho\mu^{-i}(\widehat{s}^i, s^{-i})$. By assumption, $\nabla_{s^i}\mu^i(\widehat{s}^i, s^{-i})^\perp$ varies (continuously) in s^{-i}. Therefore, $\partial_\rho\tau^{-i}(\widehat{s}^i)$ exists for an open set of directions $\rho \in \Lambda \subset \mathbb{R}^{d^i}$. In order to conclude, we need to show that these directional derivatives are continuous in s^i. Consider $\widetilde{s}^i = \widehat{s}^i + \varepsilon\rho$ for some $\rho \in \Lambda$ and $\varepsilon \in \mathbb{R}$ sufficiently small. By the above argument, there is a neighborhood U of $\widetilde{s}^i$, such that the directional derivatives $\partial_\rho\tau^{-i}(s^i)$ for $\rho \in \Lambda \subset \mathbb{R}^{d^i}$ and $s^i \in U$ exist and are continuous in s^i. Thus, τ^{-i} is differentiable for $s^i \in U$ and, after replacing $\widehat{s}^i$ by $\widetilde{s}^i$, we can conclude. For an intuition consider

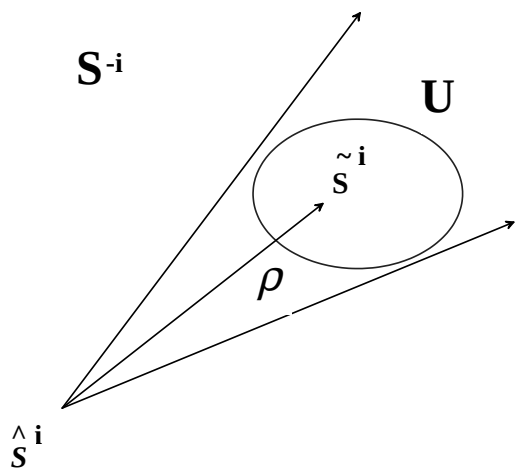

Figure 2: An illustration of $S^i \subset \mathbb{R}^2$: the directional derivatives $\partial_\rho \tau^j(\widehat{s}^i)$ exist for directions ρ inside the cone. As $\nabla_{s^i} \upsilon^i(s^i, s^j)$ is continuous in s^i, these directional derivatives also exist in a neighborhood U of $\widetilde{s}^i$ and are continuous.

Figure 2.

2) Assume now that the relative transfer τ^{-i} is discontinuous at some $\widehat{s}^i \in \operatorname{int} S^i$. We can assume w.l.o.g. that $\tau^{-i}(s^i) \in T^{-i}(s^i) := [\inf_{s^{-i}} \{\mu^{-i}(s^i, s^{-i})\}, \sup_{s^{-i}} \{\mu^{-i}(s^i, s^{-i})\}]$ for all s^i.[13] By assumption, there is a sequence of i's signals $(s_n^i)_{n\in\mathbb{N}}$ such that $\lim_n s_n^i = \widehat{s}^i$ but such that $\tau^{-i}(s_n^i)$ does not converge to $\tau^{-i}(\widehat{s}^i)$. Modulo taking a subsequence, we can assume that $\lim_n \tau^{-i}(s_n^i) = \tau^{-i}(\widehat{s}^i) + \varepsilon$, for $\varepsilon > 0$, say. Consider $\widetilde{S}^{-i} := \{s^{-i} \in S^{-i} : \mu^{-i}(\widehat{s}^i, s^{-i}) \in \left(\tau^{-i}(\widehat{s}^i) + \frac{\varepsilon}{4},\ \tau^{-i}(\widehat{s}^i) + \frac{\varepsilon}{2}\right)\}$.[14] These types $s^{-i} \in \widetilde{S}^{-i}$ of agent $-i$ prefer k when the relative payment is $\tau^{-i}(\widehat{s}^i)$, but prefer l when the relative payment is $\tau^{-i}(\widehat{s}^i) + \varepsilon$. Therefore, $\psi(\widehat{s}^i, s^{-i}) = k$, but $\psi(s_n^i, s^{-i}) = l$ for large enough n.[15] As $\lim_n s_n^{-i} = \widehat{s}^{-i}$, we can apply the taxation principle to agent i to ob-

[13] If $\tau^{-i}(s^i) < \inf_{s^{-i}} \{\upsilon^{-i}(s^i, s^{-i})\}$, say, we have $0 < \upsilon^i(s^i, s^{-i}) + \tau^{-i}(s^i)$ for all s^{-i}, and agent $-i$ will "choose" outcome k, no matter what her signal s^{-i} is. This is still the case, after we change $\tau^{-i}(s^i)$ to $\inf_{s^{-i}} \{\upsilon^{-i}(s^i, s^{-i})\}$.

[14] Note that $S^i(\varepsilon)$ is not empty. Taking $\tau^{-i}(s_n^i) \in T^{-i}(s_n^i)$ to the limit, yields that $\tau^{-i}(\widehat{s}^i) + \varepsilon \in T^{-i}(\widehat{s}^i)$. Together with $\tau^{-i}(\widehat{s}^i) \in T^{-i}(\widehat{s}^i)$, this yields $[\tau^{-i}(\widehat{s}^i), \tau^{-i}(\widehat{s}^i) + \varepsilon] \subset T^{-i}(\widehat{s}^i) = [\inf_{s^{-i}} \{\upsilon^{-i}(s^i, s^{-i})\}, \sup_{s^{-i}} \{\upsilon^{-i}(s^i, s^{-i})\}]$.

[15] Specifically, n such that: $\upsilon^{-i}(s_n^i, s^{-i}) < \upsilon^{-i}(\widehat{s}^i, s^{-i}) + \frac{\varepsilon}{4} \leq \tau^{-i}(\widehat{s}^i) + \frac{3\varepsilon}{4} < \tau^{-i}(s_n^i)$

tain $\mu^i(\widehat{s}^i, s^{-i}) - \tau^i(s^{-i}) = 0$, for all $s^{-i} \in \widetilde{S}^{-i}$ (recall that μ^i is continuous).

We now show that the gradients $\nabla_{s^i}\mu^i(\widehat{s}^i, s^{-i})$ are co-directional for all $s^{-i} \in \widetilde{S}^{-i}$. This proves the desired result since $\widetilde{S}^{-i}$ is open, and since it contains the manifolds $\left\{s = (\widehat{s}^i, s^{-i}) : \mu^{-i}(s) = \tau^{-i}(\widehat{s}^i) + \frac{\varepsilon}{3}\right\}$.

Assume that this is not the case for $s'^{-i}, s''^{-i} \in \widetilde{S}^{-i}(\varepsilon)$. We assume w.l.o.g. that $\mu^{-i}(\widehat{s}^i, s'^{-i}) < \mu^{-i}(\widehat{s}^i, s''^{-i})$. By Lemma 7.1, there is $\widetilde{s}^i$, arbitrarily close to $\widehat{s}^i$, with $\mu^i(\widetilde{s}^i, s''^{-i}) + \tau^i(s''^{-i}) < 0 < \mu^i(\widetilde{s}^i, s'^{-i}) + \tau^i(s'^{-i})$. Thus, $\psi(\widetilde{s}^i, s'^{-i}) = k$ and $\psi(\widetilde{s}^i, s''^{-i}) = l$. However, for $\widetilde{s}^i$ close enough to $\widehat{s}^i$, continuity of μ^i yields $\mu^i(\widetilde{s}^i, s'^{-i}) < \mu^i(\widetilde{s}^i, s''^{-i})$. This yields a contradiction to the monotonicity of ψ and concludes the argument. ■

Proof for Example 3.4. If (ψ, t) is a non-trivial incentive compatible ex-post mechanism with continuous relative transfers τ^i, condition (5) must be satisfied: there is $\widehat{s} \in I$, $\Delta := \mu^i(\widehat{s})$ and $(y_k, y_l)^T \in \mathbb{R}^2$, such that for all $s \in I^i(\widehat{s})$

$$\begin{pmatrix} a_k^i + b_k^i s_k^{-i} \\ -a_l^i - b_l^i s_l^{-i} \end{pmatrix} \text{ and } \begin{pmatrix} b_k^{-i} s_k^{-i} - y_k \\ -b_l^{-i} s_l^{-i} - y_l \end{pmatrix} \text{ are collinear}$$

For this to be true at some s, the cross product of these vectors must vanish, implying the following condition :

$$\left(a_k^i + b_k^i s_k^{-i}\right)\left(-b_l^{-i} s_l^{-i} - y_l\right) - \left(-a_l^i - b_l^i s_l^{-i}\right)\left(b_k^{-i} s_k^{-i} - y_k\right) = 0. \tag{16}$$

We now argue, that the above condition can be satisfied for all s in the set $I^i(\widehat{s})$ only if the coefficients a, b satisfy the algebraic condition (7).

The one-dimensional indifference set $I^i(\widehat{s})$ can be parametrized by $s_k^{-i} = \frac{\Delta}{a_k^{-i} + b_k^{-i}\widehat{s}_k^i} + \frac{a_l^{-i} + b_l^{-i}\widehat{s}_l^i}{a_k^{-i} + b_k^{-i}\widehat{s}_k^i} s_l^{-i}$, where $\Delta = \mu^{-i}(\widehat{s})$. As $a_k^{-i}, b_k^{-i}, a_l^{-i}, b_l^{-i} \neq 0$, we can assume w.l.o.g. that $a_k^{-i} + b_k^{-i}\widehat{s}_k^i \neq 0$, and $a_l^{-i} + b_l^{-i}\widehat{s}_l^i \neq 0$. Substituting for s_k^{-i} in condition (16), we see that this equation can only hold on all of $I^i(\widehat{s})$ if the coefficient of the quadratic term in s_l^{-i} vanishes, i.e. if $\frac{a_l^{-i} + b_l^{-i}\widehat{s}_l^i}{a_k^{-i} + b_k^{-i}\widehat{s}_k^i}\left(-b_k^i b_l^{-i} + b_l^i b_k^{-i}\right) = 0$. This implies condition (7). Finally, for the case of discontinuous transfers τ^i, condition (6) reduces here to $b_k^i = b_l^i = 0$, so that condition (7) is satisfied. ■

7.2 Generic Impossibility

We turn now to the proof of the genericity assertion, Proposition 4.3. Write $d = d^1 + d^2$, and let $\mathcal{P}^{2dr}$ be the space of polynomials on $\mathbb{R}^d$ of degree at most $2dr$. We need the following lemma, whose simple proof is left to the reader.

Lemma 7.2 *Let $s_1, \ldots, s_r$ be distinct points in $\mathbb{R}^d$ and let $\{a_i : 1 \leq i \leq r\}$ and $\{a_{ij} : 1 \leq i \leq r, 1 \leq j \leq d\}$ be families of real numbers. There is a polynomial $P \in \mathcal{P}^{2dr}$ such that for all i, j:*

$$\begin{aligned} P(s_i) &= a_i \\ \frac{\partial P}{\partial x^j}(s_i) &= a_{ij} \end{aligned}$$

Recall that we fixed $r > \frac{2d^1+1}{d^1-1}$ and defined $p = dr + 2d^1 + 1 - 2d^1 r$. For each i, let $\pi^i : \mathbb{R}^d \to \mathbb{R}^{d^i}$ be the projection. We will derive both parts of Proposition 4.3 from the following finite dimensional Proposition.

Proposition 7.3 *Let $\mathcal{L} \subset C^{k+1}(S, \mathbb{R}^2)$ be any finite-dimensional subspace that contains $\mathcal{P}^{2dr} \times \mathcal{P}^{2dr}$, let $\mathcal{M}$ be any translate of $\mathcal{L}$ in $C^{p+1}(S, \mathbb{R}^2)$, and let $\mathcal{M}_0 = \mathcal{H}^{p+1} \cap \mathcal{M}$. There are subsets $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4 \subset \mathcal{M}_0$ that are residual and have full Lebesgue measure in $\mathcal{M}_0$ and enjoy the following properties.*

1. *If $(\mu^1, \mu^2) \in \mathcal{M}_1$ then there do not exist $\widehat{s} \in \operatorname{int} S$ and $y \in \mathbb{R}^{d^1}$ such that $\nabla_{s^1}\mu^1(s)$ and $\nabla_{s^1}\mu^2(s) - y$ are collinear for every $s \in \tilde{I}^1(\widehat{s})$.*
2. *If $(\mu^1, \mu^2) \in \mathcal{M}_2$ then there do not exist $\widehat{s} \in \operatorname{int} S$ and $y \in \mathbb{R}^{d^2}$ such that $\nabla_{s^2}\mu^2(s)$ and $\nabla_{s^2}\mu^1(s) - y$ are collinear for every $s \in \tilde{I}^2(\widehat{s})$.*
3. *If $(\mu^1, \mu^2) \in \mathcal{M}_3$ then there do not exist $\widehat{s} \in \operatorname{int} S$, $y \in \mathbb{R}^{d^1}$, and a non-empty open set $Q \subset S^2$ such that $\nabla_{s^1}\mu^1(\widehat{s}^1, q)$ and y are collinear for every $q \in Q$.*
4. *If $(\mu^1, \mu^2) \in \mathcal{M}_4$ then there do not exist $\widehat{s} \in \operatorname{int} S$, a vector $y \in \mathbb{R}^{d^2}$, and a non-empty open set $Q \subset S^1$ such that $\nabla_{s^2}\mu^2(\widehat{s}^2, q)$ and y are collinear for every $q \in Q$.*

Moreover, the intersection $\mathcal{M}^* = \bigcap \mathcal{M}_i$ is also residual and has full Lebesgue measure in $\mathcal{M}_0$, and every pair $(\mu^1, \mu^2) \in \mathcal{M}^*$ enjoys the four properties above.

Proof. Write $(\operatorname{int} S)^{(r)}$ for the open subset of $(\operatorname{int} S)^r$ consisting of distinct r-tuples. To construct $\mathcal{M}_1$, write

$$V = (\operatorname{int} S)^{(r)} \times \mathbb{R}^r \times \mathbb{R}^{d^1} \times \mathbb{R}^{d^1} \times \mathbb{R}$$

For each $n = 1, \ldots, r$ define

$$\phi_n : \mathcal{M}_0 \times V \to \mathbb{R}^{d^1} \times \mathbb{R}^{d^1} \times \mathbb{R} = \mathbb{R}^{2d^1+1}$$

by

$$\begin{aligned} &\phi_n(\mu^1, \mu^2; s_1, \ldots, s_r; \lambda_1, \ldots, \lambda_r; y, w, c) \\ &\quad = \big(\nabla_{s^1}\mu^1(s_n) - \lambda_n[\nabla_{s^1}\mu^2(s_n) - y],\ \pi 1(s_n) - w, \mu^2(s_n) - c\big) \end{aligned}$$

Finally, write

$$\Phi = (\phi_1, \ldots, \phi_r) : \mathcal{M}_0 \times V \to \mathbb{R}^{(2d^1+1)r}$$

Because the components of Φ are either linear functions or evaluations of first derivatives of $(p+1)$-times continuously differentiable functions, Φ itself is p-times continuously differentiable. Using Lemma 7.2, it is easy to check that for every $(\mu^1, \mu^2; v) \in \mathcal{M}_0 \times V$ the directional derivatives of Φ in directions in $\mathcal{P}^{2dr} \times \mathcal{P}^{2dr} \times V$ span $\mathbb{R}^{(2d^1+1)r}$. In particular, for each $(\mu^1, \mu^2; v) \in \mathcal{M}_0 \times V$ the differential $D\Phi$ is onto. Hence, the transversality theorem (see Mas-Colell (1985) for instance) provides a subset $\mathcal{M}_1 \subset \mathcal{M}_0$ that is residual and of full measure such that, for each $(\mu^1, \mu^2) \in \mathcal{M}_1$, the set

$$J(\mu^1, \mu^2) = \{v \in V : \Phi(\mu^1, \mu^2; v) = 0\}$$

is either empty or is a manifold of dimension

$$dr + r + d^1 + d^1 + 1 - (2d^1 + 1)r = dr + 2d^1 + 1 - 2d^1 r$$

To see that $\mathcal{M}_1$ has the desired property (1), suppose not, so that there exist $\widehat{s} \in \operatorname{int} S$ and $y \in \mathbb{R}^{d_1}$ such that $\nabla_{s^1}\mu^1(s)$ and $\nabla_{s^1}\mu^2(s) - y$ are collinear for each $s \in \tilde{I}^1(\widehat{s})$. If $z_1, \ldots, z_r$ are distinct points of $\tilde{I}^1(\widehat{s})$ then we can find $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ such that

$$\nabla_{s^1}\mu^1(z_i) = \lambda_i \left(\nabla_{s^1}\mu^2(z_i) - y\right)$$

Hence

$$\big(z_1, \ldots, z_r; \lambda_1, \ldots, \lambda_r; y, \pi^1(\widehat{s}), \mu^2(\widehat{s})\big) \in J(\mu^1, \mu^2)$$

Equivalently, $\tilde{I}^1(\widehat{s})^{(r)}$ is a subset of the projection of $J(\mu^1, \mu^2)$ into $(\operatorname{int} S)^{(r)}$. Because $\tilde{I}^1(\widehat{s})$ has dimension $d^2 - 1$ and projection does not raise the dimension of a manifold, it follows that $J(\mu^1, \mu^2)$ must have dimension at least $(d^2 - 1)r$. However, our computation of the dimension of $J(\mu^1, \mu^2)$ implies that

$$dr + 2d^1 + 1 - 2d^1 r \geq (d^2 - 1)r$$

and equivalently, that

$$\frac{2d^1 + 1}{d^1 - 1} \geq r$$

Since this contradicts our choice of r, we conclude that $\mathcal{M}_1$ has the desired property. To construct $\mathcal{M}_2$ we proceed exactly as above, except that we reverse the roles of μ^1, μ^2 . The constructions of $\mathcal{M}_3, \mathcal{M}_4$ use a variant of this same construction. For $\mathcal{M}_3$, write

$$V = \operatorname{int} S^1 \times (\operatorname{int} S^2)^{(r)} \times \mathbb{R}^{d^1} \times \mathbb{R}^r$$

For each $n = 1, \ldots, r$ define

$$\phi_n : \mathcal{M}_0 \times V \to \mathbb{R}^{d^1}$$

by

$$\phi_n(\mu^1, \mu^2; \widehat{s}^1; q_1, \ldots, q_r; y; \lambda_1, \ldots, \lambda_r) = \nabla_{s^1} \mu^1(\widehat{s}^1, q_n) - \lambda_n y$$

Finally, write

$$\Phi = (\phi_1, \ldots, \phi_r) : \mathcal{M}_0 \times V \to \mathbb{R}^{d^1 r}$$

As above, we use the transversality theorem to find a residual set of full measure $\mathcal{M}_3 \subset \mathcal{M}_0$ such that if $(\mu^1, \mu^2) \in \mathcal{M}_3$ then

$$J(\mu^1, \mu^2) = \{v : (\widehat{s}^1, q_1, \ldots, q_r; y; \lambda_1, \ldots, \lambda_r) : \Phi(\mu^1, \mu^2; v) = 0\}$$

is a manifold of dimension

$$2d^1 + (d^2 - d^1)r + r - d^1 r = 2d^1 + r + (d^2 - d^1)r$$

We claim that if $(\mu^1, \mu^2) \in \mathcal{M}_3$ then there does not exist $\widehat{s}^1 \in \operatorname{int} S^1$, $y \in \mathbb{R}^{d^1}$ and an open set $Q \subset \operatorname{int} S^2$ such that $\nabla_{s^1} \mu^1(\widehat{s}^1, q)$ and y are collinear for

each $q \in Q$. To see this, we argue exactly as before: if such existed then the dimension of $J(\mu^1, \mu^2)$ would be at least as large as rd^2, whence

$$2d^1 + r + (d^2 - d^1)r \geq r$$

and

$$r \leq \frac{2d^1}{d^1 - 1} < \frac{2d^1 + 1}{d^1 - 1}$$

This contradicts our choice of r, so we conclude that $\mathcal{M}_3$ has the desired property. To construct $\mathcal{M}_4$ we proceed exactly as above, except that we reverse the roles of μ^1, μ^2. Finally, $\mathcal{M}^*$ is residual and of full measure because it is the intersection of a finite number of sets with these properties. ■

With Proposition 7.3 in hand, we turn to the proof of Proposition 4.3.

Proof for Proposition 4.3. We begin by constructing $\mathcal{G}^{p+1}$ as the intersection of four sets $\mathcal{W}_1, \ldots, \mathcal{W}_4$, corresponding to the various properties, and then use Proposition 7.3 to show that $\mathcal{G}^{p+1}$ has the desired properties.

To construct $\mathcal{W}_1$ and $\mathcal{W}_2$ we proceed in the following way. First choose and fix an increasing sequence of compact sets $L_1, L_2, \ldots,$ whose union is $\operatorname{int} S^1$. For each index m, let $C(m)$ be the set of pairs $(\mu^1, \mu^2) \in \mathcal{H}^{p+1}$ for which there exist $\widehat{s} \in L_m$, $y \in \mathbb{R}^{d^1}$ with $|y| \leq m$, and a subset $Z \subset \tilde{I}^1(\widehat{s})$ such that:

- for every $z \in Z$ there is a $\lambda \in \mathbb{R}$ such that $\mu^1(z) - \lambda[\mu^2(z) - y] = 0$ and $|\lambda| \leq m$
- the projection of Z into some d^2-1-dimensional subspace of $\mathbb{R}^d$ contains a ball of radius at least $1/m$.

It is straightforward to check that each $C(m)$ is a closed subset of $\mathcal{H}^{p+1}$, so the complement $\mathcal{H}^{p+1} \setminus C(m)$ is open. Set

$$\mathcal{W}_1 = \bigcap_{m=1}^{\infty} \left[\mathcal{H}^{p+1} \setminus C(m)\right]$$

We construct $\mathcal{W}_2$ in exactly the same way, except that the roles of μ^1, μ^2 are reversed.

To construct $\mathcal{W}_3$ and $\mathcal{W}_4$, we proceed as follows. For each index m, let $Q(m)$ be the set of pairs $(\mu^1, \mu^2) \in \mathcal{H}^{p+1}$ for which there exist $\widehat{s} \in L_m$, $y \in \mathbb{R}^{d^1}$ with $|y| \leq m$, and a ball $B \subset S^2$ such that:

- for every $b \in B$ there is a $\lambda \in \mathbb{R}$ such that $\mu^1(\widehat{s}^1, b) - \lambda y = 0$ and $|\lambda| \leq m$
- the radius of B is at least $1/m$

It is easy to see that $Q(m)$ is closed, and hence that $\mathcal{H}^{p+1} \setminus Q(m)$ is open. Set

$$\mathcal{W}_3 = \bigcap_{m=1}^{\infty} \left[\mathcal{H}^{p+1} \setminus Q(m)\right]$$

We construct $\mathcal{W}_4$ in exactly the same way, except that the roles of μ^1, μ^2 are reversed.

Set $\mathcal{G}^{p+1} = \bigcap \mathcal{W}_i$. By definition, $\mathcal{G}^{p+1}$is the countable intersection of open sets, and, in particular, is a Borel set. To see that $\mathcal{G}^{p+1}$ is finitely prevalent in $\mathcal{H}^{p+1}$, define $\mathcal{L} = \mathcal{P}^{2dr}$ and let $\mathcal{M}$ be any translate of $\mathcal{L}$. The construction of $\mathcal{G}^{k+1}$ and Proposition 7.3 guarantee that

$$\left(\mathcal{H}^{p+1} \setminus \mathcal{G}^{p+1}\right) \cap \mathcal{M} \subset \mathcal{M}^*$$

Hence Proposition 7.3 implies that $\mathcal{H}^{p+1} \setminus \mathcal{G}^{p+1}$ meets every translate of $\mathcal{L}$ in a set of Lebesgue measure 0. By definition, therefore, $\mathcal{H}^{p+1} \setminus \mathcal{G}^{p+1}$ is finitely shy in $\mathcal{H}^{p+1}$, and $\mathcal{G}^{p+1}$ is finitely prevalent in $\mathcal{H}^{p+1}$.

To see that $\mathcal{G}^{p+1}$ is residual in $\mathcal{H}^{p+1}$, let $F \subset C^{p+1}(S, \mathbb{R}^2)$ be any finite dimensional subspace that contains $\mathcal{P}^{2dr}$. It follows from Proposition 7.3 that $\mathcal{G}^{p+1} \cap F$ has full Lebesgue measure in $\mathcal{H}^{p+1} \cap F$; in particular, $\mathcal{G}^{p+1} \cap F$ is dense in $\mathcal{H}^{p+1} \cap F$. Because $C^{p+1}(S, \mathbb{R}^2)$ is the union of finite dimensional subspaces that contain F_0 , we conclude that $\mathcal{G}^{p+1}$ is dense in $\mathcal{H}^{p+1}$. Because our construction guarantees that $\mathcal{G}^{p+1}$ is the countable intersection of open sets, we conclude that it is residual in $\mathcal{H}^{p+1}$, as desired.

To construct $\mathcal{G}^1$ we proceed in almost the same way, except that we work in $\mathcal{H}^1$ instead of in $\mathcal{H}^{p+1}$. For each index m, let $C(m)$ be the set of pairs $(\mu^1, \mu^2) \in \mathcal{H}^1$ for which there exist $y \in \mathbb{R}^{d^1}$ with $|y| \leq m$ and a subset $Z \subset L_m$ such that

- for every $z \in Z$ there is a $\lambda \in \mathbb{R}$ such that $\mu^1(z) - \lambda[\mu^2(z) - y] = 0$ and $|\lambda| \leq m$
- the projection of Z into some d^2-1-dimensional subspace of $\mathbb{R}^d$ contains a ball of radius at least $1/m$.

It is straightforward to check that each $C(m)$ is a closed subset of $\mathcal{H}^1$, so the complement $\mathcal{H}^1 \setminus C(m)$ is open. Set

$$\mathcal{V}_1 = \bigcap_{m=1}^{\infty} \left[\mathcal{H}^1 \setminus C(m)\right]$$

We construct $\mathcal{V}_2$ in exactly the same way, except that the roles of μ^1, μ^2 are reversed. For each index m, let $Q(m)$ be the set of pairs $(\mu^1, \mu^2) \in \mathcal{H}^1$ for which there exist $\widehat{s}^1 \in L_m$, $y \in \mathbb{R}^{d^1}$ with $|y| \leq m$, and a ball $B \subset S^2$ such that

- for every $b \in B$ there is a $\lambda \in \mathbb{R}$ such that $\mu^1(\widehat{s}^1, b) - \lambda y = 0$ and $|\lambda| \leq m$
- the radius of B is at least $1/m$.It is easy to see that $Q(m)$ is closed, and hence that $\mathcal{H}^1 \setminus Q(m)$ is open.

Set

$$\mathcal{V}_3 = \bigcap_{m=1}^{\infty} \left[\mathcal{H}^1 \setminus Q(m)\right]$$

We construct $\mathcal{V}_4$ in exactly the same way, except that the roles of μ^1, μ^2 are reversed. Now set $\mathcal{G}^1 = \bigcap \mathcal{V}_i$. By definition, $\mathcal{G}^1$ is the countable intersection of open sets. In order to show that it is residual in $\mathcal{H}^1$, we need only show it is dense. To this end, view $C^{p+1}(S, \mathbb{R}^2)$ as a subset of $C^1(S, \mathbb{R}^2)$, and note that $\mathcal{G}^{p+1} \subset \mathcal{G}^1$ and $\mathcal{H}^{p+1} \subset \mathcal{H}^1$. Malgrange (1966) shows that $C^{p+1}(S, \mathbb{R}^2)$ is dense in $C^1(S, \mathbb{R}^2)$. Because $\mathcal{H}^{p+1}$ is open in $C^{p+1}(S, \mathbb{R}^2)$, it follows that $\mathcal{H}^{p+1}$ is dense in $\mathcal{H}^1$. Our above construction shows that $\mathcal{G}^{p+1}$ is dense in $\mathcal{H}^{p+1}$, and hence in $\mathcal{H}^1$. Because $\mathcal{G}^{p+1} \subset \mathcal{G}^1$ it follows that $\mathcal{G}^1$ is dense in $\mathcal{H}^1$. By construction, $\mathcal{G}^1$ is the countable intersection of open sets, so that, as asserted, it is residual in $\mathcal{H}^1$. ■

Remark 7.4 *The relevant property required of the space of polynomials of degree at most* $2dr$ *is embodied in Lemma 7.2: given* r *distinct points, we can find polynomials whose values and first partials can be specified arbitrarily at those points. Any other space with this property would do as well. Note, however, that the space of separable relative valuation functions does* ***not*** *have this property: If* μ *is a separable relative valuation function and the first* d^1 *coordinates of* s_1 *and* s_2 *coincide then*

$$\frac{\partial \mu}{\partial x^i}(s_1) = \frac{\partial \mu}{\partial x^i}(s_2)$$

for $1 \leq i \leq d^1$.

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