Collusion and Heterogeneity of Firms*

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Abstract

We examine the impact of heterogeneous discounting on collusion in dynamic Bertrand competition. We show exactly when collusion can be sustained and how collusion would be organized efficiently with heterogeneous discounting. First we show that collusion is possible if and only if the average discount factor exceeds a certain threshold, with or without capacity constraints. Next we identify a dynamic pattern of market share that characterizes efficient collusion and obtain the unique long-run prediction despite the presence of multiple equilibria. In the long run, the most patient firm and the most impatient firm tend to dominate the market.

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1 Introduction

To understand when and how collusion arises with heterogeneous discounting, we study a dynamic Bertrand competition model, in which firms discount future profits at different discount rates, and examine the impact of heterogeneous discounting on collusion. The model of dynamic Bertrand/Cournot competition is the standard framework to analyze and understand collusion (Tirole, 1988). The logic behind these models is simple: firms are willing to collude because they fear a "price war" in the future. Clearly, the discount rate is the most critical parameter that determines the effectiveness of such an intertemporal incentive scheme.

Almost all of the models of this kind, however, assume symmetric discounting. In fact, it is often assumed that firms are completely symmetric in every aspect. Symmetric models are useful for understanding a variety of issues associated with collusion due to its tractability. Nevertheless, the symmetry assumption is unrealistic and, thus, limits the scope of applications of such models, especially when symmetry is imposed on such a crucial parameter of the model.[1] This motivates us to introduce heterogeneity of discount rates to the dynamic model of collusion.

There are at least two reasons to believe that future profit is discounted differently by different firms. The first one is the cost of capital. If a firm faces a higher interest rate than do other firms for any reason (e.g., asymmetric information), then the firm would value profits in the short run more than would the other firm. The second reason is heterogeneity of managers' discount rates. Even if the cost of capital is the same across firms, the managers who run these firms may discount future profits differently. According to Stein (1989), managers focus on the short term when their salaries depend heavily on current stock prices. Shleifer and Vishny (1990) argue that short horizons of investors lead to short horizons of managers who are averse to underpricing of their equity. This type of aversion may occur when low equity prices increase the probability of replacement. Narayanan (1985) emphasizes that managerial career concerns may result in a focus on short-term horizons. Specifically,

[1] Fershtman and Pakes (2000) emphasize the importance of heterogeneity among firms in the same market.

Narayanan (1985) states that, if the manager possesses private information unavailable to investors and his ability is unknown, then he might choose quicker-return projects to increase his wage. To the extent that these considerations (compensation scheme, investors' horizon, managerial career concerns, manager's private information) vary across firms, future profit should be discounted differently by different firms.

Formally, our model is an infinitely repeated Bertrand (price-setting) game by firms with heterogeneous discount rates and constant marginal costs. We assume that all of the firms produce the same product and that the firm that charges the lowest price must serve the entire market, as in the standard Bertrand game. When two or more firms charge the same (lowest) price, we allow those firms to divide the market in a flexible way through communication.[2] Thus, firms can collude in two dimensions: price and market share.[3]

Our main findings consist of two parts. For the first part, we characterize exactly when collusion is possible with heterogeneous discounting. We show that the *average discount factor* is the key variable for determining the *possibility* of collusion. Specifically, we show that the monopoly price (indeed, any profitable price strictly above the marginal cost) can be sustained with n firms if and only if the average discount factor exceeds $\frac{n-1}{n}$. If the average discount factor falls strictly below this critical threshold, then the competitive outcome prevails in every period in *any* equilibrium. Thus, heterogeneity in discounting does not discourage collusion, per se. Because the allocation of market shares is flexible, a more patient firm is willing to give up more market shares to more impatient firms, whose incentive constraints are then relaxed. Thus, the distribution of discount rates matters, in general. In our Bertrand setting, it turns out that the first moment of the distribution determines completely the possibility of collusion.

We can extend this result to the case where the firms face capacity constraints. In this case, the critical level of the average discount factor depends on the price to be supported.

[2]Genesove and Mullin (2001) document actual examples of communication for the sugar refining cartel, even though they do not appear to be related directly to the way in which communication works in our model.

[3]Benoit and Krishna (1987) and Staiger and Wolak (1992) study a dynamic oligopoly model in which firms choose capacities, then prices.

The average discount factor needs to be larger to support a collusive price that is closer to the monopoly price. It may be the case that firms cannot collude at the monopoly price but can collude at a price lower than the monopoly price. As we focus on asymmetric discounting and keep every other part of the model symmetric intentionally, we assume symmetric capacity constraints for the most part. Nevertheless, we discuss asymmetric capacity constraints and illustrate the possibility that asymmetry in capacity constraints facilitates collusion in the presence of asymmetric discounting.

For the second part, we examine how collusion should be organized. We provide an almost complete characterization of all collusive equilibria that, from the firms' perspective, are Pareto-efficient. First, we show that the equilibrium prices are always set at the monopoly price in most of the efficient collusive equilibria. If this is not the case, then the equilibrium prices must be increasing monotonically and quickly until they reach the monopoly price and stay there forever.[4] Thus, it is almost without loss of generality to focus on efficient collusive equilibria with the monopoly price, with the possible exception of the first few periods. Then, we provide a complete characterization of the dynamics of market shares in all efficient collusive equilibria with the monopoly price. Our result paints a very different picture of collusion, relative to the one with symmetric discounting. The standard model with symmetric firms usually focuses on stationary equilibria, whereby the prices and the market shares are constant over time. In our setting, almost all stationary equilibria are inefficient due to heterogeneous discounting. Efficiency would be improved by an intertemporal transfer of market shares between firms with different discount rates. More specifically, we show that, in *any* efficient collusive equilibrium with the monopoly price, each firm's equilibrium market share dynamics can be described by three phases. In the first phase, a firm has no share of the market, leaving the market to more impatient firms. In the second phase, the firm enters the market and gains all of the residual market shares after leaving the "minimum" level of market share to more impatient firms, which corresponds to the smallest market share for those firms that can be supported by a stationary collusive equilibrium with the monopoly

[4] Some efficient collusive equilibrium with very asymmetric payoffs may require the market price to be lower than the monopoly price initially (see the three-firm example in Section 4).

price. The final phase for the firm starts when a more patient firm enters the market. In this phase, the firm's market share drops to its minimum level and stays there forever. Firms go through these three phases in increasing order of patience. Thus, each firm's market share is hump-shaped over time, first going up, then going down. Two exceptions are the most impatient firm and the most patient firm. The market share of the most impatient firm decreases monotonically over time (i.e., there is no first phase), and the market share of the most patient firm increases monotonically over time (i.e., there is no final phase).

One important implication of our result is that every efficient collusive equilibrium converges to the same (and unique) stationary efficient collusive equilibrium in the long run. Specifically, in any efficient collusive equilibrium, every firm's market share converges to the market share associated with the worst stationary efficient collusive equilibrium with the monopoly price, except for the most patient firm.[5] Hence, *our model delivers a unique prediction regarding the equilibrium market share in the long run without invoking any equilibrium selection criterion.* With symmetric firms, because how to share the market is irrelevant for efficiency, there are many efficient stationary equilibria with different market shares. With asymmetric discounting, however, efficiency imposes a sharp restriction on how the market should be allocated intertemporally. As a consequence, even though there are many efficient equilibria, the long-run market share must be the same across all efficient collusive equilibria.

Another important implication of our results is that the distribution of the long-run market shares across firms is U-shaped with respect to the order of patience. The most impatient firm and the most patient firm tend to occupy a larger share of the market eventually, but for very different reasons. The most impatient firm needs a large market share for the sake of incentive. The most patient firm gains a large market share eventually for the sake of efficiency. In a sense, the degree of heterogeneity in discount rates is magnified endogenously in the long run if it is measured using stable market shares as weights. Thus, if we focus on any observable characteristics of firms that are correlated with their discount rates, then we

[5] The time to reach this particular stationary efficient collusive equilibrium is bounded across all efficient collusive equilibria for a given profile of discount rates.

would find that very different firms become eventually dominant in the same market even though they produce exactly the same product.

From a more theoretical perspective, we provide insight into the theory of repeated games with unequal discounting. As reviewed briefly next, most of the available results for repeated games with unequal discounting focus on the limiting case in which players are infinitely patient. In our setting, we can characterize the equilibrium behavior in all efficient equilibria for a given profile of discount factors.

Related Literature

There are good reasons why symmetric models have been so popular in the literature on dynamic oligopolistic competition. First, there is the issue of equilibrium selection due to the existence of many equilibria in repeated games. The model of dynamic Bertrand competition is no exception. With symmetric firms, it might make more sense to focus on the symmetric collusive equilibrium, possibly as a focal point. But it is not at all clear which equilibrium would be selected when firms are asymmetric. Second, the theory of repeated games with unequal discounting is still in its development. For these reasons, there are not many works that study collusion among firms with heterogeneous discounting.

One notable exception is Harrington (1989), who shows that a stationary collusive equilibrium with possibly asymmetric market share can be sustained with asymmetric discounting if and only if the average discount factor exceeds the critical threshold $\frac{n-1}{n}$. Our first result builds on and further develops this result. We provide a complete characterization of the possibility of collusion by considering all equilibria, including nonstationary ones, and by introducing capacity constraints. We wish to emphasize that it is particularly important to study nonstationary equilibria with heterogeneous discounting as impatient firms and patient firms are willing to "trade" their profits over time. In fact, our efficiency result shows that almost all stationary equilibria are inefficient with asymmetric discounting. Another difference between our work and that of Harrington (1989) is that we obtain a unique equilibrium prediction for the long run. To cope with the issue of multiple stationary equilibria with different market share configurations, Harrington (1989) uses the Nash bargaining solution to select one stationary equilibrium (also see Harrington, 1991). In this article, we show that the long-run equilibrium behavior is the same across all efficient collusive equilibria. Thus, we do not need to apply any equilibrium selection criterion other than efficiency to select an equilibrium as far as the long-run outcome is concerned.[6]

Many studies examine the effect of asymmetric production technology on collusion. Compte, Jenny, and Rey (2002) and Lambson (1994) introduce asymmetric capacity constraints to the infinitely repeated Bertrand game.[7] Vasconcelos (2005) introduces asymmetric marginal costs to the infinitely repeated Cournot game. Mason, Phillips, and Nowell (1992) studied dynamic duopoly games with asymmetric costs and found that the symmetry of marginal costs facilitates collusion.[8]

The seminal contribution to the theory of repeated games with unequal discounting is Lehrer and Pauzner (1999), who studied a general two-player repeated game with asymmetric discounting. They characterize the set of feasible payoffs and show that it is larger than the convex hull of the underlying stage game payoffs.[9] They also characterize the limit equilibrium payoff set as discount factors go to 1 with a fixed log ratio. In particular, they show that there is some individually rational and feasible payoff that cannot be sustained in equilibrium no matter how patient the players are.

There are some recent developments in the theory of repeated games with unequal discounting. For the case of perfect monitoring, Chen (2008) and Gueron, Lamadon, and Thomas (2011) examine a folk theorem for a class of examples for which even a weak form of full dimensionality is violated. Further, Chen and Takahashi (2012) prove a folk theorem for

[6]Andersson (2008) introduces an instantaneous alternating offer bargaining game that precedes the infinitely repeated Bertrand game in Harrington (1989) and selects the equilibrium price and the equilibrium price market share endogenously but does not consider any nonstationary equilibrium for the repeated Bertrand game.

[7]Brock and Sheinkman (1985) are the first to study the infinitely repeated Bertrand game with symmetric capacity constraints. Lambson (1987) studies a broad class of symmetric Bertrand supergames with capacity constraints.

[8]Many articles –such as Osbourne and Pitchik (1983) and Schmalensee (1987)– study the effect of asymmetry of firms on the division of surplus within the framework of cooperative game. Since they adopt static models, they do not address asymmetry in discount rates.

[9]Chen and Fujishige (2013) provide a different characterization of the set of feasible payoffs for two-player repeated games.

a class of games that satisfy a certain dynamic version of full dimensionality. For the case of imperfect public monitoring, Sugaya (2015) proves a folk theorem with full dimensional payoffs and some conditions on the monitoring structure.

Fong and Surti (2009) study repeated prisoner dilemma games with differential discounting and with side payments. They provide a necessary and sufficient condition for the average discount rates to support a particular class of (almost) stationary equilibrium whereby the players cooperate in every period. They do not consider all equilibria, however, and their analysis is restricted to the two-player case. Miller and Watson (2013) study repeated games in which the stage game is preceded by a bargaining phase and then a transfer phase in each period. They provide a recursive characterization of the equilibrium payoff set for some refinement of subgame perfect equilibrium and extend it to the case with heterogeneous discount factors.

This article is organized as follows. We describe the model in detail in the next section. In Section 3, we characterize the possibility of collusion in terms of the critical average discount factor. In Section 4, we provide a characterization of all efficient collusive equilibria. We introduce capacity constraints and extend our collusion possibility result in Section 5. We discuss other possible extensions in Section 6. Then, Section 7 concludes. All proofs are relegated to the appendix.

2 Model of Repeated Bertrand Competition with Heterogeneous Discounting

In this section, we describe the basic structure of our model, an infinitely repeated Bertrand game with differential discounting. In what follows, we first define the stage game, then construct the infinitely repeated game.

The main features of the stage game are as follows. The players are $n \geq 2$ firms represented by the numbers $\mathcal{I} = \{1, 2, \ldots, n\}$, which produce the same homogeneous product. The demand of the product is given by a continuous function $D : \mathbb{R}_+ \to \mathbb{R}_+$. Each firm has a linear cost function $C_i : \mathbb{R}_+ \to \mathbb{R}_+$ given by $C_i(q_i) = cq_i$ with constant marginal cost $c \geq 0$, where q_i indicates the quantity produced by firm i. We assume that D is decreasing on $\mathbb{R}_+$ and there exists a unique monopoly price $p^m > c$ that maximizes $\pi(p) = D(p)\,(p - c)$, which is increasing on $[c, p^m]$. Let $\pi^m = D(p^m)(p^m - c)$ be the monopoly profit.

At the beginning of a stage game, firms choose prices and make a "suggestion" about how to allocate the market in the event of a draw in prices. If a firm charges a price that is higher than a price charged by another firm, then the firm's market share is 0. The firm that charges the lowest price, which we call the *market price* and is denoted by $\tilde{p}$, must produce enough products to meet the market demand. In case more than one firm charges the lowest price, the market is allocated among those lowest-price firms according to their suggestions. Formally, firm i's pure action is given by a 2-tuple $a_i = (p_i, r_i) \in A_i$, where p_i is the price choice, and r_i reflects firm i's request of market share in the event of a tie. Hence, $A_i = \mathbb{R}_+ \times [0, 1]$ is the set of pure actions available for firm i. The set of pure action profiles is $A = \prod_{i \in \mathcal{I}} A_i$. Firm i's profit function $\pi_i : A \to \mathbb{R}$ can be written as

$$\pi_i(a) = \begin{cases} D(p_i)(p_i - c) & \text{if } p_i < p^*_{-i}, \\ \frac{r_i}{R^*} D(p_i)(p_i - c) & \text{if } p_i = p^*_{-i} \text{ and } R^* \neq 0, \\ \frac{1}{|\widehat{\mathcal{I}}|} D(p_i)(p_i - c) & \text{if } p_i = p^*_{-i} \text{ and } R^* = 0, \\ 0 & \text{if } p_i > p^*_{-i}, \end{cases}$$

where $p^*_{-i} = \min_{j \neq i} p_j$, $\widehat{\mathcal{I}} = \{i \in \mathcal{I} : p_i = \min_{j \in \mathcal{I}} p_j\}$, and $R^* = \sum_{j \in \widehat{\mathcal{I}}} r_j$.

We emphasize that this is just one way to model flexible market sharing in a non-cooperative way. The details of this particular mechanism are not important. Essentially, what is needed is that the firms agree on how to share the market on the equilibrium path. There are many other ways to model flexible market sharing without affecting any of our results.[10]

Given the stage game described above, we now define the infinitely repeated game. We

[10] Athey and Bagwell (2001) adopt a similar mechanism where market shares are allocated flexibly via communication.

adopt the standard discrete time model in which the above stage game is played in each of the periods $t \in \mathbb{N}$. The distinguishing feature of our dynamic Bertrand competition model is that firms have different discount factors, given by $\delta_i \in [0, 1)$ for $i \in \mathcal{I}$.

The set of possible histories in period t is given by $H^t = A^{t-1}$, where A^0 indicates the initial history, and A^t denotes the t-fold product of A. A period t-history is thus a list of $t-1$ action profiles. We suppose perfect monitoring throughout, i.e., all firms observe every action profile chosen in the past. Setting $H = \cup_{t \in \mathbb{N}} H^t$, a pure strategy for firm i is defined as a mapping $\sigma_i : H \to A_i$ and, consequently, a strategy profile is given by $\sigma = (\sigma_i)_{i \in \mathcal{I}}$. We say that a firm *enters the market* in period t when the firm's market share is 0 until period $t-1$ and becomes strictly positive for the first time in period t.

Each strategy profile σ induces an infinite sequence of action profiles $\mathbf{a}(\sigma) = (a^t(\sigma))_{t \in \mathbb{N}} \in A^\infty$, where $a^t(\sigma) \in A$ denotes the action profile induced by σ in period t. We call the sequence $\mathbf{a}(\sigma)$ outcome path (or, more simply, outcome) generated by a strategy profile σ. Finally, for a given strategy profile σ, and its corresponding outcome path $\mathbf{a}(\sigma) = (a^t(\sigma))_{t \in \mathbb{N}}$, the discounted payoff for firm i at time t is given by

$$U_{i,t}\left(\mathbf{a}(\sigma)\right) = \sum_{\tau=t}^{\infty} \delta_i^{\tau-t} \pi_i(a^\tau(\sigma)).$$

In the following sections, we will focus on subgame perfect equilibrium, and we will limit our attention to pure strategy equilibria. We often suppress actions and state the equilibrium conditions in terms of the firms' profits. For simplicity, we denote firm i's profit and the joint profit in period t on the equilibrium path by $\pi_{i,t}$ and $\pi_t = \sum \pi_{i,t}$, respectively. If a sequence of profit profiles $(\pi_{i,t})_{t \in \mathbb{N}}$, $i \in \mathcal{I}$, is generated by an equilibrium, then they must satisfy the following incentive constraints in every period:

$$\pi_t \leq U_{i,t} = \pi_{i,t} + \delta_i U_{i,t+1}, \;\; t \in \mathbb{N}$$

On the other hand, it is clear that any sequence of profit profiles that is feasible and satisfies this condition can be generated by some equilibrium. Note that we can use the worst equilibrium with 0 profit after any unilateral deviation without loss of generality. Thus, we can use the above condition as the equilibrium condition.

3 Critical Average Discount Factor for Collusion

In this section, we derive a necessary and sufficient condition to sustain a collusive equilibrium outcome. We say that the firms are colluding when there is at least one period in which the equilibrium outcome is not a competitive one, i.e., when there is at least one firm that makes a positive profit in some period. We formalize this as follows.

Definition 1. *An outcome $\mathbf{a} = (a^t)_{t \in \mathbb{N}}$ is considered a **collusive outcome** if and only if there exists $t' \in \mathbb{N}$ such that $\pi_i(a^{t'}) > 0$ for some $i \in \mathcal{I}$. A **collusive equilibrium** is a subgame perfect equilibrium that generates a collusive outcome. A p-**collusive equilibrium** for $p > c$ is a collusive equilibrium in which every firm chooses p in every period on the equilibrium path. A p-collusive equilibrium is **stationary** if the equilibrium market share of each firm does not change over time.*

Then we can obtain the following sharp characterization regarding collusive equilibria: there exists a collusive equilibrium if and only if the average discount factor exceeds some critical threshold. Furthermore, a p^m-collusive equilibrium exists whenever there exists a collusive equilibrium. If the average discount factor is strictly below this threshold, then the competitive outcome (price equals marginal cost) prevails in every period in any equilibrium.

Theorem 3.1. *There exists a collusive equilibrium if and only if*

$$\frac{\sum_{i \in \mathcal{I}} \delta_i}{n} \geq \frac{n-1}{n}.$$

Whenever there exists a collusive equilibrium, there exists a stationary p-collusive equilibrium for any $p \in (c, p^m]$.

When the firms are symmetric, there is a collusive equilibrium if and only if $\delta \geq \frac{n-1}{n}$. Our result is a substantial generalization of this well-known result to the case of heterogeneous discounting.

The intuition about why the average discounting matters is as follows. An impatient firm has a stronger incentive to break collusion. Thus, an impatient firm needs to be assigned a larger market share to stay in collusion. In contrast, a patient firm has a stronger incentive to keep collusion and, thus, is willing to give up its market share to more impatient firms to sustain collusion. In this way, the distribution of discounting matters for the success of collusion. In our setting, it turns out that the first moment of the distribution determines the possibility of collusion completely.

The first part of this theorem is straightforward, as shown in Harrington (1989). Suppose that every firm chooses p, and firm i's market share is α_i in every period. Such a stationary outcome can be sustained in equilibrium if and only if the following incentive constraint is satisfied:

$$\pi \leq \frac{\alpha_i \pi}{1-\delta_i}.$$

By dividing both sides by π, multiplying both sides by $1-\delta_i$, and summing up these inequalities across the firms, we obtain the above inequality on the average discount factor. Conversely, it is clear that such α_i can be found when the average discount factor satisfies the above inequality.

A much more difficult part of the proof, which is a more substantial contribution of this article, is to show that no collusive equilibrium exists when the average discount factor is less than $\frac{n-1}{n}$, even if nonstationary equilibria are considered. In nonstationary equilibrium, it is possible to transfer market shares over time. Such intertemporal transfer could be a Pareto-improvement for the firms due to asymmetric discounting; hence, such transfer may facilitate collusion even if collusion cannot be sustained in a stationary equilibrium. It turns out, however, that such transfer does not work. To improve efficiency, it is necessary to let less patient firms gain more market shares first and to let more patient firms gain more shares later. Intuitively, such an arrangement is in conflict with less patient firms' incentive constraints in later periods.

Here is a sketch of our formal proof of the above impossibility result. Firm i's incentive constraint in period t is given by the equality

$$U_{i,t} = \pi_{i,t} + \delta_i U_{i,t+1} = \pi_t + \eta_{i,t}$$

where $\eta_{i,t} \geq 0$ is a slack variable (firm i's incentive constraint is binding in period t if and only if $\eta_{i,t} = 0$). Because this equality holds in every period, we can replace $U_{i,t+1}$ with $\pi_{t+1} + \eta_{i,t+1}$ to obtain

$$\pi_{i,t} + \delta_i \pi_{t+1} = \pi_t + \eta_{i,t} - \delta_i \eta_{i,t+1}.$$

By summing these equalities across the firms, we obtain the following equation regarding the sequence of joint profits:

$$\pi_{t+1} = \frac{n-1}{\sum_{i \in \mathcal{I}} \delta_i} \pi_t + \frac{1}{\sum_{i \in \mathcal{I}} \delta_i} \sum_{i \in \mathcal{I}} u_{i,t},$$

where $u_{i,t} = \eta_{i,t} - \delta_i \eta_{i,t+1}$.

The coefficient of π_t is larger than 1 if and only if the average discount factor is less than $\frac{n-1}{n}$. In fact, we can show that, when the joint profit is strictly positive in some period, the sequence $\{\pi_t : t \in \mathbb{N}\}$ must diverge to infinity, which is a contradiction. To prove this formally, however, we need to examine the behavior of the term $\sum_{i \in \mathcal{I}} u_{i,t}$ carefully. This result will be generalized to the case with capacity constraints in Section 5.

4 Characterization of Efficient Collusive Equilibria

One implication of the previous result is that we can focus on stationary collusive equilibria without loss of generality to examine the *possibility* of collusion. This result, however, does not tell us the best way to collude for the firms. In fact, except for one stationary collusive equilibrium, every stationary collusive equilibrium turns out to be inefficient from the firms' perspective because the firms would benefit from "trading" market shares over time due to the heterogeneity of their patience. In this section, we characterize the structure of efficient collusive equilibria with heterogeneous discounting.[11]

In this section, we assume that $0 < \delta_1 < \delta_2 < \cdots < \delta_{n-1} < \delta_n < 1$ for the sake of simplicity. Nevertheless, our results can be extended naturally to the case with equal discounting for some of the firms, as explained at the end of this section. We also assume, throughout this section, $\frac{\sum_{i=1}^n \delta_i}{n} > \frac{n-1}{n}$, which guarantees the existence of a stationary p^m-collusive equilibrium by Theorem 3.1. In fact, there is a continuum of market shares that can be supported by stationary p^m-collusive equilibria in this case.[12] Let $\widehat{\pi}_i$ be firm i's per-period profit in the stationary p^m-collusive equilibrium that is worst from firm i's viewpoint. Specifically, $\widehat{\pi}_i$ is defined by the following binding incentive compatibility condition $\frac{\widehat{\pi}_i}{1-\delta_i} = \pi^m$. This implies that firm i's market share is $1 - \delta_i$ in this stationary equilibrium. Note that the market share/profit per period is larger for more impatient firms and that the total discounted payoff in the worst stationary p^m-collusive equilibrium is exactly π^m for any firm.

Efficient Collusive Equilibrium with Non-Monopoly Price

With symmetric discounting, every efficient collusive equilibrium must be a p^m-collusive equilibrium. Even with asymmetric discounting, if we restrict attention to equilibria with stationary market share, then it is without loss of generality to focus on p^m-collusive equilibrium, as Theorem 3.1 shows. However, this is not the case in general. The price sometimes needs to be below the monopoly price for some efficient collusive equilibrium. This is because each firm's incentive constraint can be relaxed by reducing the total profit. For example, if the total profit is reduced by ϵ by lowering the price, then each firm's profit in the same period can be reduced by ϵ without violating its incentive constraint because the gain from a deviation is the same. If we take ϵ-profit away from $n-1$ firms in this way and transfer them to the remaining firm, then this firm's net gain would be $(n-2)\epsilon$, which is positive with

[11] A collusive equilibrium is efficient if there is no other collusive equilibrium that is Pareto-improving for the firms.

[12] Almost all such stationary p^m-collusive equilibria are inefficient because no firm's incentive constraint is binding.

three firms or more. The following example shows that we can apply this argument to the best stationary p^m-collusive equilibrium to construct an equilibrium with a non-monopoly price whereby firm n's profit is higher than its profit in any p^m-collusive equilibrium.

Example with $n = 3$ Set $n = 3$. We maintain the assumption that $0 < \delta_1 < \delta_2 < \delta_3 < 1$ and $\delta_1 + \delta_2 + \delta_3 > 2$. Consider the following sequence of payoff profiles. In period 1, $\pi_{i,1} = \pi_1 - \delta_i \pi^m$ for $i = 1, 2$, and $\pi_{3,1} = (\delta_1 + \delta_2)\pi^m - \pi_1$ for some joint profit $\pi_1 \in [\delta_2 \pi^m, \pi^m]$. From the second period on, the best stationary p^m-collusive equilibrium for firm 3 is played: $\pi_{1,t} = \widehat{\pi}_1$, $\pi_{2,t} = \widehat{\pi}_2$, and $\pi_{3,t} = (\delta_1 + \delta_2 - 1)\pi^m$ for $t = 2, 3, \ldots$. The incentive constraints of Firms 1 and 2 are binding by construction. Note that Firm 3's deviation gain in period 1 is $2\pi_1 - (\delta_1 + \delta_2)\pi^m$, which is increasing in π_1 and identical to the deviation gain in the stationary p^m-collusive equilibrium that is best for Firm 3 (hence worst for Firm 1 and 2) when $\pi_1 = \pi^m$. Thus, Firm 3's incentive constraint is satisfied as well. Theorem 4.1, which is the main theorem in this section, shows that Firm 3 gains $(\delta_1 + \delta_2 - 1)\pi^m$ in every period in the p^m-collusive equilibrium that is best for Firm 3. Thus, this equilibrium is better than any p^m-collusive equilibrium for Firm 3 if π_1 is set smaller than π^m (with a price lower than p^m). It can be shown that, indeed, the best efficient collusive equilibrium for Firm 3 is obtained when $\pi_1 = \delta_2 \pi^m$.[13]

As the above example shows, the equilibrium price may need to be strictly below the monopoly price in some efficient collusive equilibrium. The following two propositions show, however, that we do not lose much from focusing on efficient collusive equilibrium with monopoly price in every period, which we call **p^m-efficient collusive equilibrium**. The first proposition shows that every efficient collusive equilibrium must become a p^m-efficient collusive equilibrium within a finite number of periods. Further, the equilibrium prices must be strictly increasing at a certain rate until they reach and stay at the monopoly price; hence, the time to reach p^m-efficient collusive equilibrium is bounded uniformly across all efficient collusive equilibria. Therefore, it is without loss of generality to focus on p^m-efficient

[13] A complete characterization of efficient collusive equilibrium for the case of $n = 3$ is available upon request.

collusive equilibrium as long as we are not so concerned with the very first few periods of efficient collusive equilibria.

Proposition 4.1. *For any efficient collusive equilibrium, there exists $T \geq 1$ such that the joint profit $\pi_t > 0$ is monotonically increasing over time up to period T with $\frac{1}{\delta_n} \leq \frac{\pi_{t+1}}{\pi_t}$ for $t = 1, \ldots, T-2$ and $\pi_t = \pi^m$ for any $t \geq T$. Further, this T is bounded across all efficient collusive equilibria.*

The next proposition shows that every efficient collusive equilibrium with a non-monopoly price must be very asymmetric. More specifically, in such an equilibrium, there must be some firm i whose total discounted profit is lower than the profit in the worst stationary p^m-collusive equilibrium, which is $\pi^m = \frac{\widehat{\pi}_i}{1-\delta_i}$.

Proposition 4.2. *Every efficient collusive equilibrium in which every firm's total discounted payoff is at least as large as π^m must be a p^m-efficient collusive equilibrium.*

Finally, we observe that every efficient collusive equilibrium must be a p^m-efficient collusive equilibrium for the special case of $n = 2$. By Proposition 4.1, if the market price is not p^m in some period, then the first period price must be strictly below p^m. If this is an efficient collusive equilibrium, then the incentive constraints must be binding for both firms in the first period. Otherwise, it would be possible to generate a Pareto-improving outcome in the first period without violating any incentive constraint by increasing the joint profit and sharing the marginal gain equally. This implies that each firm's total discounted profit must be exactly $\pi_1 (< \pi^m)$, the joint profit in the first period. This equilibrium, however, is Pareto dominated by any stationary p^m-collusive equilibrium, where every firm's total discounted profit is at least π^m.

Market Share Dynamics in p^m-Efficient Collusive Equilibrium

Now we provide a complete characterization of all p^m-efficient collusive equilibria. We first consider the simplest case with only two firms to illustrate our main result. With asymmetric discounting, efficiency can be improved by having more patient firms to "lend" some market share initially to less patient firms. It is not difficult to see that the first best allocations for the firms are characterized as follows:

- The market price is always the monopoly price p^m.
- Firm 1 (more impatient firm) gains the whole market share up to some period $t-1$.
- Firm 1 and Firm 2 share the market in some way in period t.
- Firm 2 gains the whole market share from period $t+1$ on.

Clearly, this is not an equilibrium outcome, as Firm 1's incentive constraint is violated after period $t+1$. Hence, no first best allocation can be achieved by any equilibrium. To keep Firm 1's incentive to collude, some market share must be left for Firm 1. Our result shows that every efficient collusive equilibrium takes the following slightly different form, with two firms to resolve this trade-off between efficiency and incentive. Note that we can focus on p^m-efficient collusive equilibrium without loss of generality in this case, as briefly discussed after Proposition 4.2 .

- The market price is always the monopoly price p^m.
- Firm 1 gains the whole market share up to some (not too late) period $t-1$.
- Firms 1 and 2 share the market in period t in such a way that Firm 1's profit is at least as large as $\widehat{\pi}_1$.
- Firm 1 gains $\widehat{\pi}_1$ forever and Firm 2 gains all the rest from period $t+1$ on.

This is very intuitive. A first best outcome is being approximated as closely as possible, subject to the incentive constraint of Firm 1, which needs to "pay back" in a later stage of the game.[14]

The following theorem provides a general version of this result for the case with n firms. Note that our assumption $\frac{\sum_{i=1}^{n}\delta_i}{n} > \frac{n-1}{n}$ implies $\pi^m - \sum_{i=1}^{n}\widehat{\pi}_i > 0$.

[14]Hence Firm 2's incentive constraint is not binding in the long run.

Theorem 4.1. *Every p^m-efficient collusive equilibrium has the following structure: There exists $1 = t_1 \le t_2 \le \cdots \le t_n$ such that, for every i, a sequence of equilibrium profits $\{\pi_{i,t}\}_t$ satisfy the following properties (the description for t_{i+1} applies for firm i in period t_i when $t_i = t_{i+k}$ for any $k \ge 1$).*

1. *Firm i enters the market in period t_i with $\pi_{i,t_i} \in [0, \pi^m - \sum_{h=1}^{i-1} \widehat{\pi}_h]$*
2. $\pi_{i,t} = \pi^m - \sum_{h=1}^{i-1} \widehat{\pi}_h$ *for* $t = t_i + 1, \ldots, t_{i+1} - 1$
3. $\pi_{i,t} \in [\widehat{\pi}_i, \sum_{h=1}^{i-1} \widehat{\pi}_h]$ *for* $t = t_{i+1}$
4. $\pi_{i,t} = \widehat{\pi}_i$ *for* $t > t_{i+1}$
5. $U_{i,1} \ge \pi^m$ *for all* i

Conversely, if there exist $(t_1, t_2, \ldots, t_n)$ and a sequence of profit profiles $\{\pi_{i,t}\}_{i,t}$ such that the above properties are satisfied for every $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}} \pi_{i,t} = \pi^m$ for every t, then there exists a p^m-efficient collusive equilibrium that generates them.

The equilibrium dynamics of each firm's market share is roughly divided into three phases, which can be described as follows. Firm i's market share is initially 0 while more impatient firms are gaining profits. Then, Firm i enters the market and enjoys the maximum market share subject to the constraint that all less patient firms gain their worst stationary p^m-collusive equilibrium profit $\widehat{\pi}_h, h = 1, \ldots, i-1$ in every period. As soon as a more patient firm enters the market, Firm i's per-period profit is slashed down to $\widehat{\pi}_i$, with market share $1 - \delta_i$, and stays there forever. Thus, each firm's market share/profit first goes up, then goes down. Two exceptions are the most impatient firm (Firm 1), whose market share is monotonically decreasing over time, and the most patient firm (Firm n), whose market share is monotonically increasing over time. In the long run, Firm i's market share converges to $1 - \delta_i$ for $i = 1, \ldots, n-1$, and Firm n's market share converges to $1 - \sum_{i=1}^{n-1} (1 - \delta_i) = \sum_{i=1}^{n-1} \delta_i - (n-2) > 1 - \delta_n > 0$ in every p^m-efficient collusive equilibrium.[15]

[15] Firm i's continuation payoff is exactly π^m in the long run for $i = 1, \ldots, n-1$, as the incentive constraint is binding. The most patient firm's continuation payoff is larger than π^m.

Note that every continuation equilibrium of p^m-efficient collusive equilibrium is a p^m-efficient collusive equilibrium. Hence, p^m-efficient collusive equilibrium exhibits a nice renegotiation-proof property on the equilibrium path.[16]

The proof of Theorem 4.1 is based on the following two intuitive equilibrium properties, which we establish in the appendix.

- If a firm has not entered the market (i.e., never gained a positive market share), then every firm that is more patient should not have entered the market.
- Once a firm enters the market, every firm that is more impatient must receive its worst stationary p^m-collusive equilibrium profit in every period from the next period on.

All the above properties of p^m-efficient collusive equilibria can be deduced from these two properties.

We would like to emphasize the following features of the equilibrium market share dynamics.

Unique Equilibrium Selection in the Long Run Every efficient collusive equilibrium eventually becomes a p^m-efficient collusive equilibrium by Proposition 4.1. Hence, Theorem 4.1 implies that every firm's market share must converge to the same level in the long run for any efficient collusive equilibrium. More specifically, every efficient collusive equilibrium converges to the unique stationary efficient collusive equilibrium outcome, which is the worst stationary p^m-collusive equilibrium for Firm $i = 1, \ldots, n-1$ and the best one for Firm n.Therefore, our model delivers a unique prediction in the long run without invoking any equilibrium selection criterion other than efficiency. This stands in contrast to other studies that rely on stationary equilibria. As we have observed, there is usually a continuum of stationary collusive equilibria; thus, one needs to apply some equilibrium selection rule to choose an equilibrium, as in Harrington (1989).

[16] Of course, it is not fully renegotiation-proof due to the perfect competition equilibrium off the equilibrium path.

Heterogeneity of Firms in the Long Run There are two firms that would potentially dominate the market in the long run. As Firm $i(< n)$'s market share is $1 - \delta_i$, which is decreasing in patience, the market share of the most impatient firm tends to be large in the long run. Yet, the market share of the most patient firm can be quite large because it gains all of the residual market share in the end.[17] Hence, the distribution of long-run market shares can be U-shaped with respect to the patience of the firms. If we evaluate the degree of heterogeneity by using the market shares as weights, then the heterogeneity of patience in the market would be magnified in the long run. The reasons that those two firms would occupy a larger market share than others are very different. The most impatient firm requires a large market share for the sake of incentive. The most patient firm gains a large market share for the sake of efficiency.

Monotonicity of Market Share Dynamics Our result sheds light on the structure of the dynamics of efficient equilibrium in repeated games with unequal discounting, especially in regard to the behavior of middle-patient players. Lehrer and Pauzner (1999) show that every efficient equilibrium for two-player repeated games with unequal discounting exhibits the following type of monotonicity: The more impatient player's continuation payoff decreases over time, and the more patient player's payoff increases over time. Not much is known about the structure of equilibrium when there are more than two players. In our model, the least patient firm's payoff/market share and the most patient firm's payoff/market share exhibit the same monotonicity.[18] Further, the middle-patient firms' payoffs/market shares are hump-shaped. They go up first and then go down and stay constant eventually.

It is useful to compare our long-run market share to the one in Harrington (1989)'s study, which is derived by applying the Nash bargaining solution to the set of stationary subgame perfect equilibrium payoffs. In Harrington (1989), k most impatient firms receive the worst stationary equilibrium market share $1 - \delta_i$ for $i = 1, \ldots, k$, and other $n - k$ firms

[17] The most patient firm's long-run market share is larger than the most impatient firm's long-run market share if and only if $\sum_{i=1}^{n-1} \delta_i + \delta_1 > n - 1$.

[18] However, observe that Firm 3's continuation profit decreases over time in our 3-firm example. Hence, this monotonicity does not hold for efficient collusive equilibrium with a non-monopoly price.

share the remaining market equally, where k is the minimum number such that the incentive constraints of these $n-k$ firms are satisfied. Our long-run outcome is quite different from this allocation, as the most patient firm plays a very special role as the residual claimant. The difference is most stark when every firm, even the most impatient one, is very patient. In this case, a completely equal market sharing would be obtained with $k=0$ in Harrington (1989), but our long-run stationary market share would be very asymmetric. As $1-\delta_i$ is small for every firm, Firm n's market share would be close to 1, and every other firm's market share is very small. Our solution would coincide with the one in Harrington (1989) when $k=n-1$. It can be shown by simple calculation that this holds if and only if the remaining long-run market share for Firm n in our model (i.e. $1-\sum_{i=1}^{n-1}(1-\delta_i)$) is as large as $1-\delta_n$, but strictly smaller than $1-\delta_{n-1}$: the worst stationary equilibrium market share for Firm $n-1$. This is satisfied when the average discount factor is close to the threshold for Theorem 3.1.

We close this section by describing how Theorem 4.1 would be extended to the case where not all discount factors are distinct. To be concrete, suppose that $\delta_i = \delta_{i+1} = \delta'$ for some $i < n$ and all the other discount factors are distinct. Then we can treat these two firms as one firm with discount factor δ', and their joint profit/market share behaves exactly as characterized by Theorem 4.1. In the second phase and the transition periods, their market shares can be asymmetric as long as their sum follows our characterization and each firms incentive constraints before the final phase is satisfied. The long-run market share is still $1-\delta'$ for both firms for any efficient collusive equilibrium when $i+1 < n$. If $i+1=n$ (i.e., there are two most patient firms), then the long-run market shares for them are not unique. They could be any (α_{n-1}, α_n) such that (1) $\sum_{i=1}^{n-2}(1-\delta_i) + \alpha_{n-1} + \alpha_n = 1$ (i.e., they share the residual market share in the long run) and (2) $\min\{\alpha_{n-1}, \alpha_n\} \geq 1-\delta'$ (i.e., their incentive constraints in the final phase are satisfied).[19]

[19] Note that the long-run market share of Firm n would jump up at δ_n as δ_{n-1} increases from $\delta_n - \epsilon$ to $\delta_n + \epsilon$.

5 Capacity Constraint and Heterogeneous Discounting

We can extend our collusion possibility result in Section 3 to the case in which firms face capacity constraints. Suppose that each firm can produce at most K units at constant marginal cost c. To avoid complications, we assume $D(c) \leq (n-1)K$, which means that $n-1$ firms can meet the entire demand at the competitive price. Under this assumption, 0 profit outcome can be still sustained as an equilibrium outcome, hence, used as a punishment, as before.[20] We restrict attention to *collusive equilibrium with uniform price*, whereby every firm charges the same price within each period.[21] When there is no capacity constraint (i.e., $K = \infty$), every collusive equilibrium is essentially a collusive equilibrium with a uniform price. As such, this notion generalizes collusive equilibrium to the case in which firms are capacity constrained. Clearly, p-collusive equilibrium is an example of a uniform-price equilibrium, but uniform-price equilibrium is more general, as the equilibrium prices can change over time.

Let $R(p,K) = \frac{\min\{D(p),K\}}{D(p)}$ be the ratio of the maximum market demand that one firm can steal by charging a price slightly below the market price p. Because $D(p)$ is decreasing, $R(p,K)$ is nondecreasing in p.

The following result is a generalization of our collusion possibility result to the case with capacity constraints. Note that $R(p,K) = 1$ when K is large enough.

Theorem 5.1. *There exists a collusive equilibrium with uniform price if and only if*

$$\frac{\sum_{i\in\mathcal{I}}\delta_i}{n} \geq \frac{n - \frac{1}{R(p,K)}}{n}.$$

for some $p \in (c, p^m]$. Further, there exists a stationary p-collusive equilibrium for $p \in (c, p^m]$

[20] In general, capacity constraints have an ambiguous effect on collusion. They make price-cutting less profitable, whereas they may make punishment less harsh. This assumption means that the second effect is absent, hence guarantees that the introduction of capacity constraints makes collusion easier.

[21] With capacity constraints, we typically need to specify a rationing rule in case the demand exceeds the total supply, as the choice of rationing rule matters in general (Davidson and Deneckere, 1986). However the choice of rationing rule is irrelevant for our analysis as we focus on uniform-price equilibria and assume $D(c) \leq (n-1)K$.

if and only if this inequality is satisfied at p.[22]

Theorem 5.1 implies that the set of prices that can be sustained in collusive equilibrium depends on the size of the capacity constraint. The set of p for which a stationary p-collusive equilibrium exists is given by some interval $(c, \widehat{p}]$, where $\widehat{p}$ depends on K and may be strictly below the monopoly price p^m. Without capacity constraint, p^m-collusive equilibrium exists whenever there exists a collusive equilibrium. With capacity constraint, even if the monopoly price cannot be supported, a lower price may be supported in collusive equilibrium. This is because the proportion of market share that a firm can steal by price-cutting decreases as the market price decreases.

6 Discussion and Extension

Asymmetric Capacity Constraints

In this study, we examine the impact of asymmetric discounting on collusion. Several studies –such as those by Compte et. al. (2002) and Lambson (1994)– examine the impact of asymmetric capacity constraints on collusion. A general message that emerges from these articles is that asymmetry in capacity constraints makes it more difficult to sustain collusion.

To isolate the effect of asymmetric discounting, we intentionally kept every other part of our model symmetric. This does not mean, however, that we cannot allow for asymmetric technology. We can introduce asymmetric capacity constraints to our model and replicate some of our results in a straightforward way. Let K_i be Firm i's capacity constraint and $R_i(p, K_i) = \frac{\min\{D(p), K_i\}}{D(p)}$. Then, we define Firm i's *effective discount factor* by $\delta_i(p, K_i) = 1 - (1 - \delta_i) R_i(p, K_i)$. Note that a smaller capacity constraint makes the effective discount factor higher. Then it is straightforward to prove the following theorem.

Theorem 6.1. *There exists a p-collusive equilibrium with $p \in (c, p^m]$ if and only if*

$$\frac{\sum_{i \in \mathcal{I}} \delta_i(p, K_i)}{n} \geq \frac{n-1}{n}.$$

[22] It follows easily from the proof that this inequality is in fact necessary and sufficient for the existence of (not necessarily stationary) p-collusive equilibrium.

Whenever a p-collusive equilibrium exists, there exists a stationary one.

This theorem is not as general as our previous theorems –such as Theorem 3.1 and Theorem 5.1– as we restrict attention to p-collusive equilibrium.[23] Nonetheless, this result already has an interesting implication due to the way in which asymmetric discounting and asymmetric capacity constraints interact in our model: *Asymmetric capacity constraints may facilitate collusion when firms differ in their patience.* Intuitively, this is because a smaller capacity would mitigate the incentive problem of an impatient firm. The above theorem shows that collusion is more easily sustained when $\sum_{i \in \mathcal{I}} (1 - \delta_i) R_i(p, K_i)$ is smaller. We can make this number smaller by increasing one unit of capacity of a patient firm and decreasing one unit of capacity of an impatient firm. Thus, a certain kind of asymmetry in capacity constraints may facilitate collusion with asymmetric discounting.

More General Model of Discounting

We assume that discount factors are fixed throughout the game. Another way to model asymmetric discounting would be to assume that discount factors are random and changing over time. It would be interesting to examine whether and how our results could be extended to such a model with a kind of flexible market sharing rule in our model. For example, Dal Bo (2007) considers a repeated Bertrand model where the common discount factor is a i.i.d. random variable across time.[24] We expect that we can obtain results similar to ours, for example, if discount factors are i.i.d. across time and players. As an another example, Bagwell and Staiger (1997) consider a model in which demand growth alternates between fast- and slow-growth phases, according to a Markov process, and prove that collusive prices are weakly procyclical when growth rates are positively correlated through time. A reinterpretation of this growth dynamics is that the discount factor moves between high and low values. It would be interesting to consider a model in which growth phases can differ across firms.[25]

[23] Due to the dependence of $\delta_i(p, K_i)$ on p, it is not straightforward to extend the statement to all collusive equilibria.

[24] Dal Bo (2007) assumes the equal sharing rule.

[25] To model this formally, we need to introduce differentiated products so that each firm has its own market.

We can go further to explicitly model the factors (e.g., imperfection in the credit market) that generate asymmetric discounting endogenously. This would be an interesting and challenging problem, but we expect that our intuition about the trade-off between incentive and efficiency is still robust and valid even in such a model. This topic is left for future research.

Quantity Competition

We can use the Cournot game instead of the Bertrand game as the stage game in our model. If we do so, our threshold result would need to be modified substantially because it depends heavily on the structure of the Bertrand game. We believe, however, that we can obtain a similar efficiency result if the firms can transfer money over time in some way. This is another interesting topic for future research.

7 Conclusion

In this study, we examine collusive behavior among firms with heterogeneous discounting and with or without capacity constraints. We find that it is possible to sustain collusion if and only if the average discount factor is at least as large as $\frac{n-1}{n}$, where n is the number of firms.

We also succeed in characterizing the structure of efficient collusive equilibria with heterogeneous discounting. In any efficient collusive equilibrium, the joint profit must be strictly increasing over time until it reaches the monopoly profit level within finite time and stay there forever. Hence, every efficient collusive equilibrium becomes an efficient collusive equilibrium with the monopoly price (i.e., p^m-efficient collusive equilibrium). In every p^m-efficient collusive equilibrium, every firm (except for the most impatient firm) has no market share initially, gains a large market share and sustains it for a while, then loses some of its market share to more patient newcomers and holds on to the "minimum" level of market share afterward (except for the most patient firm). Every efficient collusive equilibrium converges to the unique stationary efficient collusive equilibrium in the long run, where the equilibrium price is p^m, Firm i's market share is $1 - \delta_i$ for every $i < n$ and $\sum_{i<n} \delta_i - (n-2)$ for Firm n in every period. This implies that the market share of the most impatient firm and the most patient firm tends to be large in the long run.

Our findings have several empirical implications for researchers who aim to estimate the demand function, as well as the marginal cost, using available data.[26] First, as we suggest that there is a unique stationary equilibrium after a finite number of periods, we can avoid equilibrium selection issues that generate identification problems. Second, we predict that there are $n-1$ binding constraints in the long run, and these equality constraints can be employed as overidentifying restrictions to test the model or improve the efficiency of the estimators. Third, we allow heterogeneous discounting and do not impose any functional restriction on the demand function beyond continuity and strict monotonicity; thus, researchers can implement a nonparametric approach without imposing the (widely used) assumption of homogeneous discounting.

Regarding the empirical predictions of our findings, we have suggested a time path for each firm's market share and order of entry that depends on how firms discount future profits. Predictions about this time path could be tested using data that includes a proxy of the discount factors. Narayanan (1985) suggests that the more experienced the manager, the better the precision with which his or her ability is known, and this reduces the manager's incentive to choose short-term projects. Discount factors should then be positively correlated with the manager's experience, which is an observable characteristic of the firm. According to our findings, the length of the dormant period should increase with the manager's experience. For instance, if we observed entry decisions made by colluding firms across different geographic markets, we would expect that the firm with the least experienced manager enters the markets first. Testing this prediction using data is left for future empirical research.

[26] A dataset may include observations of prices, entry decisions (number of stores), and market shares across different geographic markets and time periods.

A Appendix: Proofs

Proof of Theorem 3.1

Suppose that $\frac{\sum_{i\in\mathcal{I}}\delta_i}{n} \geq \frac{n-1}{n}$ holds. Let $\alpha_i' = 1-\delta_i > 0$. Then the inequality implies

$$\begin{aligned}\sum_{i\in\mathcal{I}}\alpha_i' &= n - \sum_{i\in\mathcal{I}}\delta_i \\ &\leq 1.\end{aligned}$$

So we can define $\alpha_i = \alpha_i' + \epsilon_i$ for some $\epsilon_i \geq 0$ for each i in such a way that $\alpha_i, i \in \mathcal{I}$ sums up to 1. Now we show that the market share $\alpha = (\alpha_1, \ldots, \alpha_n)$ can be sustained in a stationary p-collusive equilibrium for any $p \in (c, p^m]$. By definition, α_i satisfies

$$\frac{\alpha_i(p-c)}{1-\delta_i} \geq (p-c)$$

for any $p \geq c$. But this is exactly firm i's one-shot deviation constraint for a p-collusive equilibrium given such profile of market shares.

Next we show that every firm's profit must be always 0 in any equilibrium when $\frac{\sum_{i\in\mathcal{I}}\delta_i}{n} < \frac{n-1}{n}$. To derive a contradiction, assume that there exists a collusive equilibrium price in which the market price in some period is strictly above c. Let $\{p_t\}_{t=1}^{\infty}$ be the sequence of equilibrium prices. Without loss of generality, we suppose that the joint profit is strictly positive in the first period of the game, i.e. $\pi_1 = (p_1 - c)D(p_1) > 0$.

For each $i \in \mathcal{I}$, there exists a bounded nonnegative sequence $\{\eta_{i,t} : t \in \mathbb{N}\}$ defined by $\eta_{i,t} := U_{i,t} - \pi_t$, which is just firm i's incentive constraint with respect to a deviation of price-cutting against p_t. As $U_{i,t} = \pi_{i,t} + \delta_i U_{i,t+1}$, we have that

$$\pi_{i,t} + \delta_i U_{i,t+1} = \pi_t + \eta_{i,t},$$

and therefore

$$\pi_{i,t} + \delta_i\left(\pi_{t+1} + \eta_{i,t+1}\right) = \pi_t + \eta_{i,t},$$

or equivalently,

$$\pi_{i,t} = \pi_t + \eta_{i,t} - \delta_i \left[\pi_{t+1} + \eta_{i,t+1}\right].$$

Adding up this equality across $i \in \mathcal{I}$ and denoting $s_* = \sum_{i\in\mathcal{I}} \delta_i$, we obtain

$$\pi_t = n\pi_t - s_*\pi_{t+1} + \sum_{i\in\mathcal{I}} \left(\eta_{i,t} - \delta_i\eta_{i,t+1}\right),$$

or more shortly,

$$\pi_{t+1} = \frac{n-1}{s_*}\pi_t + \frac{1}{s_*}\sum_{i\in\mathcal{I}} u_{i,t}, \tag{1}$$

where $u_{i,t} = \eta_{i,t} - \delta_i\eta_{i,t+1}$.

By assumption, the coefficient of π_t is larger than 1. Therefore

$$\begin{aligned} \pi_{t+1} &\geq \pi_t + \frac{1}{s_*}\sum_{i\in\mathcal{I}} u_{i,t} \\ &\geq \pi_1 + \frac{1}{s_*}\sum_{i\in\mathcal{I}}\sum_{\tau=1}^{t} u_{i,\tau}, \end{aligned}$$

for every $t \in \mathbb{N}$.

Now consider the series $\sum_{\tau=1}^{t} u_{i,\tau}$ for $i \in \mathcal{I}$, and observe that it can be written as

$$\sum_{\tau=1}^{t} u_{i,\tau} = \eta_{i,1} + (1-\delta_i)\sum_{\tau=2}^{t} \eta_{i,\tau} - \delta_i\eta_{i,t+1}.$$

We know that $\eta_{i,\tau}$ is nonnegative and bounded above. This implies that the series $\sum_{\tau=2}^{t} \eta_{i,\tau}$ must be either unbounded above or converging to a finite nonnegative number. However, if $\sum_{\tau=2}^{\infty} \eta_{i,\tau}$ is unbounded above for some $i \in \mathcal{I}$, then $\sum_{\tau=1}^{t} u_{i,\tau}$ is unbounded above, hence π_t must be unbounded above as $t \to \infty$ by the above inequality. This is a contradiction. Hence $\sum_{\tau=2}^{\infty} \eta_{i,\tau}$ must be finite for all $i \in \mathcal{I}$. Then $\eta_{i,t}$ as well as $u_{i,t}$ must converge to zero for all $i \in \mathcal{I}$. If $\eta_{i,\tau} = 0$ for all $i \in \mathcal{I}$ and $\tau \in \mathbb{N}$, it follows immediately that $\sum_{i\in\mathcal{I}}(\sum_{\tau=1}^{t} u_{i,\tau}) = 0$, hence $\pi_t > 0$ for any $t \geq 1$. On the other hand, suppose that $\eta_{i,t_i} = b_i > 0$ for some $i \in \mathcal{I}$

and $t_i \in \mathbb{N}$. As $\eta_{i,t}$ converges to 0, there exists $T_i > t_i$ such that $|\eta_{i,t}| < b_i(1-\delta_i)/(2\delta_i)$ for all $t \geq T_i$. As a result, we have that

$$\sum_{\tau=1}^{t} u_{i,\tau} \geq \eta_{i,1} + (1-\delta_i)b_i - (1-\delta_i)\frac{b_i}{2} > 0,$$

when $t > T_i$. Hence there is $T \in \mathbb{N}$ such that $\sum_{i\in\mathcal{I}}\sum_{\tau=1}^{t} u_{i,\tau} \geq 0$ for all $t \geq T$. Consequently, $\pi_t \geq \pi_1 > 0$ holds again for every $t \geq T$.

Now we derive a contradiction by showing that π_t explodes. Pick $\tilde{t} \in \mathbb{N}$ such that $\gamma^{\tilde{t}}\pi_1 > 2M$, where $\gamma = \frac{n-1}{s_*} > 1$ and M is any number that is larger than π^m. Because $\mathcal{I}$ is a finite set and $u_{i,t}$ converges to zero for each $i \in \mathcal{I}$, there exists $\tilde{T} \in \mathbb{N}$ (independent of i) such that $\tilde{T} > T$ and $|u_{i,t}| < (s_* M)/(n\tilde{t}\gamma^{\tilde{t}})$ for all $i \in \mathcal{I}$ and $t \geq \tilde{T}$.

The following inequality holds for any $t \geq 1$ as a straightforward implication:

$$\begin{aligned} \pi_{\tilde{T}+t} &= \gamma\pi_{\tilde{T}+t-1} + \frac{1}{s_*}\sum_{i\in\mathcal{I}} u_{i,\tilde{T}+t-1} \\ &\geq \gamma\pi_{\tilde{T}+t-1} - \frac{M}{\tilde{t}\gamma^{\tilde{t}}}, \end{aligned}$$

and by induction, we can prove that

$$\pi_{\tilde{T}+t} \geq \gamma^t\pi_{\tilde{T}} - \sum_{\tau=0}^{t-1}\frac{M}{\tilde{t}\gamma^{\tilde{t}-\tau}},$$

for every $t \in \mathbb{N}$. Finally, after replacing $t = \tilde{t}$ in the previous inequality, we obtain the desired result:

$$\begin{aligned} \pi_{\tilde{T}+\tilde{t}} &\geq \gamma^{\tilde{t}}\pi_{\tilde{T}} - \sum_{\tau=0}^{\tilde{t}-1}\frac{M}{\tilde{t}\gamma^{\tilde{t}-\tau}} \\ &> \gamma^{\tilde{t}}\pi_1 - \sum_{\tau=0}^{\tilde{t}-1}\frac{M}{\tilde{t}} \\ &> 2M - M. \end{aligned}$$

The second inequality follows by $\pi_{\tilde{T}} \geq \pi_1$ and $\gamma > 1$, whereas the last one by $\gamma^{\tilde{t}}\pi_1 > 2M$. And obviously, this is a contradiction because a per-period joint profit cannot exceed M

given any price. ■

Proof of Proposition 4.1

Take any efficient collusive equilibrium. We first note that the joint profit must be strictly positive in every period on the equilibrium path. If $\pi_t = 0$, it is possible to construct a more efficient equilibrium by skipping period t.

Suppose that $\pi_{t+1} \in (0, \pi^m)$ for some $t \geq 1$. We assume $\frac{\pi_t}{\delta_n} > \pi_{t+1}$ for some $t \geq 1$ and derive a contradiction. Define $\mathcal{I}_{t+1} \subset \mathcal{I}$ as follows.

$$\mathcal{I}_{t+1} := \{i \in \mathcal{I} : U_{i,t+1} = \pi_{t+1}\}$$

This is the set of firms with binding incentive constraints in period $t+1$. It consists of at least two firms (otherwise the joint profit could be increased to improve efficiency) and is a proper subset of $\mathcal{I}$ (otherwise the continuation equilibrium from period $t+1$ is Pareto-dominated by any stationary p^m-collusive equilibrium where every firm gains π^m).[27]

First consider the case where there exists Firm $i' \notin \mathcal{I}_{t+1}$ with $\pi_{i',t+1} > 0$. As $\pi_{i,t} \geq \pi_t - \delta_i U_{i,t+1}$ holds by the incentive compatibility constraint in period t, $\pi_{i,t} \geq \pi_t - \delta_n \pi_{t+1} > 0$ holds for all $i \in \mathcal{I}_{t+1}$. Consequently, it is feasible to perturb the profits for firms in $\mathcal{I}_{t+1}$ and Firm i' in the following way : $\pi'_{i,t} = \pi_{i,t} - \delta_i \varepsilon$ and $\pi'_{i,t+1} = \pi_{i,t+1} + \varepsilon$ for $i \in \mathcal{I}_{t+1}$; whereas $\pi'_{i',t} = \pi_{i',t} + \sum_{i \in \mathcal{I}_{t+1}} \delta_i \varepsilon$ and $\pi'_{i',t+1} = \pi_{i',t+1} - (|\mathcal{I}_{t+1}| - 1)\varepsilon$ for i', because $\pi_{i',t+1} > 0$ and $\pi_{t+1} < \pi^m$ (hence the joint profit in period $t+1$ can be increased). Clearly this new allocation is incentive compatible for $\varepsilon > 0$ small enough. Note that $\sum_{i \in \mathcal{I}_{t+1}} \delta_i > |\mathcal{I}_{t+1}| - 1$ holds because $\sum_{i \in \mathcal{I}} \delta_i \geq n - 1$ holds by Theorem 3.1 and $\delta_i < 1$. Hence this allocation Pareto-dominates the initial one. This is a contradiction.

Next consider the case where there does not exist such i'. Then $\pi_{t+1} = \sum_{i \in \mathcal{I}_{t+1}} \pi_{i,t+1}$

[27] This is why the equilibrium price must be always p^m in any efficient collusive equilibrium with two firms. See the discussion after Proposition 4.2.

must hold. Therefore, there is $j \in \mathcal{I}_{t+1}$ such that $\pi_{j,t+1} > (1-\delta_j)\pi_{t+1}$, otherwise,

$$\pi_{t+1} = \sum_{i \in \mathcal{I}_{t+1}} \pi_{i,t+1} \leq \pi_{t+1} \sum_{i \in \mathcal{I}_{t+1}} (1-\delta_i) = \pi_{t+1}(|\mathcal{I}_{t+1}| - \sum_{i \in \mathcal{I}_{t+1}} \delta_i),$$

but this is smaller than π_{t+1} as $\sum_{i \in \mathcal{I}_{t+1}} \delta_i > |\mathcal{I}_{t+1}| - 1$, a contradiction. As a result, $\pi_{t+2} \leq U_{j,t+2} = \frac{U_{j,t+1} - \pi_{j,t+1}}{\delta_j} < \pi_{t+1}$. The first inequality is derived from the incentive constraint in period $t+2$, whereas the second one from the fact that $\pi_{j,t+1} > (1-\delta_j)\pi_{t+1}$ and $U_{j,t+1} = \pi_{t+1}$. Then, $\pi_{t+1} > \pi_{t+2}$. Note that can proceed in a similar manner to obtain $\pi_{t+k} > \pi_{t+k+1}$ for $k \geq 2$, otherwise we would obtain the first kind of contradiction. Then the profits of all firms can be scaled up by the same factor from period $t+1$ without disturbing any incentive constraint. This contradicts the efficiency assumption. Hence $\frac{1}{\delta_n} \leq \frac{\pi_{t+1}}{\pi_t}$ must hold whenever $\pi_{t+1} < \pi^m$. As the joint profit must be strictly positive in period 1 (and in every period), this establishes a lower bound of the growth rate of the joint profits before they reach the monopoly profit. Therefore, for any efficient collusive equilibrium, we can find period T after which the joint profit is always the monopoly profit.

Finally we prove that this T is bounded across all efficient collusive equilibria. For any given T, the joint profit is at most $\delta_n^{T'-1}\pi^m$ for the first $T - T'$ periods for any $T' \leq T$ due to the above result. Hence, given T, firm i's total discounted profit is bounded above by $\frac{1-\delta_i^{T-T'}}{1-\delta_i}\delta_n^{T'-1}\pi^m + \delta_i^{T-T'}\frac{\pi^m}{1-\delta_i}$. If T is large, we can take large enough T' to make this value smaller than π^m for any i. Such profit profile is Pareto-dominated by any stationary p^m-collusive equilibrium. ■

Proof of Proposition 4.2

Consider any efficient collusive equilibrium where every firm's total discounted payoff is at least as large as π^m. Suppose that the equilibrium price is strictly below the monopoly price in period 1. Then every firm's incentive constraint is not binding in period 1 as $\pi^m > \pi_1$. This is a contradiction because then it is possible to increase the joint profit and make every firm better off without violating the incentive constraint of any firm. It is clear that the equilibrium price should not exceed the monopoly price in period 1. Hence the equilibrium price must be the monopoly price in period 1 in any such efficient collusive equilibrium. Then Proposition 4.1 implies that the price must be always p^m along the equilibrium path. ■

Proof of Theorem 4.1

We prove this theorem through a series of lemmata.

Lemma A.1. *For any p^m-efficient collusive equilibrium, if firm i's incentive constraint is not binding in period $t > 1$, then $\pi_{j,t-1} = 0$ for every $j > i$.*

Proof. Suppose not, i.e. there exists some p^m-efficient collusive equilibrium where firm i's incentive constraint is not binding in period $t > 1$ and $\pi_{j,t-1} > 0$ for some $j > i$. Then there is a period $t' \geq t$ such that firm i's incentive constraint is not binding for $t, t+1, \ldots, t'$ and $\pi_{i,t'} > 0$. We can find such t', otherwise we have $\pi_{i,t} = \pi_{i,t+1} = \cdots = 0$ (if $\pi_{i,t} = 0$, then $U_{i,t+1} > \frac{\widehat{\pi}_i}{1-\delta_i}$ hence i's incentive constraint in period $t+1$ is not binding. If $\pi_{i,t+1} = 0$, then $U_{i,t+2} \ldots$). Such a path is not sustainable.

Now perturb the profit of firms i and j as follows:

$$\begin{aligned}
\pi'_{i,t-1} &= \pi_{i,t-1} + \varepsilon, \\
\pi'_{j,t-1} &= \pi_{j,t-1} - \varepsilon, \\
\pi'_{i,t'} &= \pi_{i,t'} - \varepsilon', \\
\pi'_{j,t'} &= \pi_{j,t'} + \varepsilon'.
\end{aligned}$$

We are basically exchanging firm j's market share in period $t-1$ with firm i's market share in period t', keeping every other firm's profit at the same level. As $\delta_i < \delta_j$, $\pi_{j,t-1} > 0$ and $\pi_{i,t'} > 0$, we can pick $\varepsilon, \varepsilon' > 0$ so that firm j's continuation payoff in every period increases and firm i's continuation payoff in period $t-1$ increases. So this allocation Pareto-dominates the original one. Firm j's incentive constraints are clearly satisfied. Firm i's incentive constraints in period $t-1$ is satisfied by construction. We can take $\varepsilon, \varepsilon' > 0$ small enough so that firm i's incentive constraint from period t to t' is still not binding. So we can construct a more efficient collusive equilibrium in this case, which is a contradiction. □

Lemma A.2. *For any p^m-efficient collusive equilibrium, if $\pi_{i,t} < \widehat{\pi}_i$, then $\pi_{j,t} = 0$ for every $j > i$.*

Proof. If $\pi_{i,t} < \widehat{\pi}_i$, then $U_{i,t+1} > \frac{\widehat{\pi}_i}{1-\delta_i}$. Hence firm i's incentive constraint is not binding in period $t+1$. Then $\pi_{j,t} = 0$ for every $j > i$ by Lemma A.1. □

Lemma A.3. *For any p^m-efficient collusive equilibrium, (a): $\pi_{1,t} \geq \widehat{\pi}_1$ for every $t \geq 1$ and (b): firm 1's incentive constraint is binding in period t' if and only if $\pi_{1,t'+k} = \widehat{\pi}_1$ for any $k = 0, 1, 2, \ldots$.*

Proof. If $\pi_{1,t} < \widehat{\pi}_1$ for any t, then $\pi_{j,t} = 0$ for every $j = 2, 3, \ldots, n$ by Lemma A.2, which contradicts $\sum_i \pi_{i,t} = \pi^m$. This proves (a).

Firm 1's incentive constraint is binding in period t' if and only if $U_{1,t'} = \pi^m = \frac{\widehat{\pi}_1}{1-\delta_1}$. Given (a), this holds if and only if $\pi_{1,t'+k} = \widehat{\pi}_1$ for $k = 0, 1, 2, \ldots$. □

The next lemma shows that, by induction, a similar property holds for every firm.

Lemma A.4. *For any p^m-efficient collusive equilibrium, suppose that $\pi_{h,t+k} = \widehat{\pi}_h$ for every $k = 0, 1, 2, \ldots$ and every $h = 1, 2, \ldots, i$ for some t and some $i < n$. Then (a): $\pi_{i+1,t+k} \geq \widehat{\pi}_{i+1}$ for every $k = 0, 1, 2, \ldots$ and (b): firm $i+1$'s incentive constraint is binding in period $t' \geq t$ if and only if $\pi_{i+1,t'+k} = \widehat{\pi}_{i+1}$ for every $k = 0, 1, 2, \ldots$.*

Proof. Suppose that $\pi_{i+1,t+k} < \widehat{\pi}_{i+1}$ for any k. Then $U_{i+1,t+k+1} > \frac{\widehat{\pi}_{i+1}}{1-\delta_{i+1}}$. Hence firm $i+1$'s incentive constraint is not binding in period $t+k+1$. Then $\pi_{j,t+k} = 0$ for every $j > i+1$ by Lemma A.1. However, $\sum_h \pi_{h,t+k} = \sum_{h=1}^{i+1} \pi_{h,t+k} = \sum_{h=1}^{i} \widehat{\pi}_h + \pi_{i+1,t+k} < \sum_{h=1}^{i+1} \widehat{\pi}_h < \pi^m$, which is a contradiction. This proves (a).

As for (b), Firm $i+1$'s incentive constraint is binding in period $t' \geq t$ if and only if $U_{i+1,t'} = \frac{\widehat{\pi}_{i+1}}{1-\delta_{i+1}}$. As in Lemma A.3, this holds if and only if $\pi_{i+1,t'+k} = \widehat{\pi}_{i+1}$ for every $k = 0, 1, \ldots$. □

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. By Lemma A.2, if firm i does not enter the market in the first period, then every firm $j > i$ does not enter the market. So suppose that firm $1, 2, \ldots, h^1$ enters the market in the first period. If $h^1 = 1$, firm 1 occupies the whole market and gains π^m. If $h^1 \geq 2$, Lemma A.1 and Lemma A.3 implies that firm 1's profit must be $\widehat{\pi}_1$ forever from the second period on. Then, by induction, Lemma A.1 and Lemma A.4 implies that firm i's profit must be $\widehat{\pi}^i$ forever from the second period on for every $i < h^1$. Then $\pi_{i,1} \geq \widehat{\pi}_i$ must hold for every $i < h^1$ in the first period for their incentive constraints to be satisfied in period 1. Of course $\sum_{i \leq h^1} \pi_{i,1} = \pi^m$. This corresponds to the case in which $t_1 = t_2 = \ldots = t_{h^1} = 1$.

Let t_2 be the first period when any firm $i > h^1$ enters the marker. Between period $1(= t_1)$ and period t_2, firm $i < h^1$ gains $\widehat{\pi}_i$ every period and firm h^1 gains the rest: $\pi^m - \sum_{i<h^1} \widehat{\pi}_i (> \widehat{\pi}_{h^1})$. In period t_2, firm $h^1 + 1, \ldots, h^2$ enters the market by Lemma A.2. Lemma A.1 and Lemma A.4 implies that firm i's profit must be $\widehat{\pi}_i$ forever from period $t_2 + 1$ for $h^1 \leq i < h^2$. Hence, as in the previous step, these firms' profit in period t_2 must satisfy $\pi_{i,t_2} \geq \widehat{\pi}^i$ and $\sum_{i<h^1} \widehat{\pi}_i + \sum_{h^1 \leq i \leq h^2} \pi_{i,t_2} = \pi^m$. This corresponds to the case in which $t_{h^1+1} = \ldots = t_{h^2} = t^2$.

This argument can be repeated until Firm n enters the market, after which firm i's profit is always $\widehat{\pi}_i$ for any $i < n$ and Firm n's profit is always $\pi^m - \sum_{i<n} \widehat{\pi}_i$. This proves Property 1-4. Property 5 ($U_{i,1} \geq \pi^m$) is just the incentive constraint in the first period, so it must hold for any firm i in equilibrium.

Conversely, suppose that there exist $(t_1, t_2, \ldots, t_n)$ and a sequence of profit profiles $\pi_{i,t}$ that satisfy Property 1-5 and $\pi^m = \sum_i \pi_{i,t}$ for all t. We know that firm i's incentive constraint is binding for any $t \geq t_{i+1} + 1$. It is also clear that firm i's incentive constraint for any $t \leq t_{i+1}$ follows from the incentive constraint in the 1st period (Property 5) because $U_{i,1}$ is the smallest among $U_{i,t}$ for $t \leq t_{i+1}$. Hence this sequence of profit profiles is generated by a p^m-collusive equilibrium.

Finally, we show that this equilibrium is efficient. By Proposition 4.2, if there exists a more efficient equilibrium, it must be a p^m-efficient collusive equilibrium. So we just need to check that there is no Pareto-superior p^m-efficient collusive equilibrium. Suppose that there exists a Pareto-superior p^m-efficient collusive equilibrium, which must satisfy Property 1-5.

Let $(\widetilde{t}_1, \widetilde{t}_2, \ldots, \widetilde{t}_n)$ be the corresponding critical periods and $\widetilde{\pi}_{i,t}$ be the associated sequence of profit profiles. As this equilibrium is more efficient, Firm 1's total discounted profit must be weakly larger. Then it must be the case that either $t_2 < \widetilde{t}_2$ (i.e. Firm 2 enters the market later) or $t_2 = \widetilde{t}_2$ and $\pi_{1,t_2} \leq \widetilde{\pi}_{1,t_2}$. Similarly, either $t_3 < \widetilde{t}_3$ or $t_3 = \widetilde{t}_3$ with $\pi_{2,t_3} \leq \widetilde{\pi}_{2,t_3}$ must hold for Firm 2's profit to be as large as before. Note that if the inequality holds as an equality for the latter case, then both $t_2 = \widetilde{t}_2$ and $\pi_{1,t_2} = \widetilde{\pi}_{1,t_2}$ must hold for Firm 2 to be weakly better off. This argument can be repeated for Firm $3, \ldots, n$. That is, either $t_i < \widetilde{t}_i$ or $t_i = \widetilde{t}_i$ with $\pi_{i-1,t_i} \leq \widetilde{\pi}_{i-1,t_i}$ must hold for every $i = 2, ..., n$. Furthermore, $t_{i+1} = \widetilde{t}_{i+1}$ with $\pi_{i,t_{i+1}} = \widetilde{\pi}_{i,t_{i+1}}$ implies $t_i = \widetilde{t}_i$ with $\pi_{i-1,t_i} = \widetilde{\pi}_{i-1,t_i}$ for any $i < n$.

Suppose that $t_n < \widetilde{t}_n$. Then clearly Firm n is strictly worse off, hence a contradiction. Next suppose that we have $t_n = \widetilde{t}_n$ with $\pi_{n-1,t_n} \leq \widetilde{\pi}_{n-1,t_n}$. As Firm n must be weakly better off, $\pi_{n-1,t_n} = \widetilde{\pi}_{n-1,t_n}$ holds. Then $t_i = \widetilde{t}_i$ with $\pi_{i-1,t_i} = \widetilde{\pi}_{i-1,t_i}$ must hold for $i = 2, \ldots, n-1$ as observed above, hence every firm's profit is unchanged. This is another contradiction. Therefore, there does not exist any Pareto-superior p^m-efficient collusive equilibrium. □

Proof of Theorem 5.1

The proof is similar to the proof of Theorem 3.1, so we just point out where the proof needs to be modified. First, suppose that the inequality is satisfied for some $p \in (c, p^m]$. Define $\alpha_i' = (1-\delta_i)R(p, K) \geq 0$. Then the inequality implies $\sum_i \alpha_i' \leq 1$. Thus we can find a market share vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \geq (1-\delta_i)R(p, K)$ for each i. This is identical to the following one shot deviation constraint of firm i as before:

$$\frac{\alpha_i(p-c)D(p)}{1-\delta_i} \geq (p-c)\min\{D(p), K\}.$$

Hence we can construct a p-collusive equilibrium with the stationary market share $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Next we show that there does not exist any collusive equilibrium with uniform price when $\frac{\sum_{i\in\mathcal{I}}\delta_i}{n} < \frac{n-\frac{1}{R(p,K)}}{n}$ for every $p \in (c, p^m]$. Suppose that there exists such an equilibrium and let $\{p_t\}_{t=1}^{\infty}$ be the sequence of equilibrium prices. Without loss of generality, we can assume that $p_t \in [c, p^m]$ for every t and $p_1 > c$.

This time $\eta_{i,t}$ can be defined as $\eta_{i,t} := U_{i,t} - \pi^*(p_t)$, where $\pi^*(p_t) = (p_t - c)\min\{D(p_t), K\} = \pi(p_t)R(p_t, K)$ is the maximum deviation gain given p_t and may be smaller than $\pi(p_t) = (p_t - c)D(p_t)$ due to the capacity constraint.

Following the same steps as in the proof of Theorem 3.1, we obtain the following equation:

$$\pi(p_{t+1})R(p_{t+1}, K) = \frac{n - \frac{1}{R(p_t,K)}}{s_*}\pi(p_t)R(p_t, K) + \frac{1}{s_*}\sum_{i\in\mathcal{I}} u_{i,t},$$

where $u_{i,t} = \eta_{i,t} - \delta_i\eta_{i,t+1}$. This reduces to equation (1) if $R(p_t, K)$ is always one, i.e. there is no capacity constraint.

By assumption, $\frac{n-\frac{1}{R(p,K)}}{s_*} > 1$ for every $p \in (c, p^m]$, hence $\frac{n-\frac{1}{R(c,K)}}{s_*} \geq 1$ by the continuity of R. As before, we can show that there is $T \in \mathbb{N}$ such that $\sum_{i\in\mathcal{I}}(\sum_{j=1}^{t} u_{i,t}) \geq 0$ for all $t \geq T$, hence $\pi(p_t)R(p_t, K) \geq \pi(p_1)R(p_1, K) > 0$ for every $t \geq T$. As $p_t \in [c, p^m]$ and $\pi(p)R(p, K)$ is increasing in $p \in [c, p^m]$, it follows that $p_t \geq p_1 > c$ for every $t \geq T$. As $\frac{n-\frac{1}{R(p_1,K)}}{s_*} > 1$, $\frac{n-\frac{1}{R(\tilde{p}_t,K)}}{s_*}$ is bounded below by some $\gamma > 1$ across all $t \geq T$. Thus we have the following inequality for $t \geq T$:

$$\pi(p_{t+1})R(p_{t+1}, K) \geq \gamma\pi(p_t)R(p_t, K) + \frac{1}{s_*}\sum_{i\in\mathcal{I}} u_{i,t}. \tag{2}$$

Then we can derive a contradiction by showing that $\pi(p_t)R(p_t, K)$ (instead of π_t) explodes as t goes to infinity, using (2), as we did in the proof of Theorem 3.1. ■

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