Endogenous Monitoring*

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December 2, 2004

Abstract

In the standard model of dynamic interaction, players are assumed to observe public signals according to some exogenous distributions for free. We deviate from this assumption in two directions to model monitoring structure in a more realistic way. We assume that signals are *private* rather than *public* and that each player needs to actively monitor the other player with some costs. In each stage, a player decides whether to monitor the other player with some costs in addition to which action to take. We first provide a class of strategies which approximate efficiency and examine its interesting properties, among them are (1) each player monitors the other player randomly like "random auditing" to reduce monitoring costs and (2) *players cheat and monitor at the same time* in their cooperative phase. In particular, this implies that *cheating may happen (randomly) during collusion for the sake of efficiency.*

Then we discuss multi-task partnership games with endogenous monitoring, where two players play H games (tasks) instead of one. The additional twist is that we allow each player to choose freely which tasks to monitor. Our main result is that, *how large the monitoring cost per task is,* the efficient outcome can be approximated as players become patient *when there are enough many tasks.* This result suggests that the size of a partnership may tend to be large when monitoring is not free.

*We would like to thank the audience at 2004 SED conference at Florence. Obara gratefully aknowledges support from NSF Grant #0417833.

1 Introduction

In any long term relationship, the nature of available information is critical for successful cooperation or collusion. For example, any kind of trigger strategy based on retaliation can be useful only when there exists a reliable signal which helps a player to detect the other players' cheating. Thus many researchers have been led to investigate what effects different information structure may have on long term relationships. Many works has shown that information structure indeed has a significant impact on the nature of long term relationships. One of the most famous such examples is a paper by Green and Porter [10], which showed that price war is necessary for collusion with imperfect monitoring unlike perfect monitoring case.

In this paper, we also propose to study an information structure which has not been explored thoroughly before in the context of long term relationships, namely, *endogenous (costly) monitoring*. The standard assumption is that players receive public signals according to some exogenous distributions for free. There are at least two important aspects of real information processing which are missing from this picture. First, signals are often *private* rather than *public*. It is rarely the case that every player knows what the other players know. Second, the process of acquiring information and the costs of monitoring are not present. We explicitly assume that players can obtain reasonably accurate private signals by paying some costs in addition to free private/public signals available. Players play a given stage game and, at the same time, also decide whether to monitor the other players with some costs or not. We also assume that monitoring reveals enough (imperfect) information about the other player's monitoring activity as well.

We first provide a class of strategies which approximate efficiency and describe some of its interesting properties. First, we find that a player monitors the other player randomly like "random auditing" to reduce monitoring costs. If no monitoring is done, proper incentive for cooperation is not provided. On the other hand, if monitoring occurs every period, the efficiency is not obtained due to monitoring costs. Thus *monitoring needs to be done randomly to approximate the efficient outcomes.* Second, *players cheat and monitor at the same time* in their cooperative phase. In particular, this implies that *cheating may happen (randomly) during collusion for efficiency reason.*[1] Note that, as observed by Ellison [6] and many others,

[1] This can also happen for a model where cheating provides better information about the other player's cheating (Kandori and Obara [11]). Our extra contribution here is to identify a more natural setting where such mixing behavior is optimal. (it's less clear

standard theory predicts cheating does not occur during cooperative phase of collusion, but cheating does happen all the time. Third, it is crucial for our construction that the gain from cheating is larger than the monitoring cost. In another word, we need that the *monitoring cost is "financed" by a deviation gain on the spot* when they deviate and monitor at the same time. Finally, our equilibrium is robust in the sense that it is independent of exogenous information structure. What information is available to players besides explicit monitoring is totally irrelevant to our construction.

Then we discuss multi-task partnership games with endogenous monitoring, where two players play H games (tasks) instead of one. Each player can monitor the other player with some costs as before. The additional twist is that we allow each player to choose freely which tasks to monitor. Our main result is that, *how large the monitoring cost per task is,* the efficient outcome can be approximated *when there is a large enough number of tasks* as players become patient. Our result is likely to hold for very general class of stage games although we model each task as a prisoners' dilemma game for the sake of simple exposition. This result suggests that the size of a partnership tends to be large when active monitoring is important.

In equilibrium, players randomize between "cooperating for all the tasks" and "deviating for all the tasks and monitoring *some randomly selected subset of tasks*". Remember that one of the important conditions in the basic model is that the monitoring cost can be financed by deviation gain. This condition turns out to be more easily satisfied with many tasks because the players need to monitor only a certain number of tasks independent of H while they deviate for all H tasks. This equilibrium is based on the idea of Ely and Välimäki [8] or Kandori and Obara [11], but more complicated than the examples in those papers.

This multi-task model can be also interpreted as a model of *multimarket contact* if we regard the two players as two big firms competing in many separate markets. Edwards [5] argued that bigger conglomerates may have a better ability to sustain implicit collusion between them. His claim has been supported by many empirical works. Our result with endogenous monitoring may provide a theoretical foundation for this claim.[2]

when [11]'s assumption is reasonable). This insight relies on a separation of *cheating action* and *monitoring action*, which just happens to be one action in [11]. Note also that our reasoning is a bit more twisted than [11]. We first argue that players need to randomly monitor the other players in the cooperative phase, then go on to show that it is useful to finance the cost of monitoring by cheating, thus obtaining random cheating behavior in the cooperative phase.

[2]Matsushima [12] also demonstrated that multimarket contact may help to sustain

We describe the model and some preliminary results in the next section. In Section 3, we introduce an equilibrium which we call "two state machine equilibrium" and discuss its interesting properties. Section 4 is devoted to multi-task partnership games. The last section discusses related literature.

2 The Model and Some Preliminary Results

2.1 The Model

Let us suppose that there are two players $i = 1, 2$. An action of player i is

$$a_i = (e_i, m_i) \in A_i \equiv E_i \times \{M, N\}.$$

The first element e_i is the strategy of the stage game in the usual sense, and we call it "*effort*" to avoid confusion. The second element m_i represents the monitoring activity. $m_i = M$ represents "to monitor", while N represents "not to monitor". We assume that the monitoring activity entails cost $K > 0$ and perfectly reveals both (1) the rival's action and (2) monitoring activity. Both assumptions are basically made for the sake of simplicity. We can allow almost perfect private monitoring of actions, for which our results holds approximately. (2) is even less crucial than (1). Indeed we only need that *monitoring activity reveals enough information about the rival's monitoring activity* as we will argue.

The stage game payoff is

$$g_i(a) = \begin{cases} u_i(e_1, e_2) - K & \text{if } m_i = M \\ u_i(e_1, e_2) & \text{if } m_i = N \end{cases}$$

Let $\omega_i = (y_i, \widehat{a}_j) \in \Omega_i \equiv Y_i \times \{A_j \cup \{0\}\}$ be player i's signal, where $\widehat{a}_j = a_j$ with probability one if monitoring action $m_i = M$ is taken, and otherwise $\widehat{a}_j = 0$ with probability one. Free signals y_i can be either pubic or private. Player i's strategy is a mapping from all t-period private histories $h^i_t = ((a_{i,1}, \omega_{i,1}), ..., (a_{i,t-1}, \omega_{i,t-1}))$ to A_i. We assume that players' payoffs are given by average discounted stage game payoffs, and employ sequential equilibrium as the equilibrium concept.

Remark

- Note that monitoring is in general essential for sustaining any level of cooperation. An extreme example would be a stage game for which

implicit collusion using a model with imperfect public monitoring.

y_i is almost uninformative. For such games, you can only support a repetition of the stage game Nash equilibrium without monitoring. On the other hand, if players monitor the other players every period, this game is reduced to one of the games with *perfect monitoring*, thus every payoff profile (minus monitoring cost) can be supported by Folk Theorem [9].[3] However, it may not be wise to monitor every period because it is costly. This suggests that we need to find more creative way of monitoring than such crude one to approximate fully efficient outcomes.

2.2 Kandori and Obara (2003)

In this subsection, we briefly describe some of the results from Kandori and Obara [11], which are relevant to this project. In [11], we proposed an equilibrium called "two state machine" for repeated games with two players. It consists of two states; "Reward state" R and "Punishment state" P. In the beginning of the game, players are at R. They play a behavior strategy α^R in R and move to P with different probabilities according to different realizations of action-signal pair. Similarly, they play a behavior strategy α^P in P and move back to R in a similar way. The trick is to choose transition probabilities so that players have the incentive to choose α^R and α^P in respective states.

We found a necessary and sufficient condition for such machine to be an equilibrium. It consists of a few lines of linear (in)equalities;

(LI) For $i, j = 1, 2$, there exist $x_i^R : \Omega_i \times A_j^R \to [0, \infty)$ and $x_i^P : \Omega_i \times A_j^P \to [0, \infty)$ such that

$$\forall a_i \in A_i^* \quad V_i^R = g_i(a_i, \alpha_j^R) - E[x_i^R(\omega_j, a_j)|a_i, \alpha_j^R] \tag{1}$$

$$\forall a_i \notin A_i^* \quad V_i^R \geqq g_i(a_i, \alpha_j^R) - E[x_i^R(\omega_j, a_j)|a_i, \alpha_j^R] \tag{2}$$

$$\forall a_i \in A_i^* \quad V_i^P = g_i(a_i, \alpha_j^P) + E[x_i^P(\omega_j, a_j)|a_i, \alpha_j^P], \tag{3}$$

$$\forall a_i \notin A_i^* \quad V_i^P \geqq g_i(a_i, \alpha_j^p) + E[x_i^P(\omega_j, a_j)|a_i, \alpha_j^P], \text{ and} \tag{4}$$

$$V_i^R > V_i^P. \tag{5}$$

where A_i^* is the union of the support of α_i^R and α_i^P.

[3] However, if monitoring is almost perfect but not perfect, some information remain private through the game. Many simple equilibria based on perfect information cease to be an equilibrium even with such slight perturbation of information structure. Our equilibrium is robust to any such perturbation of information structure.

Here we give only a brief intuition as the space is limited. You can regard $x_i^R(\omega_j, a_j)$ as punishments and $x_i^P(\omega_j, a_j)$ as rewards for player i. (1) and (2) implies that, if player j is in state R (hence playing α_j^R), player i's total payoff is V_i^R independent of the action in A_i^* player i choose, and it is lower than V_i^R if player i plays anything else. Similarly, (3) and (4) implies that, if player j is in state P, player i's total payoff is V_i^P independent of the action in A_i^*, and it is lower than V_i^P otherwise. In conclusion, these inequalities guarantee that player i is always indifferent among any action in A_i^* given that player j is playing this two state machine. This in turn implies that this two state machine is indeed a best response to itself. Note that, since a player does not need any information about the other player's state, this two state machine equilibrium is an equilibrium even though many information is not public.

Condition (LI) reveals that there is a certain restriction on α^R and α^P that can be used for a two state machine equilibrium:

Proposition 1 *The (potentially mixed) actions used in a two-state machine equilibrium α_i^R and α_i^P, and their support A_i^* must satisfy the **separation condition***

$$\min_{a_i \in A_i^*} g_i(a_i, \alpha_j^R) > \max_{a_i \in A_i} g_i(a_i, \alpha_j^P) \tag{6}$$

Proof. Condition (1) and the non-negativity of x_i^R implies $g_i(a_i, \alpha_j^R) \geq V^R$ for all $a_i \in A_i^*$. In contrast, (3), (4), and the non-negativity of x_i^P shows $V^P \geq g_i(a_i, \alpha_j^P)$ for all $a_i \in A_i$. Then we can obtain $\min_{a_i \in A_i^*} g_i(a_i, \alpha_j^R) \geq V^R > V_i^P > \max_{a_i \in A_i} g_i(a_i, \alpha_j^P)$. ■

The separation condition is necessary for a two-state machine equilibrium, but it is also sufficient under "good observability". "Good observability" roughly means that, for any j's action $a_j \in A_j$, there exists an action-signal pair (ω_i, a_i) for $i \neq j$ such that ω_i is very unlikely to be observed when a_i is chosen but a_j is NOT chosen (that is, $\frac{\Pr ob(\omega_i | (a_i, a_j))}{\Pr ob(\omega_i | (a_i, a_j'))}$ is very large for any $a_j' \neq a_j$).

Proposition 2 *The **separation condition** is sufficient for existence of two state machine equilibrium which uses α_i^R,α_i^P and achieves $V_i^R = \min_{a_i \in A_i^*} g_i(a_i, \alpha_j^R)$, $i = 1, 2$ under good observability.*

Proof. See Kandori and Obara [11]. ■

3 Endogenous Monitoring

Now we apply the above result to repeated games with endogenous monitoring to obtain our first result. The key is to treat a pair (e_i, m_i) as an action a_i in (LI).

Theorem 3 *Let e^* be an efficient profile and suppose that the following* ***separation conditions*** *hold for $i = 1, 2$, $j \neq i$, with some "punishing actions" e_i^P;*

$$\max_{e_i} u_i(e_i, e_j^*) - K \geq u_i(e^*), \tag{7}$$

$$u_i(e_i^P, e_j^*) \geq u_i(e^*), \text{ and} \tag{8}$$

$$u_i(e^*) > \max_{e_i} u_i(e_i, e_j^P). \tag{9}$$

Then, $(u_1(e^*), u_2(e^*))$ is approximately attained as the discount factor tends to unity.

Proof. See Appendix. ∎

The proof is a relatively straightforward adaptation of the above results from [11] To provide a flavor of the result, below we describe an explicit example of equilibria we use in the proof.

Suppose that the stage game is given by Prisoners' Dilemma game, whose payoff

$$u_i(e_i) = E[\pi_i(e_i, y_i)|e_i]$$

is represented by the following payoff table:

	C	D
C	$1, 1$	$-l, 1+d$
D	$1+d, -l$	$0, 0$

(10)

where $d, l > 0$ (D is a dominant strategy) and $2 > 1 + d - l$ ((C, C) is efficient). We interpret C as e_i^* and D as e_i^P in terms of the above sufficient condition.

We consider the following two state machine; (1) In state R, player i chooses (C, N) with probability $(1 - \varepsilon)$ and (D, M) with probability ε. He moves to P only when D is observed, with probability one for (D, N) and probability $\rho^R \in [0, 1]$ for (D, M). (2) In state P, player i chooses (D, M), and moves back to R with probability ρ_C^P when C is observed and with probability ρ_D^P when (D,M) is observed (stay in P when (D,N) is observed).

The parameters $\left(\varepsilon,\rho^R,\rho_C^P,\rho_D^P\right)$ are chosen so that the following inequalities are satisfied for some $V^R > V^P$

$$
\begin{aligned}
V^R &= (1-\delta)\,u\left(C,\alpha_\varepsilon^R\right)+\delta V^R && (11)\\
V^R &= (1-\delta)\left(u\left(D,\alpha_\varepsilon^R\right)-K\right)+\delta\left\{\left(1-\varepsilon\rho^R\right)V^R+\varepsilon\rho^R V^P\right\} && (12)\\
V^R &\geq (1-\delta)\,u\left(D,\alpha_\varepsilon^R\right)+\delta\left\{(1-\varepsilon)\,V^R+\varepsilon V^P\right\} && (13)\\
V^P &= (1-\delta)\,(-l)+\delta\left\{\varepsilon\rho_C^P V^R+\left(1-\varepsilon\rho_C^P\right)V^P\right\} && (14)\\
V^P &= (1-\delta)\,(-K)+\delta\left\{\varepsilon\rho_D^P V^R+\left(1-\varepsilon\rho_D^P\right)V^P\right\} && (15)\\
V^P &\geq \delta V^P && (16)
\end{aligned}
$$

where the incentive constraints for (C,M) are omitted as they are trivial. This system of inequalities basically corresponds to (LI).[4]

When $d \geq K$ ((7)), we can find a solution for (11), (12), (14), and (15) for any small ε as $\delta \to 1$ and (13) is automatically satisfied for large δ. It is clear that this two state machine is a sequential equilibrium with such parameters (Ely and Välimäki [8], [11]). Moreover, the equilibrium payoff is $V^R = u\left(C,\alpha_\varepsilon^R\right)$ from (11), which converges to 1 as $\varepsilon \to 0$.

3.1 Examples

We examine when the separation conditions (7), (8) and (9) are satisfied through some examples.

Example 1: **Cournot Competition**.

Consider a standard symmetric cournot competition model with two firms (for example, with an inverse demand function $D_i(q,\varepsilon)$ with $\frac{\partial D_i(q)}{\partial q_l}, \frac{\partial D_i(q)}{\partial q_j} <$ 0 and a smooth convex cost function $C_i(q_i)$). We assume strategic substitution. Let q^{CN} and q^* be the unique Cournot-Nash equilibrium and the most collusive symmetric output profile respectively. Let $q_i^D = \arg\max_{q_i} \pi\left(q_i, q_j^*\right) = D_i\left(q_i,q_j^*\right)q_i - C_i(q_i)$. We can check separability condition by setting $e^* = q^*$, $e_i^P = q_i^{CN}$, and $e_i^D = q_i^D$. (7) is satisfied if monitoring cost is

[4] For example, (13) can be transformed to

$$V^R \geq u\left(D,\alpha_\varepsilon^R\right)-\frac{\varepsilon\delta}{(1-\delta)}\left(V^R-V^P\right)$$

where $\frac{\delta}{(1-\delta)}\left(V^R-V^P\right)$ corresponds to $x\left((D,N),(D,M)\right)$.

small enough because. (9) is satisfied as the right hand side is at most the Cournot-Nash equilibrium payoff and $\pi_i(q^*) > \pi_i\left(q^{CN}\right)$. (8) holds because

$$0 \leq \pi_i\left(q^{CN}\right) - \pi_i\left(q_i^*, q_j^{CN}\right) \leq \pi_i(q_i^{CN}, e_j^*) - \pi_i(q^*)$$

where strategic substitution is used in deriving the second inequality.

Example 2: **Bertrand Competition** (**Secret Price Cutting)**

Consider a simple symmetric Bertrand model with linear demand functions $D_i(p) = \alpha - \beta p_i + \gamma p_j + \varepsilon_i$ and linear cost functions $C(q_i) = cq_i$ $(\alpha > c > 0, \beta > \gamma > 0)$. The unique Nash equilibrium is given by $p^{BN} = \left(\frac{\alpha+\beta c}{2\beta-\gamma}, \frac{\alpha+\beta c}{2\beta-\gamma}\right)$ and the efficient price profile is $p^* = \left(\frac{\alpha+(\beta-\gamma)c}{2(\beta-\gamma)}, \frac{\alpha+(\beta-\gamma)c}{2(\beta-\gamma)}\right)$. (7) is clearly satisfied if the monitoring cost is small. It is easy to verify that the set of p_i which satisfies $\pi_i\left(p_i, p_j^*\right) \geq \pi_i(p^*)$ (for (8)) is an interval $\left[\underline{p}, p^*\right]$. So we can find the punishment action e_i^P to satisfy (8) and (9) if and only if $\underline{p} = p^* - \frac{\gamma(p^*-c)}{\beta}$ satisfies $\pi_i(p^*) > \max_{p_i} \pi_i\left(p_i, \underline{p}\right)$. Whether this condition is satisfied or not depends on parameters. For example, $(\alpha, \beta, \gamma) = (1, 2, 1)$ satisfies this condition.

Note that efficiency can be achieved without monitoring if the joint distribution of private signals (quantities) satisfies a certain type of conditional independency as shown in Matsushima [13]. Our contribution lies in the case where such restriction is not satisfied.

Remark.

- Note that in the price competition case efficiency is achieved if the joint distribution of quantities satisfies a certain condition (a weaker version of conditional independence: see Matsushima's paper about Secret Price Cuts). Our contribution lies in the case where such restriction is not satisfied.

3.2 Imperfect Monitoring of Monitoring

As we mentioned, perfect monitoring of monitoring is not necessary. Suppose that each player observes the rival's effort level perfectly, but receives an imperfect signal z_i which only depends on the rival's monitoring activity. As long as there is some information contained in z_i, we can pick a function $z_i \to f(z_i) \in [0, 1]$ such that $0 \leq E[f(z_i)|m_j = M] < E[f(z_i)|m_j = N] \leq 1$.[5] Let $E[f_i(z_i)|m_j = N] = \eta_N$, $E[f(z_i)|m_j = M] = \eta_M$. We can modify the above two state machine as follows. In R, players move to P with

[5] We assume symmetry here for the sake of simplicity.

probability $\rho^R f(z_i)$ after D is observed. In P, players move back to R with probability $\rho^P_D (1 - f(z_i))$ when D is observed.

Now we need to replace (12), (13), (15), and (16) with the following four inequalities;

$$
\begin{aligned}
V^R &= (1-\delta)\left(u\left(D, \alpha^R_\varepsilon\right) - K\right) + \delta\left\{\left(1 - \varepsilon\rho^R\eta_M\right)V^R + \varepsilon\rho^R\eta_M V^P\right\} && (17)\\
V^R &\geq (1-\delta)\,u\left(D, \alpha^R_\varepsilon\right) + \delta\left\{\left(1 - \varepsilon\rho^R\eta_N\right)V^R + \varepsilon\rho^R\eta_N V^P\right\} && (18)\\
V^P &= (1-\delta)(-K) + \delta\left\{\varepsilon\rho^P_D(1-\eta_M)V^R + \left(1 - \varepsilon\rho^P_D(1-\eta_M)\right)V^P\right\} && (19)\\
V^P &\geq \delta\left\{\varepsilon\rho^P_D(1-\eta_N)V^R + \left(1 - \varepsilon\rho^P_D(1-\eta_N)\right)V^P\right\} && (20)
\end{aligned}
$$

We can show that (18) and (20) are automatically satisfied if $\frac{\eta_N}{\eta_M}$ is large enough, that is, each player's monitoring is enough informative about the rival's monitoring activity ($\frac{\eta_N}{\eta_M} = \infty$ when monitoring is perfectly observed). Again, when $d \geq K$, we can find a solution for (11), (13), (17), and (19) for any small ε as $\delta \to 1$, thus obtaining efficiency.[6]

4 Multi-task Partnership Games

In this section, we study multi-task partnership game with endogenous monitoring, where two players play many games (tasks) instead of one. In the previous model, there are only two options for players; "monitor" or "not to monitor". Here we assume that each player can also decide which task to monitor. Note that the level of cooperation they can achieve may be affected in a nontrivial way by the number of the tasks in which they are involved. There are two opposing effects from increasing the number of the tasks. First, if there are more tasks, the stake for the partnership is larger, which might facilitate more cooperation. On the other hand, monitoring gets more difficult and costly as the number of different tasks is increasing. We are interested in whether more dependence (more tasks) leads to more cooperation or vice versa. Below we illustrate that the efficient outcome is asymptotically achieved (so the first effect dominates the second) as the number of the tasks increases. Moreover, this result holds *however large the monitoring cost per task is.* This suggests that the "size" of a partnership tends to be large when it is difficult to obtain useful (public) information about the partner's behavior without active monitoring.

[6] Note that almost perfect monitoring of monitoring is not necessary for our efficiency result, while (almost) perfect monitoring of the effort level is crucial for the (almost) efficiency result.

Let $h = 1, ..., H$ be an index for different tasks. We assume that each task corresponds to the above Prisoner's dilemma game for simplicity, but it can be generalized to a very broad class of games. Player i's effort vector is given by $e_i = (e_i^1, e_i^2, ..., e_i^H)$. The vector of outcomes observed by player i is $y_i = (y_i^1, ..., y_i^H)$. The signal for each task depend on the effort profile in that task only, independently over the tasks. We assume that, if player i monitors task h, i observes (1) the rival's effort in task h and (2) whether the rival monitors task h. As we claimed before, these assumptions are basically made for simplicity. Formally, player i's signal is represented as $\omega_i = (y_i, \widehat{a}_j)$, where $\widehat{a}_j = \left\{ \widehat{a}_j^1, ..., \widehat{a}_j^H \right\}, \widehat{a}_j^h \in A_j \cup \{0\}$. It is assumed that the cost per task is K.

First, let's try to apply the previous theorem directly to the current situation as a benchmark. If the counterparts of the separation conditions (7)-(9), namely

$$\begin{aligned} H(1+d) - KH &\geq H \\ H(1+d) &\geq H \\ H &> 0 \end{aligned} \tag{21}$$

are satisfied, the efficient payoff is approximately achieved by a strategy which *cheats & monitor in all the tasks simultaneously* with a small probability. The last two conditions are by definition satisfied. The first ("financing") condition is equivalent to

$$d \geq K \tag{22}$$

Note that this is equivalent to the condition with one task case and independent of the number of the tasks. Thus there is no advantage or disadvantage of having many tasks with such a strategy profile.

Nonetheless, we can approximate the efficient outcome by employing a slightly different strategy which *cheats in all the tasks and monitor some randomly selected subset of tasks simultaneously* with a small probability. Moreover we can do so *how large the monitoring cost per task is, as long as there are enough tasks*. Notice the subtlety involved in our construction of equilibrium two state machine. Suppose that players are playing some two state machine and indifferent between the full cooperation and cheating & monitoring in all the tasks as in the above benchmark case. Then one of the optimal strategy for each player is to cheat & monitor in all the tasks after every history. However, this implies that each player's discounted average payoff cannot exceed $H(1+d) - KH$, which is far below the efficient payoff

H if $K > d$. Therefore it should be suboptimal to cheat and monitor in all the tasks to approximate the efficient outcome. On the other hand, every task needs to be monitored with some probability. No task without monitoring contributes to the efficiency for obvious reason. This observation compels us to construct a two state machine where players monitor in some randomly selected subset of tasks simultaneously with a small probability.

Note that the strategy we propose is much more complicated than a simple two state machine in Ely and Välimäki [8] or examples in Kandori and Obara [11] in the sense that playing certain actions are not optimal. It is not a trivial matter to incorporate such strict incentive compatibilities while explicitly constructing two state machine[7], and that is indeed the main technical contribution of the following theorem.

Theorem 4 *For any level of monitoring cost K, there exists $\underline{H}$ such that, for any $H \geq \underline{H}$, the efficient outcome (H, H) is approximately attained as the discount factor tends to unity.*

Proof. See Appendix. ■

4.1 Generalization

While we used a rather special class of game; Prisoners' Dilemma, our efficiency result survives in much more general settings. The above result can be generalized in two directions. First, the stage game can be any two person normal form game as long as appropriate **separation conditions** are satisfied. Second, the cost of monitoring can be represented by a convex functions $C_i(h).i = 1, 2$ rather than linear functions. Let HG be a multi-task partnership game where each task of the H tasks corresponds to normal form game G. We can obtain the following more general theorem.

Theorem 5 [8] *Let G be any two person normal form game and e^* be an efficient profile of G. Suppose that the following **separation conditions** hold for $i = 1, 2$, $j \neq i$, with some "punishing actions" e_1^P, e_2^P;*

$$d_i = \max_{e_i} u_i(e_i, e_j^*) - u_i(e^*) > 0, \text{ and}$$

[7] Although such strict incentive compatibility can be easily incorporated into two state machines as is clear from (LI) or a general formulation by Ely, Horner, Olszewski [7], there are not many explicit examples. Matsushima [13] is one example of explicit two state machines for which strict incentive compatibility plays an important role.

[8] Note that this result is already very close to Nash reversion Folk theorem as e^* does not need to be an efficient action profile.

$$u_i(e^*) > \max_{e_i} u_i(e_i, e_j^P).$$

Suppose also that the cost functions for monitoring satisfies $hd_i - C_i'(h) > 0, i = 1, 2$ *as* $h \to \infty$.[9] *Then there exists* $\underline{H}$ *such that, for any* $H \geq \underline{H}$, *the efficient out come* $((Hu_1(e^*), Hu_2(e^*)))$ *for* HG *is approximately attained as the discount factor tends to unity.*

This result implies that efficiency is obtained for the Bertrand stage game in Example 2 without any restriction on the parameters as long as there are many markets in which the two firms compete.

4.2 Multimarket contact

The multi-task model can be reinterpreted as a model of *multimarket contact* if we regard the two players as two big firms competing in H separate markets. Edwards [5] argued that bigger conglomerates may have a better ability to sustain implicit collusion between them. His claim has been supported by many empirical works. Our result with endogenous monitoring may provide a theoretical foundation for this claim once a theorem for general two person stage games is developed more.

5 Related Literature

We discuss related literature briefly. The most closely related works are Ben-Porath and Kahneman [4] and Miyagawa, Miyahara, and Sekiguchi [14]. The first paper proves a folk theorem for general discounted repeated games with communication when perfect monitoring is possible with some costs. While our result is not ready to be applied to games with more than two players, we do not allow communication among players. The second paper proves a folk theorem for a class of stage games (without communication) when players are patient enough. There are two main differences between our work and their work. First, they focus on the limit case where monitoring cost is almost negligible, while we deal with the fixed level of monitoring cost. Second, they assume that there is no signal about monitoring activity. On the other hand, we assume that players can observe some informative signal about the other players' monitoring activity while monitoring himself.

Our work is also related to works on repeated games with private monitoring. The basic idea behind two state machine was first proposed by

[9] Note that this condition is trivially satisfied when $C(H)$ is linear, the case treated above.

Piccione [16] in the context of repeated prisoners' dilemma with imperfect private monitoring. It was further simplified and developed to the current style of two state machine by Ely and Välimäki [8], Obara [15], and Kandori and Obara [11]. In particular, our Proposition in Section 2 is borrowed directly from [11].

As for applications and extensions, we have already mentioned Matsushima [12] for multimarket contact. There are a few attempts to model collusion with private information such as Aoyagi [1], Athey, Bagwell and Sanchirico [3] and Athey and Bagwell [2]. But they are different from our model in that they use communication extensively.

Appendix.

Proof of Proposition 3.

Proof. The proof is a direct application of our Linear Inequality characterization of two-state machines. We use R and P to denote "reward" and "punishment" states. Let A_i^Z be the support of α_i^Z for $Z = R, P$. Set $A_i^R = \left\{(e_i^*, N), \left(e_i^D, M\right)\right\}$, and $\alpha_i^R(e_i^*, N) = 1 - \varepsilon$, $\alpha_i^R(e_i^D, M) = \varepsilon$, where $e_i^D \in \arg\max_{e_i} u_i(e_i, e_j^*)$ (i.e. cheat and monitor at the same time with a small probability). Similarly, set $A_i^P = \left\{\left(e_i^P, N\right), \left(e_i^D, M\right)\right\}$, and $\alpha_i^P(e_i^P, N) = 1 - \varepsilon$, $\alpha_i^P(e_i^D, M) = \varepsilon$. Then these α_i^R and $\alpha_i^P, i = 1, 2$ satisfy separation condition. Since the quality of monitoring is very good when monitoring action is used $\left(\frac{\Pr ob\left(\omega_i = (y_i, a_j) | \left(\left(e_i^D, M\right), a_j\right)\right)}{\Pr ob\left(\omega_i = (y_i, a_j) | \left(\left(e_i^D, M\right), a_j'\right)\right)} = \infty \text{ for any } \left(a_j', a_j\right) \text{ such that } a_j' \neq a_j\right)$, the separation condition is also sufficient. Hence there exists a two state machine equilibrium with such α_i^R and α_i^P to achieve $V_i^R = \min_{a_i \in A_i^*} g_i(a_i, \alpha_j^R), i = 1, 2, j \neq i$, which is approximately $u_i(e^*)$ if we choose ε very small.

The following argument, which follows a general line of proof of Proposition 2, shows how to find $x(\cdot)$ to satisfy (LI) for such α_i^R and $\alpha_i^P, i = 1, 2$. For simplicity, suppose that (7) and (8) holds with strict inequality. First, we can set $x_i^R((y_i, \widehat{a}_i), a_j) = 0$ when either player j is choosing the efficient action $\left(a_j = (e_j^*, N)\right)$ or he deviates and monitors at the same time, but finds that player i is not cheating $\left((\widehat{a}_i, a_j) = \left((e_j^*, N), (e_j^D, M),\right)\right)$. This implies that V_i^R is $g_i\left(e_i^*, e_j^R\right)$, which is approximately efficient. For $(\widehat{a}_i, a_j) = \left((e_i^D, M), (e_j^D, M)\right)$ or $\left((e_i^P, N), (e_j^D, M)\right)$, we can choose $x_i^R(\cdot)\,(\geqq 0)$ so that

$$\forall a_i \in A_i^* \quad V_i^R = g_i\left(e_i^*, e_j^R\right) = g_i(a_i, \alpha_j^R) - E[x_i^R(\omega_j, a_j) | a_i, \alpha_j^R]$$

This can be done because the conditions (7) and (8) hold strictly and ε is small. Finally, for any other $\widehat{a}_i$ with $a_j = (e_i^D, M)$, $x_i^R(\cdot)$ can be set very large so that (2) is satisfied. In this way, we can find x_i^R to satisfy both (1) and (2) given (7) and (8).

In a similar way, we can arrange $x_i^P(\cdot) \geqq 0$ so that

$$\forall a_i \in A_i \quad V_i^P = \max_{e_i} u_i(e_i, e_j^P) = g_i(a_i, \alpha_j^P) + E[x_i^P(\omega_j, a_j) | a_i, \alpha_j^P]$$

, hence both (3) and (4) are trivially satisfied.

Then, the last condition (5) is satisfied because $V_i^R = g_i\left(\left(e_i^*, N\right), \alpha_j^R\right) > \max_{e_i} u_i(e_i, e_j^P) = V_i^P$ by (9) as $g_i\left(\left(e_i^*, N\right), \alpha_j^R\right) \approx u_i\left(e^*\right)$. ■

Proof of Theorem 4.

Proof. Fix an integer G so that $d > \frac{K}{G}$ for $i = 1, 2$. We use the following two state machine. Let $CN = \left(\left(C, N\right), ..., \left(C, N\right)\right)$ and DM^G be playing D for all the tasks and monitoring some G tasks. At state R, player i plays CN with probability $1-\varepsilon$ and play one of DM^G with equal probability with probability ε.[10] Let $d_i^\varepsilon = u_i\left(D, \alpha_j^R\right) - u_i\left(C, \alpha_j^R\right)$ be i's deviation gain for each task when player j is in state R. At state P, player i just plays DM^G.

We define x^R as follows;

$$
\begin{aligned}
& x_i^R((y_j, \widehat{a}_i), a_j) \\
= & \begin{cases}
0 & \text{if (1) } a_j = CN \text{ or (2) } a_j = DM^G \text{ and observe only } (C, N) \\
X_R - Y_R' & \text{if (1) } a_j = DM^G \text{ and (2) observe } (D, M) \\
& \quad 1 \sim G-1 \text{ times and } (D, N) \text{ for the rest} \\
X_R + Y_R'' & \text{if (1) } a_j = DM^G \text{ and (2) observe } (D, M) \ G \text{ times} \\
X_R & \text{if (1) } a_j = DM^G \text{ and (2) observe only } (D, N) \\
X_R + Y_R''' & \text{otherwise}
\end{cases}
\end{aligned}
$$

Since $x^R = 0$ when CN is played, this machine clearly approximates the efficient payoff profile (H, H) as $\varepsilon \to 0$. We show that (LI) is satisfied for an appropriate choice of X_R, Y_R', Y_R'', Y_R'''. We omit subscript from now on.

First of all, we only need to deal with deviations which play either (D, M) or (D, N) for each task. Any other deviation can be deterred by setting Y_R''' large enough.

Then the part of (LI) for x_R is satisfied if the following conditions are met;

$$X_R, Y_R', Y_R'', Y_R''', X_R - Y_R' \geqq 0 \tag{23}$$

$$
\begin{aligned}
& Hd^\varepsilon - Km \\
\leqq \ & \varepsilon\left\{X_R - \left(1 - p_0\left(m\right) - p_G\left(m\right)\right)Y_R' + p_G\left(m\right)Y_R''\right\} \text{ for all } m \\
= \ & \varepsilon\left\{X_R - \left(1 - p_0\left(m\right) - p_G\left(m\right)\right)Y_R' + p_G\left(m\right)Y_R''\right\} \text{ for } m = G
\end{aligned}
\tag{24}
$$

[10] Each set of G tasks are chosen with equal probability; $\varepsilon\frac{(H-G)!G!}{H!}$.

where $p_h(m)$ is the probability that exactly h monitoring activities are found out by the other player when you pick (D, M) for m tasks and (D, N) for $H - m$ tasks. More explicitly, they are given as follows;

$$\begin{aligned} p_0(m) &= \begin{cases} \frac{(H-m)\ldots(H-m-G+1)}{H(H-1)\ldots(H-G+1)} & \text{for } m \leq H-G+1 \\ 0 & \text{for } m \geq H-G+1 \end{cases} \\ p_G(m) &= \begin{cases} 0 & \text{for } m \leq G-1 \\ \frac{m\ldots(m-G+1)}{H(H-1)\ldots(H-G+1)} & \text{for } m \geq G-1 \end{cases} \end{aligned}$$

The left hand side of (24) is the gain from playing DM^m instead of CN. The right hand side is the expected loss from such deviations. These conditions mean that players are indifferent between CN and DM^G and prefer DM^G to $DM^m, m \neq G$ as required by (LI).

Let $f(m) = \varepsilon \{X_R - (1 - p_0(m) - p_G(m)) Y'_R + p_G(m) Y''_R\} + Km$. A sufficient condition for (24) is

$$Hd^{\varepsilon} = f(G) \tag{25}$$

$$G = \arg \min_{0 \leq m \leq H} f(m) \tag{26}$$

Note that f is not differentiable at $m = G - 1$ and $H - G + 1$. Since $f''(m) > 0$ for any $m \neq G - 1$ and $H - G + 1$, The following conditions are enough for (26) to be satisfied.

$$\varepsilon \left\{p'_0(G) Y'_R + p'_G(G) \left(Y'_R + Y''_R\right)\right\} + K = 0 \text{ (First order condition)} \tag{27}$$

$$\varepsilon p'_0(G-1) Y'_R + K \leq 0 \tag{28}$$

$$\varepsilon p'_G(G-1) \left(Y'_R + Y''_R\right) + K \geq 0 \tag{29}$$

It is easy to see that (29) is automatically satisfied. In the following, we find X_R, Y'_R, Y''_R, Y'''_R to satisfy (23), (25), (27) and (28).

We can solve for X_R and Y'_R in terms of Y''_R from (25), and (27)

$$\begin{aligned} X_R &= \left\{-\frac{(1 - p_0(G) - p_G(G)) p'_G(G)}{p'_0(G) + p'_G(G)} - p_G(G)\right\} Y''_R \\ &\quad + \frac{1}{\varepsilon}\left\{Hd^{\varepsilon} - Km - \frac{(1 - p_0(G) - p_G(G)) K}{p'_0(G) + p'_G(G)}\right\} \\ Y'_R &= -\frac{p'_G(G)}{p'_0(G) + p'_G(G)} Y''_R - \frac{1}{\varepsilon} \frac{K}{p'_0(G) + p'_G(G)} \end{aligned}$$

The coefficient of Y''_R and the constant for Y'_R is strictly positive for large H because $\frac{p'_G(G)}{p'_0(G)} \to 0$ as $H \to \infty$, $p'_0(G) < 0$ and $p'_G(G) > 0$. Note also that the coefficient of Y''_R is smaller for X_R. These imply that X_R, Y'_R and Y''_R which satisfy (23) can be found by choosing appropriate Y''_R if and only if the constant of X_R is larger than the constant of Y'_R, that is,

$$\begin{aligned} Hd^{\varepsilon} - KG - \frac{(1 - p_0(G) - p_G(G))K}{p'_0(G) + p'_G(G)} &\geq -\frac{K}{p'_0(G) + p'_G(G)} \\ d^{\varepsilon} - \frac{KG}{H} + \frac{(p_0(G) + p_G(G))K}{H\left(p'_0(G) + p'_G(G)\right)} &\geq 0 \qquad (30) \end{aligned}$$

Since we can show that $\frac{(p_0(G)+p_G(G))}{H\left(p'_0(G)+p'_G(G)\right)} \to -\frac{1}{G}$ as $H \to \infty$,[11] if we choose H large enough, $d - \frac{KG}{H} + \frac{(p_0(G)+p_G(G))K}{H\left(p'_0(G)+p'_G(G)\right)} > 0$ holds by assumption. Then (30) is satisfied for any small enough ε for any such large H.

Since Y'_R is at least $-\frac{1}{\varepsilon}\frac{K}{p'_0(G)+p'_G(G)}$, (28) is also satisfied if

$$\frac{p'_0(G-1)}{p'_0(G) + p'_G(G)}K + K \leq 0$$

which is indeed satisfied for large H because $\frac{p'_0(G-1)}{p'_0(G)} < -1$ and $\frac{p'_0(G)}{p'_0(G)+p'_G(G)} \downarrow 1$ as $H \to \infty$.

Take any $\underline{H}$ such that (30) and (28) are satisfied for each $H \geq \underline{H}$. Since ε can be arbitrary small, the efficient outcome can be arbitrarily approximated for each such H.

Since a similar proof works for X_P, the theorem is proved. ■

[11] Since $H\frac{p_G(G)}{p_0(G)} \to 0$ as $H \to \infty$,

$$\begin{aligned} \frac{(p_0(G) + p_G(G))}{H(p'_0(G) + p'_G(G))} &= \frac{(p_0(G) + p_G(G))}{-Hp_0(G)\left(\sum_{s=0}^{G-1}\frac{1}{H-G-s}\right) + Hp_G(G)\left(\sum_{s=0}^{G-1}\frac{1}{G-s}\right)} \\ &= \frac{\left(1 + \frac{p_G(G)}{p_0(G)}\right)}{-H\sum_{s=0}^{G-1}\frac{1}{H-G-s} + H\frac{p_G(G)}{p_0(G)}\left(\sum_{s=0}^{G-1}\frac{1}{G-s}\right)} \\ &\to -\frac{1}{G} \end{aligned}$$

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