Exchange Rates Under Robustness: An Account of the Forward Premium Puzzle

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Abstract

We characterize conditions under which a desire for robustness against model misspecification may account for the forward premium puzzle. We consider a setup where optimizing agents, who hold no misperception about the model, distort their forecasts to attain robustness against potential misspecification. In general, the robust forecast distortion is data-dependent. However, a clear bias arises in an empirically relevant case to the major currency markets. We prove that, with probability approaching one, there is forecast under-reaction if the interest rate differential is close to a random-walk and there is structured uncertainty about the volatility of the differential. Using approximate analytical solutions we show that in equilibrium, this forecast under-reaction translates into a delayed response of the exchange rate to interest rate shocks, which gives rise to a negative unconditional correlation between interest rate differential shocks and exchange rate changes, i.e., a negative Fama coefficient. We calibrate our model with empirical estimates of key parameters and generate a negative Fama coefficient under sufficient uncertainty-aversion.

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1. Introduction

We characterize conditions under which a desire for robustness against model misspecification may account, in a probabilistic sense, for the forward premium puzzle (FPP). That is, the tendency for the domestic currency to appreciate when domestic interest rates are higher than abroad. This puzzling pattern generates predictable excess returns as investors can pocket, on average, both the interest rate differential as well as the subsequent gains from appreciation.\(^1\)

Using survey data across major economies, Gourinchas and Tornell (2004) find that forecasts of interest rate differentials systematically under-react to news, and show that such distorted beliefs have the potential to account for a negative Fama coefficient. In this paper, we provide a microfoundation for such distorted beliefs. Specifically, we ask when does a desire for robustness lead optimizing agents—that hold no misperception—to distort the probability distribution of the data-generating process and to make forecasts of interest rate differentials that tend to under-react to news. We then investigate when under-reaction leads to a gradual adjustment of exchange rates to interest rate shocks and generates a negative correlation between the interest rate differential and exchange rate changes—i.e., a negative Fama coefficient.\(^2\)

We consider a setup where the differential between the interest rates of two currencies is driven by a persistent hidden state and by transitory observational noise.\(^3\) Investors are endowed with a baseline model of this process, but fear misspecification. It turns out that simply invoking robustness against any type of misspecification does not generate under-reaction. Rather, it is necessary to specify the aspects of the model that agents fear are misspecified. This requires a more refined description of model uncertainty than that contained in the so-called unstructured uncertainty.

We find that the FPP may arise only if investors fear misspecification in the equation that links noisy observations and the unobservable persistent component of the interest

\(^1\)A negative Fama coefficient is referred to in the literature a the strong form of the FPP. A Fama coefficient lower than one is the weak FPP. In this paper we will refer to the strong FPP simply as FPP. Engel (1996) and Chinn (2006) survey the literature.

\(^2\)There are several mechanisms that have been proposed to account for the FPP. Some work through the forecasts, while some work through the risk premium. The first mechanism is the focus of this paper as well as Kaminski (2003), Evans and Lyons (2002), and Gourinchas and Tornell (2004), among others.

\(^3\)This state-space setup is similar to that considered by Bansal and Yaron (2004).
rate differential, but not under either state-equation or unstructured uncertainty. Our main result is that if there is observational uncertainty and the interest rate differential is close to a random-walk, then with probability approaching one there is under-reaction and the Fama coefficient tends on average to be negative, provided uncertainty-aversion is high enough. These conditions hold in the sample of major currencies of Gourinchas and Tornell (2004).4

To give some intuition for why under-reaction of robust forecasts may arise under observational uncertainty, notice that the robust investor solves her forecasting and her portfolio problems jointly, not separately as in standard rational expectations models. In doing so, the investor asks: if the model were misspecified, what would be the costliest direction in which things could go wrong. She then weighs the benefits of dampening the effects of misspecification in this costliest direction against the costs of moving away from optimality under the baseline model. This trade-off is weighted by her degree of uncertainty aversion. In our setup, model misspecification affects the payoff through two channels: (i) the variance of excess returns and (ii) the mean of excess returns. Under observational uncertainty, the variance channel implies that the costliest misspecification occurs when the investor’s model incorrectly assigns a lower importance to transitory observation shocks than the importance they actually have. Thus, robustness against this channel implies an upward distortion of the variance of transitory shocks, which in turn generates forecast under-reaction. In contrast, the mean channel has an indeterminate sign as it is data-dependent: there is under-reaction if the news is favorable to the portfolio; there is overreaction if the news is unfavorable.5

Our main result is that if the interest rate differential is close to a random-walk, the variance channel dominates the mean channel in a probabilistic sense. Thus, under observational uncertainty, the sensitivity of robust forecasts to news converges in probability to a level which is lower than the sensitivity of the baseline Kalman filter. In other words, there is under-reaction with probability approaching one. In equilibrium, this forecast under-reaction translates into a delayed response of the exchange rate to interest rate shocks, which gives rise to a negative unconditional correlation between

4Papers that have used robust control to analyze macroeconomic issues include Brock, Durlauf and West (2007), Hansen, Sargent, Turnuhambetova and Williams (2006), and Svensson and Williams (2007), among others.
5This channel has been emphasized by Epstein and Schneider (2008) in a stock-pricing context.
interest rate differential shocks and exchange rate changes. That is, a negative Fama coefficient.\textsuperscript{6,7}

What about other classes of misspecification? When there is structured uncertainty about the unobservable persistent component of the interest rate differential, the variance channel alluded to above leads to understating the relative importance of transitory observational shocks. As a result, if a systematic distortion were to arise, it would be an overreaction, and so one cannot account for the FPP. We also consider the unstructured uncertainty case, under which investors fear misspecification of the entire interest rate differential process, but cannot pinpoint either its nature or location. The result is surprising: the sensitivity of robust forecasts equals that of the Bayesian forecasts under the baseline model. Thus, the forecast-distortion mechanism underlying the FPP is not operative. This is because the joint forecasting-portfolio problem can be separated into two independent problems, via the so-called Representation Lemma, and so the forecasting problem reduces to a standard Bayesian problem under the baseline model.

The structure of the paper is as follows. In Section 2, we present an intuitive overview. In Section 3, we present the model. In Section 4, we derive the equilibrium exchange rate. In Section 4.5, we characterize the conditions under which observation uncertainty may generate the anomalies and show simulation results. In Section 5, we consider other types of model uncertainty. In Sections 6 and 7, we present a literature review and the conclusions, respectively. Finally, the proofs can be found in the appendix at the end of this paper and in an extended appendix found on our websites.

2. Overview

This section presents a non-technical roadmap that connects our results with the international finance literature and can be skipped.

The FPP is a violation of the familiar uncovered interest parity condition. In prin-

\textsuperscript{6}The negative correlation follows because the dollar appreciation is measured as a fall in the dollar-euro exchange rate.

\textsuperscript{7}The \textit{conditional} delayed overreaction—i.e., humped-shaped—response to an unanticipated interest rate shock represents the existence of conditional \textit{momentum} in exchange rates. In the international finance literature it is known as \textit{delayed overshooting} and has been documented by Eichenbaum and Evans (1995)). It stands in contrast to Dornbusch’s (1976) overshooting path along which there is an immediate appreciation followed by a gradual depreciation.
ciple, one can account for this violation either via time-varying risk premia or via forecasting distortions. Here, we focus on the latter channel. We use an infinite horizon overlapping generations model where the differential between the domestic and foreign interest rate is the source of misspecification. The interest rate differential process, which is given by (3.1), has a transitory component \((v_t)\) that dies out after one period, and a persistent component \((x_t)\) that dies out only gradually. The investor, however, does not observe them separately.

The investor has a baseline model that corresponds to the data generating process. However, she fears misspecification in some parts of the model. If there was no fear of misspecification, the forecasts would be generated by the Kalman filter and the FPP would not arise.

We derive several change-of-measure lemmas, analogous to the Girsanov theorem, that allows us to represent the no-arbitrage condition in terms of a robust uncovered interest parity condition: the log exchange rate \(e_t\) satisfies \(E_t^{\theta^*}(e_{t+1}) - e_t = i_t - i_t^f + \zeta_t\). Although this condition has the same form as in familiar international finance models, there are important differences. Expectations \(E_t^{\theta^*}(\cdot)\) are taken under a robustly distorted probability measure, and the premium \(\zeta_t\) reflects not only risk-aversion but also aversion to model uncertainty.

The equilibrium exchange rate takes the familiar linear form (Proposition 4.2)

\[
e_t^* = -(i_t - i_t^f) - \sum_{j=1}^{\infty} E_t^{\theta^*}(i_{t+j} - i_{t+j}^f) + \alpha_t^*
\]

The implication of this equation is that the domestic currency appreciates \((e_t^*\) goes down) if there is an increase in current or forecasted interest rate differentials. The summation can be solved in closed-form because in equilibrium the forecasts are given by the sequence \(\{a_t^* x_t^\theta^*\}\), and the estimate of the persistent component of the differential \((x_t^\theta^*)\) can be computed recursively as a weighted average of the interest rate differential observation and the prior (Proposition 4.2)

\[
x_t^{\theta^*} = k^\theta^*[i_t - i_t^f] + [1 - k^\theta^*]a_t^* x_{t-1}^{\theta^*}.
\]

The “gain” \(k^\theta^*\) is endogenously determined and captures the sensitivity of forecasts to news. This gain is the critical determinant of whether or not hump-shaped dynamics
and the FPP arise in equilibrium. We will see that these anomalies can be accounted for only if the robust forecasts are less sensitive to news than the Bayesian forecasts associated with the baseline model, i.e. $k^{\theta^*} < k^{Baseline}$. In contrast, if $k^{\theta^*} \geq k^{Baseline}$, the forecast-distortion mechanism cannot account for the FPP.

It turns out that generating a robust gain $k^{\theta^*}$ lower than $k^{Baseline}$ as the outcome of investors’ optimization is not trivial. When we first tried to tackle this problem we found that simply invoking robustness against any misspecification does not generate $k^{\theta^*} < k^{Baseline}$. The need for refinement of the uncertainty set has lead us to consider three different classes of misspecification of the process (3.1): uncertainty in the observation equation, uncertainty in the persistent component of the differential, and the so called unstructured uncertainty, under which agents fear misspecification of the entire interest rate differential process but cannot pinpoint either its nature or location. We represent these classes of model misspecification via sets of underlying probability measures.

The objective function trades off optimality, under the baseline model, versus robustness against model misspecification, as in Hansen et. al. (2006). It selects in a pessimistic way the probability measure under which expectations are computed. To avoid an extreme worst-case scenario outcome, it includes a penalty for deviations from the baseline model—in addition to the standard primitive utility function. This penalty is a proportion $\lambda$ of the distance between the baseline and the agent’s models, which is captured by the relative entropy between the baseline probability measure and the robust measure used to compute expectations. The parameter $\lambda$ is key because $1/\lambda$ indexes the degree of uncertainty-aversion, which is distinct from the traditional risk-aversion.

For each type of misspecification, the results are derived in three steps. First, we define a set of probability measures that captures that type of misspecification. Second, for each set we establish a one-to-one relationship between sets of probability measures and distortions of probability distributions via change-of-measure lemmas, which are analogous to the Girsanov theorem. These lemmas allow us to convert the optimization problem over unknown probability measures into an optimization over a parameter of a pdf, which is much simpler and allows for closed-form solutions for the forecasts and portfolio strategies. Third, we derive in closed-form the equilibrium exchange rate, the slope coefficient of the Fama regression and the impulse response function to news.

Our main results are the following. First, systematic under-reaction to news (i.e.,
gain condition \( k^\theta < k^{Baseline} \) may arise under observation uncertainty (Proposition 4.2), but not under the other types of uncertainty (Propositions 5.2 and 5.5). Moreover, it arises only in a probabilistic sense provided the interest rate differential (3.1) is highly persistent (Corollary 4.3). Second, the FPP \( \beta^{Fama} < 1 \) obtains if agents are both risk-averse and uncertainty-averse; the strong FPP \( \beta^{Fama} < 0 \) obtains if, in addition, the drift \( a \) in (3.1) is high and uncertainty aversion \( 1/\lambda \) is high, but not too high (Proposition 4.5). Third, there is delayed overshooting—i.e., conditional momentum—under similar conditions as those for \( \beta^{Fama} < 0 \) (Proposition 4.4). Fourth, when there is unstructured uncertainty, robust forecasts have the same sensitivity to news as the Bayesian forecasts associated to the baseline model (Proposition 5.5). Thus, the forecast-distortion mechanism underlying the anomalies is not operative. Lastly, we do some simulations for the case of observational uncertainty, and find that empirical estimates of key parameters in the model generate a negative Fama coefficient provided there is enough uncertainty-aversion.

3. The Model

We consider a setup that incorporates robustness considerations into a minimal nominal exchange rate model. There are two one-period bonds: a dollar bond that will pay \( \exp(i_t) \) dollars next period and a euro bond that will pay \( \exp(i^e_t) \) euros. The source of model misspecification is the interest rate differential, which has two components: observation noise \( (v_t) \) and an unobservable persistent component, that is stationary, and which we will refer to as the state or the trend \( (x_t) \).

\[
\begin{align*}
  i_t - i^e_t &\equiv y_t = x_t + v_t \\
  x_t &= a x_{t-1} + w_{t-1}, \quad x_0 = 0, \quad a \in (0, 1)
\end{align*}
\]  

(3.1)

We allow for both risk and Knightian uncertainty. In the literature, risk refers to the unique probability distribution of the relevant random variables, \( \{w_j\} \) and \( \{v_j\} \) in our case, which agents either know or can learn. Knightian uncertainty refers to a potential misspecification of the model. We introduce both dimensions of uncertainty by assuming that the representative agent is endowed with a baseline model of the interest rate
differential. However, she fears *model misspecification* and takes the baseline model simply as an approximation to the data-generating model.

In order to measure the distance between models in a simple way we represent model uncertainty in terms of sets of *underlying probability measures*. To this end we will call ‘model $\theta'$ a probability measure $\theta$ defined on a measurable space $(\Omega, \mathcal{B}(\Omega))$, where $\Omega$ is a compact metric space and $\mathcal{B}(\Omega)$ is the Borel $\sigma-$algebra on the space $\Omega$. The agent is endowed with a baseline model, which we will denote by $\theta'$, but she takes $\theta'$ simply as an approximation and allows for the possibility that the true model $\theta$ lies in an *uncertainty set* $\Theta$.

In order to obtain the standard Bayes estimator in the special case where agents do not care about robustness, we assume that the baseline model $\theta'$ corresponds to the data-generating process, under which the interest rate differential follows (3.1) and the shock processes are i.i.d. normal random variables

$$w_t \sim N(0, \sigma^2_w), \quad v_t \sim N(0, \sigma^2_v)$$

(3.2)

The robust control literature considers two classes of model uncertainty: structured and unstructured. The former specifies the location and nature of the misspecification. In contrast, under unstructured uncertainty one does not have this refined information. In Section 4, we consider a structured uncertainty set that captures misspecification in the observation equation of the interest rate differential process (3.1). In Section 5 we consider structured uncertainty about the latent state variable in (3.1), as well as unstructured uncertainty. The observation uncertainty set is given by

$$\Theta^u = \left\{ \theta \in P(\Omega) : \frac{d\theta}{d\theta'} = \exp \left(-\frac{1}{2} \left( \frac{1}{\bar{\sigma}^2_v} - \frac{1}{\sigma^2_v} \right) (y_t - x_t)^2 \right) \cdot \sqrt{\frac{\sigma^2_v}{\bar{\sigma}^2_v}}, \bar{\sigma}^2_v \in [\omega, \infty), \omega > 0 \right\}$$

(3.3)

This set of probability measures is generated by letting the parameter $\bar{\sigma}^2_v$, in the Radon-Nikodim derivative, take on values on the positive real line. The lower bound $\omega$ is an arbitrarily small number that ensures the set $\Theta^u$ is closed and convex.\(^8\)

There are overlapping generations of two-period lived investors that can take any long and short positions in the dollar bond and the euro bond described above. A rep-

\(^8\)The proof that the set is closed and convex is presented in Lemma 8.3 in the extended appendix.
resentative young investor is willing to sacrifice profitability under the baseline model in exchange for some degree of robustness against misspecification—within the uncertainty set. We formalize this problem as in Hansen, et. al. (2006) by considering a utility function that pessimistically selects a robust probability measure from the uncertainty set \( \Theta^j \).

\[
U_t = \inf_{\theta \in \Theta^j} \left\{ E_t^\theta \left[ -\exp \left( -\gamma W_{t+1} \right) + \lambda \cdot \mathcal{R}(\theta || \theta') \right] \right\}, \quad \lambda \geq 0, \quad \gamma \geq 0, \quad (3.4)
\]

where \( W_{t+1} \equiv b_t \left[ (i_t - i_t^L) - (e_{t+1} - e_t) \right], \quad e \equiv \log(E) \quad (3.5) \)

\( E \) denotes the dollar-euro exchange rate, i.e., the number of dollars per euro; and \( b_t \) is the young agent’s position in the dollar bond. The bracketed term in \( W_{t+1} \) is the familiar excess rate of return of the carry trade: the dollar-euro interest rate differential minus the dollar’s depreciation rate.\(^9\)

The set \( \Theta^j \) is a closed and convex set of probability measures and \( \mathcal{R}(\theta || \theta') \) is the relative entropy of probability measure \( \theta \) with respect to the baseline measure \( \theta' \):

\[
\mathcal{R}(\theta || \theta') = \begin{cases} 
\int_\Omega \log \left( \frac{d\theta}{d\theta'} \right) d\theta & \text{if } \theta \text{ is absolutely continuous w.r.t. } \theta' \\
\infty & \text{otherwise.} 
\end{cases} \quad (3.6)
\]

The relative entropy can be thought of as the distance between the baseline model \( \theta' \) and the alternative model \( \theta \).

We consider an equilibrium concept analogous to the standard rational expectations equilibrium used in the asset pricing literature. To this end, we endow the representative agent with prior beliefs and with knowledge of the mapping from the agents’ information and beliefs onto the exchange rate (i.e., the ‘conjecture’). The representative young \( t- \)agent’s prior is that the persistent component of the interest rate differential \( (x_{t-1}) \) is Normally distributed with mean \( \hat{x}_{t-1}^\theta \) equal to the \( t-1 \)-agent’s estimate of \( x_{t-1} \), and variance \( \sigma^2_{t-1} = \frac{(a^2\sigma^2_{t-2}\sigma^2_{t}+a^2\sigma^2_{t})\sigma^2_2}{a^2\sigma^2_{t-2}+a^2\sigma^2_{t}+a^2_2} \). The conjecture is

\[
e_t^{conj} = \alpha_t + \beta_1 \hat{x}_{t-1}^\theta + \beta_2 \left( i_t - i_t^L \right), \quad (3.7)
\]

\(^9\)Consider a young representative investor that forms a zero-cost portfolio by taking a position \( b_t \) in dollar bonds and a position \(-\frac{b_t}{E_t} \) in euro bonds. Next period her payoff, in dollar terms, will be \( \exp(i_t) b_t - \frac{E_{t+1}}{E_t} \exp(i_t^L) b_t \). We show in the extended appendix that one obtains \( W_{t+1} \) by taking the Taylor expansion of this expression.
where \( \hat{x}_t^{\theta_t} \equiv E_t^{\theta_t}(x_t) \) is the estimate, under probability measure \( \theta_t \), of the mean of the interest rate differential’s persistent component, conditional on the observation \( y_t \equiv i_t - i_t^f \) and the prior \( \hat{x}_{t-1}^{\theta_{t-1}} \). In the equilibrium, agents know the value of the parameters \( \{ \alpha_t, \beta_1, \beta_2 \} \). At this stage we take them as undetermined coefficients.

During each period \( t \), the representative young \( t \)-agent solves the following portfolio-forecasting problem.

\[
\Gamma_t = \max_{b_t \in \mathcal{R}} \inf_{\theta_t \in \Theta} \left\{ E_t^{\theta_t} \left[ - \exp \left( -\gamma W_{t+1}(b_t) \right) + \lambda \mathcal{R}(\theta_t || \theta') \right] \right\}, \quad I_t = \left\{ e_t, y_t, \hat{x}_{t-1}^{\theta_{t-1}} \right\} \quad (3.8)
\]

That is, given the log exchange rate \( e_t \), the interest rate differential \( y_t \), the prior \( \hat{x}_{t-1}^{\theta_{t-1}} \) and the conjecture (3.7), the \( t \)-agent updates her estimates of \( x_t \) and \( x_{t+1} \) under the robust probability measure \( \theta_t \), that solves (3.8), and chooses her demand \( b_t(I_t, \alpha_t, \beta_1, \beta_2) \).

We close the model by imposing an exogenous supply of domestic bonds \( b_t^s \). We will use the following equilibrium concept.

**Robust Linear Equilibrium** An equilibrium is a stochastic process \( \{ \theta^*_t, b^*_t, e^*_t \} \) for \( t = 1, \ldots, T \), such that, given the log exchange rate \( e_t \) and the interest rate differential \( y_t \), the robust probability measure \( \theta^*_t \) and the bond demand \( b^*_t \) solve the \( t \)-agent’s problem (3.8), and the bond market clears:

\[
b^*_t(e^*_t, \hat{e}_{t+1}, y_t) = b^*_t. \quad (3.9)
\]

Furthermore, the equilibrium exchange rate function \( e^*_t(\hat{e}_{t+1}, y_t) \) coincides with conjecture (3.7).

### 3.1. Discussion of the Setup

The formalization of robustness via a utility function like (3.4) follows the existing macroeconomics robustness literature. The new elements in our setup are the modeling of the uncertainty sets and the derivation of a robust linear exchange rate equilibrium.

The utility function (3.4) distinguishes *risk aversion*—parametrized by \( \gamma \)—from *uncertainty aversion*, which is parametrized by \( 1/\lambda \). As we will show, both *risk aversion* and *uncertainty aversion* are necessary to account for the anomalies. To interpret this utility function we can think of nature as choosing the true probability measure \( \theta \) in a malev-
olent way, so as to minimize the agent’s utility.\textsuperscript{10} Note, however, that by increasing the distance between model \( \theta \) and the baseline model \( \theta' \) proxied by \( \mathcal{R}(\theta||\theta') \) —nature incurs a penalty \( \lambda \cdot \mathcal{R}(\theta||\theta') \).\textsuperscript{11} The greater the uncertainty aversion (\( 1/\lambda \)), the lower the penalty for deviating from the baseline model. In one extreme, if \( 1/\lambda \) were zero, (3.4) would reduce to the familiar expected utility function 

\[ -E_t^{\theta'} \exp (-\gamma W_{t+1}) \]

In this case, the agent would not allow for model misspecification because choosing any model \( \theta \) different from \( \theta' \) would make (3.4) infinite. In the other extreme, if \( 1/\lambda \) was set to infinity, the investor would choose the worst case model among all possible models. In these two extreme cases, robustness against misspecification does not generate the hump-shaped dynamics necessary for the FPP.\textsuperscript{12}

In order to investigate how robustness accounts for the FPP, we need a tractable model with a closed-form solution of the Fama regression coefficient: 

\[ \beta^{Fama} = \text{cov}(\Delta e_{t+1}, i_t - i_t^*) / \text{var}(i_t - i_t^*) \]

To this end we have considered a minimal overlapping generations setup where the \( t \)-agent inherits the \( t-1 \)-agent’s estimate of the unobservable state \( \hat{x}_{t-1}^{\theta_t} \) as her prior. What we gain from using this setup is that the forecast series has a recursive structure, 

\[ \hat{x}_t^{\theta_t} = (1 - k^{\theta_t}) a \hat{x}_{t-1}^{\theta_t} + k^{\theta_t} (i_t - i_t^*) \]

which allows for a closed-form equilibrium exchange rate that is linear in \( (i_t - i_t^*) \) and the robust estimate of the state \( \hat{x}_t^{\theta_t} \): 

\[ e_t^* = -(i_t - i_t^*) - a \hat{x}_t^{\theta_t} + \alpha_t \] (Proposition 4.2). For the purpose of accounting for the anomalies, the key aspect of this setup is that the distortion in the investors’ forecasts is carried over to the equilibrium exchange rate. The tractability of the equilibrium function \( e_t^* \) will permit the derivation in closed-form of \( \text{cov}(\Delta e_{t+1}, y_t) \) and allow us to characterize analytically the conditions under which the FPP arises. Furthermore, since the forecast distortion enters linearly and is parametrized solely by the gain \( k^{\theta_t} \), we can connect with the finding of Gourinchas and Tornell (2004) that the gain \( k \) implied by

\textsuperscript{10}Obviously, nature does not care about our forecasts. This device is simply a useful way to induce robustness.

\textsuperscript{11}We can think of relative entropy as a distance between two probability measures on \( P(\Omega) \), the set of all probability measures on \( \Omega \). It is always non-negative, and it is zero if and only if \( \theta = \theta' \), i.e., \( \theta'(A) = \theta(A) \) for all \( A \in \mathcal{B}(\Omega) \). Moreover, if we consider a new measure \( \eta = \tau \theta + (1 - \tau) \theta' \), then \( \mathcal{R}(\eta||\theta') \leq \mathcal{R}(\theta||\theta') \). Notice that the relative entropy is not a metric on the space \( P(\Omega) \) because it does not satisfy the triangle inequality.

\textsuperscript{12}The function (3.4) is related to the utility function of Gilboa and Schmeidler (1989), who show that in the presence of Knightian uncertainty, preferences can be represented by a utility function of the form: 

\[ U = \inf_{\theta \in \Theta} E^\theta (u) \]

where \( \Theta \) is a closed and convex set of underlying probability measures and \( u \) is a von Neumann-Morgenstern utility over outcomes. Epstein and Schneider (2004) consider a multiperiod setup.
interest rate survey data is lower than that associated with the data generating process, and that this under-reaction can generate a negative $\beta^{Fama}$. One could think of other more complicated setups where the $t$-agent’s prior estimate of the state differs from the $t-1$-agent’s posterior estimate. However, in these setups, $e_t$ would have to include several state estimates, and so we would lose the tractability of $e_t^*$, and the link to empirical estimates of the gain $k$.$^{13}$

The reader might recognize a discrepancy as we have described the intuition in Section 2 via distortions of the gain $k$ (equivalently, misspecifications to the probability distributions of disturbances), while the objective function we consider is defined in terms of underlying probability measures. This former representation is attractive because, arguably, distortions to means and variances of probability distributions can be estimated by an econometrician, they are intuitive, and it is the way in which the empirical evidence is presented in the literature. In contrast, underlying probability measures are abstract, unobservable concepts. However, they are conceptually more convenient because they allow us to treat different types of uncertainty in a unified framework as well as define the agent’s objective concisely. We can do this translation because we establish a one-to-one mapping between probability measures and the gain $k$. For instance, under observational uncertainty every probability measure $\theta \in \Theta^{o}$ can be represented in terms of a specific value of the variance of the observational noise, and there is a one-to-one mapping between this variance and the gain $k$. An attractive property of the uncertainty set $\Theta^{o}$ is that it captures observational uncertainty in a particularly useful way that allows for the derivation of a tractable closed-form solution.

4. Robust Linear Equilibrium Under Observation Uncertainty

We derive the equilibrium in two steps. First, taking conjecture (3.7) as given, we obtain the posterior beliefs and the demand function that solve the representative agent’s

$^{13}$For instance, if the prior equals the baseline Bayesian prior estimate ($\hat{\xi}_t^{\theta'} = a\hat{\xi}_t^{\theta'}$), we show in the last part of extended appendix that the equilibrium must include two state estimates, $e_t^* = -y_t + \beta_1^*\hat{\xi}_t^{\theta'} + \beta_2^*\hat{\xi}_t^{\theta'} + \alpha_t^*$, and the coefficients are quite complicated (i.e., $\beta_1^* = a [\beta_1^*k_{t+1}^{\theta'} + \beta_2^*k_{t+1}^{\theta'} - 1]$ and $\beta_2^* = a[\beta_1^*(1-x_{t+1}^{\theta'}) + \beta_2^*(1-k_{t+1}^{\theta'})]$ instead of $\beta_1^* = \beta_2^* = \frac{a}{1-a}$). Although underreaction carries over to the equilibrium $e_t^*$, in this more complicated setup one cannot obtain the clean closed-form solution for the Fama coefficient as in (4.14) that allows for the clear link between $k$ and the FPP established in Proposition 4.5.
problem (3.8). Then, we derive the exchange rate function that equilibrates the market and that is consistent with conjecture (3.7).

4.1. The Joint Forecasting-portfolio Problem

The investor’s problem (3.8) cannot be solved by applying the well known separation principle, under which the forecasts are derived independently of the portfolio optimization. Instead, we need to consider a joint forecasting-portfolio problem. This problem is rather complicated as the investor must optimize over a set of unknown probability measures. We convert this problem into a parametric problem in which the investor chooses a variance distortion of a probability distribution rather than a robust probability measure. We do this transformation by using change-of-measure Lemma 4.1, which is analogous to the Girsanov theorem.14 For a given random variable, the lemma: (i) establishes a one-to-one mapping between the variance of the probability distribution and the underlying probability measure; and (ii) provides a parametric formula for the relative entropy $\mathbb{R}(\theta||\theta')$.

Lemma 4.1 (Change of Measure I). If under baseline probability measure $\theta'$ the random variables in the interest rate differential process (3.1) are distributed as $x_t|I_{t-1} \sim N(a\hat{x}_{t-1},\sigma^2_u)$, $y_t|x_t \sim N(x_t,\sigma^2_v)$ and $x_{t-1} \sim N(\hat{x}_{t-1},\sigma^2_{t-1})$, then for any probability measure $\theta \in \Theta^v$:

1. The distribution of the observation satisfies $y_t|x_t \sim N(x_t,\bar{\sigma}^2_v)$, while the distributions of $x_t|I_{t-1}$ and $x_{t-1}$ are preserved.

2. The relative entropy of probability measure $\theta$ with respect to the baseline $\theta'$ equals

$$\mathbb{R}(\theta||\theta') = \frac{1}{2} \left( \frac{\bar{\sigma}^2_v}{\sigma^2_v} - \log \left( \frac{\bar{\sigma}^2_v}{\sigma^2_v} \right) - 1 \right) \quad \text{for any } \theta \in \Theta^v \quad (4.1)$$

The first part of Lemma 4.1 says that if under baseline model $\theta'$ the interest rate differential process is given by (3.1), and random variables $v_t$ and $w_t$ are normally distributed as in (3.2), then under an alternative model $\theta \in \Theta^v$ only the variance of the observation equation is distorted from $\sigma^2_v$ to $\bar{\sigma}^2_v$, while the rest of the process remains

14The Girsanov theorem applies to a change of drift. Here we are interested in a change of variance.
unchanged. Equation (4.1) says that the relative entropy between measures \( \theta \) and \( \theta' \) is zero when there is no variance distortion, and that it increases at an increasing rate as the distortion grows in either direction. This equation will prove quite useful because it defines the distance between models only in terms of the variance distortion.

**Solution to the Investor’s Problem.** Using the change-of-measure Lemma 4.1, we show in the appendix that problem (3.8) is equivalent to the following Min-Max problem.

\[
\Gamma_t = \max_{b_t \in \mathbb{R}} \inf_{\tilde{\sigma}_{v,t}^2 > 0} \left[ -\exp \left( -\gamma b_t E_t^{\theta_t} (J_{t+1}) + \frac{(\gamma b_t)^2}{2} Var_t^{\theta_t} (J_{t+1}) \right) + \frac{1}{2} \left( \tilde{\sigma}_{v,t}^2 - \log \frac{\tilde{\sigma}_{v,t}^2}{\sigma_v^2} - 1 \right) \right]
\]

(4.2)

\( J_{t+1} \equiv (i_t - i_t') - (e_{t+1} - e_t) \) is the log excess return, \( E_t^{\theta_t} (J_{t+1}) \) is the conditional mean and \( Var_t^{\theta_t} (J_{t+1}) \) is the conditional variance of \( J_{t+1} \) under probability measure \( \theta_t \in \Theta' \).

The attractive property of (4.2) is that it has converted an optimization problem over unknown probability measures to a much simpler parametric one over the variance distortion \( \tilde{\sigma}_{v,t}^2 \). We solve this joint problem by treating it as a zero-sum game between the investor and nature. The investor makes her estimate of the interest rate differential unobservable component \( \hat{x}_t^{\theta_t} \) and chooses her portfolio \( (b_t) \) accounting for the strategy of nature \( \tilde{\sigma}_{v,t}^2 = s(b_t) \). Conditional on the investor’s choice of \( b_t \), nature chooses \( \tilde{\sigma}_{v,t}^2 \)—equivalently, the probability measure \( \theta_t \)—in a malevolent way. We show in the appendix that the demand for the domestic bond has the standard form

\[
b_t^* = \frac{(i_t - i_t') - (E_t^{\theta_t} e_{t+1} - e_t)}{\gamma \cdot Var_t^{\theta_t} (J_{t+1})}.
\]

(4.3)

The numerator is the expected excess rate of return, and the denominator is the risk aversion coefficient times the variance of returns. A non-standard aspect of the demand is that the moments of returns are computed under the robust probability measure \( \theta_t \), which need not be the same as the baseline measure \( \theta' \), and is characterized in Proposition 4.2.

### 4.2. Equilibrium Exchange Rate

To derive the equilibrium we substitute the investor’s demand for the bond (4.3) in the market clearing condition (3.9) and impose the fixed point condition that the market
Proposition 4.2 (Robust Linear Equilibrium). In the presence of observational uncertainty, i.e., \( \theta_t \in \Theta^u \), a robust linear equilibrium exists if uncertainty aversion \( 1/\lambda \) is below a threshold \( 1/\lambda^u_t \) defined in (8.13).

1. The equilibrium log exchange rate is

\[
e^*_t = - \left( i_t - i_t^f \right) - \frac{a}{1 - a} \tilde{x}_t^{\theta_t} + \alpha_t^*, \quad \alpha_t^* = \alpha_{t+1}^* + \gamma b_t^* \left( a^2 k_{t+1}^{\theta_{t+1}} \sigma_{v,t}^{2*} + \sigma_w^2 + \sigma_{v,t}^{2*} \right) \left( 1 + \frac{a k_{t+1}^{\theta_{t+1}}}{1-a} \right)^2
\]

2. The robust forecast of the interest rate differential \( y_t \equiv i_t - i_t^f \) is given by the Kalman filter under the robust probability measure \( \theta_t^* \)

\[
E_t^{\theta_t^*} (y_{t+1}) = E_t^{\theta_t^*} (x_{t+1}) \equiv \tilde{x}_{t+1}^{\theta_t^*} = a \tilde{x}_t^{\theta_t^*}, \\
\text{with} \quad \tilde{x}_t^{\theta_t^*} = \left( 1 - k_t^{\theta_t^*} \right) a \tilde{x}_{t-1}^{\theta_t^*} + k_t^{\theta_t^*} (i_t - i_t^f).
\]

The ‘gain’ of the robust filter is \( k_t^{\theta_t^*} \equiv \frac{a^2 \sigma_{t-1}^{2*} + \sigma_w^2}{a^2 \sigma_{t-2}^{2*} + \sigma_w^2 + \sigma_{v,t}^{2*}} \), with \( \sigma_{t-1}^2 = \frac{(a^2 \sigma_{t-2}^{2*} + \sigma_w^2) \sigma_w^2}{a^2 \sigma_{t-2}^{2*} + \sigma_w^2 + \sigma_{v,t}^{2*}} \).

3. Under the robust measure \( \theta_t^* \), the distorted observational variance \( \sigma_{v,t}^{2*} \) satisfies

\[
\left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_{v,t}^{2*}} \right) \frac{\lambda}{2} = \left( -\gamma b_t^* \left( y_t - a \tilde{x}_{t-1}^{\theta_t^*} \right) \frac{a}{1 - a} \frac{dk_t^{\theta_t^*}}{d\sigma_{v,t}^{2*}} + \frac{(\gamma b_t^*)^2}{2} \frac{\partial Var_t^{\theta_t^*} (J_{t+1})}{\partial \sigma_{v,t}^{2*}} \right) W_t^*
\]

where

\[
\frac{dk_t^{\theta_t^*}}{d\sigma_{v,t}^{2*}} = - \frac{a^2 \sigma_{t-1}^{2*} + \sigma_w^2}{\left( a^2 \sigma_{t-2}^{2*} + \sigma_w^2 + \sigma_{v,t}^{2*} \right)}, \quad \frac{\partial Var_t^{\theta_t^*} (J_{t+1})}{\partial \sigma_{v,t}^{2*}} = \left( k_{t+1}^{\theta_{t+1}} a \frac{1}{\lambda} + 1 \right)^2 \left( \frac{a^2 (a^2 \sigma_{t-1}^{2*} + \sigma_w^2) \sigma_w^2}{\left( a^2 \sigma_{t-2}^{2*} + \sigma_w^2 + \sigma_{v,t}^{2*} \right)^2} + 1 \right)
\]

4. There is no forecast distortion (i.e., \( \hat{\sigma}_{v,t}^{2*} = \sigma_v^2 \)) if the primitive utility function is risk neutral (\( \gamma = 0 \)), there is no aversion to uncertainty (\( 1/\lambda = 0 \)) or the dollar bond is in zero net supply (\( b_t^* = 0 \)).

The equilibrium exchange rate function (4.4) has a standard form. The domestic currency appreciates—i.e., there is a fall in the number of Dollars per Euro—if there is an increase in the current interest rate differential \( i_t - i_t^f \) or in the discounted value of its forecasts: \( \sum_{i=1}^{\infty} E_{t+i}^{\theta_t^*} (y_{t+i}) = \sum_{i=1}^{\infty} \tilde{x}_{t+i}^{\theta_t^*} = \sum_{i=1}^{\infty} a^i \tilde{x}_t^{\theta_t^*} = \frac{a}{1-a} \tilde{x}_t^{\theta_t^*} \). Part 2 shows that

\[\text{the third term } \alpha_t^* \text{ can be interpreted as the long-run exchange rate. In order for } \alpha_t^* \text{ to be bounded it is necessary that } \lim_{t \to \infty} \gamma b_t^* Var_t^{\theta_t^*} (J_{t+1}) = 0.\]
the forecasts have the same form as the celebrated Kalman filter, which gives a weight \( k_t^{\theta} \) to the current interest rate differential and weight \( 1 - k_t^{\theta} \) to the prior estimate. The robust gain \( k_t^{\theta} \), however, need not be the same as the standard Bayesian gain \( k_t^{\theta'} \) associated with the baseline model \( \theta' \). This is because the variance of the observation noise is distorted under the robust probability measure \( \theta_t^* \): \( \sigma_{v,t}^{2*} \neq \sigma_v^2 \), as shown in part 3.

Part 4 shows that both risk aversion and uncertainty aversion are necessary for distorted forecasts. In particular, a desire for robustness does not generate the anomalies if investors have a risk-neutral primitive utility function. To see this note that if \( \gamma = 0 \), the RHS of (4.6) equals zero.

For the purpose of accounting for the anomalies, two key properties of the equilibrium are that: (i) the abstract robust probability measure \( \theta_t \) does not enter (4.4) directly, but only through the distorted variance \( \tilde{\sigma}_{v,t}^{2*} \); and (ii) the implied forecast distortion in (4.5) is carried over to the equilibrium exchange rate. This transformation will prove useful because it allows us to describe the robust interest parity condition simply in terms of distorted variances, which can be linked to the empirical literature on the FPP. The reason for this simplicity is that there is a one-to-one map between \( \tilde{\sigma}_{v,t}^{2*} \) and \( \theta_t \) established by Lemma 4.1.

Finally, notice that the observational variance distortion grows as uncertainty aversion \( 1/\lambda \) increases. However, for an equilibrium to exist, \( 1/\lambda \) must remain below the upper bound \( 1/\lambda^*_v \). There are two reasons for this: (i) the objective function is convex at \( \tilde{\sigma}_{v,t}^{2*} \) only if \( 1/\lambda \) is lower than a bound \( 1/\lambda^*_v \); and (ii) (4.6) generates a non-negative \( \tilde{\sigma}_{v,t}^{2*} \) provided \( 1/\lambda \) is lower than a bound \( 1/\lambda^#_v \). The bound \( 1/\lambda^v = \min\{1/\lambda^*_v, 1/\lambda^#_v\} \) is the smaller of these two bounds, where \( \lambda^*_v \) and \( 1/\lambda^#_v \) are defined in the appendix by (8.11) and (8.12) respectively.

### 4.3. Distorted Forecasts

As we will show in Section 4.5, the anomalies arise in equilibrium only if there is under-reaction to news (i.e., \( k_t^{\theta} < k_t^{\theta'} \)). Under observational uncertainty, under-reaction arises only if the robust variance is distorted upwards \( \tilde{\sigma}_{v,t}^{2*} > \sigma_v^2 \).\(^{16}\) In general, however,

\(^{16}\)Setting \( \tilde{\sigma}_{v,t}^{2*} > \sigma_v^2 \) means that the robust agent considers a probability density function for \( y_t \) with fatter tails, but the same mean, as the distribution under the baseline model.
the direction of the equilibrium distortion of the observational variance $\tilde{\sigma}_{v,t}^2$ is data-dependent, as we can see in (4.6). Here, we discuss the intuition for this indeterminacy and show that it disappears in a probabilistic sense if the interest rate differential is close to a random-walk (Corollary 4.3).

In designing her strategy the robust investor asks: if things were to go wrong, what would be the costliest direction in which they could go wrong? She then trades off the benefits of dampening the effects of misspecification in this costliest direction, against the costs of moving away from ‘optimality under the baseline model.’ This trade-off is weighted by the degree of uncertainty aversion $1/\lambda$.

Under observation uncertainty, the robust variance $\tilde{\sigma}_{v,t}^2$ affects the investor’s payoff in two ways: via the conditional variance and via the conditional mean of returns. We consider each effect separately. Since the conditional variance of returns is strictly increasing in $\tilde{\sigma}_{v,t}^2$, it is more costly from the investor’s perspective to mistakenly set the robust observation variance $\tilde{\sigma}_{v,t}^2$ too low ($\tilde{\sigma}_{v,t}^2 < \sigma_v^2$) than to mistakenly set $\tilde{\sigma}_{v,t}^2$ too high ($\tilde{\sigma}_{v,t}^2 > \sigma_v^2$). Therefore, if we disregard for a moment the effect of $\tilde{\sigma}_{v,t}^2$ on the conditional mean of returns, robustness against observation uncertainty entails distorting upwards the robust variance ($\tilde{\sigma}_{v,t}^2 > \sigma_v^2$).

How far shall the robust investor distort the robust variance $\tilde{\sigma}_{v,t}^2$? If the investor’s problem imposed hard bounds on the observational variance (i.e., $\sigma_{v,t}^2 \in [\tilde{\sigma}_{v,t}^2, \tilde{\sigma}_{v,t}^2]$), then the solution would be the upper-bound: $\tilde{\sigma}_{v,t}^2 = \sigma_v^2$. Instead of imposing hard bounds on $\tilde{\sigma}_{v,t}^2$, Problem (3.8) penalizes the fictitious malevolent nature for choosing a probability measure $\theta$ different from the baseline $\theta' : \lambda \cdot \mathcal{R}(\theta||\theta')$. This penalty grows as the distorted variance moves away in either direction from the baseline. Since the conditional variance of returns (i.e., the benefit to nature) grows with $\tilde{\sigma}_{v,t}^2$ too, the distorted variance is set at some $\tilde{\sigma}_{v,t}^2$ above the baseline $\sigma_v^2$, such that the marginal increase in $\lambda \cdot \mathcal{R}(\theta||\theta')$ is

\[\frac{\partial \mathcal{V}_{\alpha,\theta}(J_{t+1})}{\partial \tilde{\sigma}_{v,t}^2} = \left(\frac{\alpha^2}{\sigma_v^2} - 1\right) > 0.\]

\[\frac{\partial \mathcal{R}(\theta||\theta')}{\partial (\tilde{\sigma}_{v,t}^2)} = \left(\frac{\sigma_v^2}{\tilde{\sigma}_{v,t}^2}\right)^2 > 0.\]

\[\frac{\partial^2 \mathcal{R}(\theta||\theta')}{\partial (\tilde{\sigma}_{v,t}^2)^2} = \frac{1}{\sigma_v^2} \left(1 - \frac{1}{\sigma_v^2/\sigma_v^2}\right) = 0.\]
exactly offset by the marginal loss of investor’s utility from higher conditional variance of returns, as shown in (4.6).

Consider now the effect of $\tilde{\sigma}_{v,t}^2$ on the conditional mean of returns, which depends on the estimate of the state $\hat{x}_{t}^{\theta}$, that in turn depends on the gain $k_{t}^{\theta}$, which is a function of $\hat{x}_{v,t}^2$ : 

$$
\frac{\partial E_{t}^{\theta}(\hat{x}_{t+1})}{\partial \hat{x}_{v,t}^2} = -a (\beta_{1}^* + \beta_{2}^*) \frac{d \hat{x}_{t}^{\theta}}{d \hat{x}_{v,t}^2} \frac{d k_{t}^{\theta}}{d \hat{x}_{v,t}^2}. 
$$

Replacing $\beta_{1}$ and $\beta_{2}$ by their equilibrium values, $\beta_{1}^* = -1$ and $\beta_{2}^* = -\alpha$, yields $-\gamma b_{t}^* \frac{d \hat{x}_{t}^{\theta}}{d \hat{x}_{v,t}^2} \left( y_{t} - ax_{t-1}^{\theta} \right) \frac{d k_{t}^{\theta}}{d \hat{x}_{v,t}^2}$, which is the first term in the RHS of (4.6). Since $\frac{d k_{t}^{\theta}}{d \hat{x}_{v,t}^2} < 0$, this expression is positive if the news $\left( y_{t} - ax_{t-1}^{\theta} \right)$ and the dollar’s bond position $b_{t}^*$ have the same sign. In contrast, it is negative if the news and bond position have opposite signs. Intuitively, ignoring the effect on the conditional variance of returns, to be robust the investor under-reacts to good news and overreacts to bad news: the forecasts of an investor with a long position under-react to positive news and overreact to negative news. Analogously, the robust forecasts of an investor with a short position under-react to negative news and overreact to positive news. This dependence of the direction of the robust distortion on the news is the one emphasized in Epstein and Schneider (2008).

Putting together the two effects, it follows that if the news and the dollar’s bond position have the same sign, then $\tilde{\sigma}_{v,t}^2 > \sigma_{v}^2$ unambiguously. However, if the news and the dollar’s bond position have opposite signs, then the sign of $\tilde{\sigma}_{v,t}^2 - \sigma_{v}^2$ is data-dependent.

Unfortunately, when attempting to account for DO and the FPP one cannot condition on the sign of the news and bonds’ position. The next Corollary shows that this ambiguity disappears in a probabilistic sense if the interest rate differential is close to a random-walk. For the major currencies considered by Gourinchas and Tornell (2004) this is the empirically relevant case and it is the one we will consider when we rationalize the anomalies in the reminder of the paper.

**Corollary 4.3.** If there is observational uncertainty and the data generating process for the interest rate differential $i_{t} - \hat{i}_{t}^{\theta}$ is highly persistent, i.e. the drift $\alpha$ in (3.1) is close enough to one, then with probability approaching one the observational variance is distorted upwards ($\tilde{\sigma}_{v,t}^2 > \sigma_{v}^2$) and there is under-reaction to news in a robust linear equilibrium:

$$
\Pr \left( k_{t}^{\theta} (\tilde{\sigma}_{v,t}^2) < k_{t}^{\theta} (\sigma_{v}^2) \right) \uparrow 1 \quad \text{as } a \uparrow 1.
$$
The robust and Bayes gains are given, respectively, by

\[ k_\ell^0 (\hat{\sigma}_{v,t}^2) = \frac{a^2 \sigma_{t-1}^2 + \sigma_{\psi}^2}{\sigma_{t-1}^2 + \sigma_{\psi}^2 + \hat{\sigma}_{v,t}^2} \]

and

\[ k_\ell^0 (\sigma_{\psi}^2) = \frac{a^2 \sigma_{t-1}^2 + \sigma_{\psi}^2}{\sigma_{t-1}^2 + \sigma_{\psi}^2 + \hat{\sigma}_{v,t}^2}. \]

The proof in the Appendix shows that the RHS of (4.6) is a quadratic function of

\[ z \equiv \gamma b^\pi \frac{\alpha}{1 - \alpha}. \]

This function \( f(z) \) is a convex parabola as the quadratic term comes from
the derivative of the conditional variance \( Var(J_{t+1}) \), which is positive. We show that as
\( \alpha \uparrow 1 \), the roots of the equation \( f(z) = 0 \) remain bounded, while \( z \) goes to either positive
or negative infinity. Since \( f(z) \) is a convex parabola, it follows that \( f(z) \) goes to positive
infinity when \( \alpha \uparrow 1 \).

Intuitively, when the drift \( \alpha \) is close enough to one, the derivative of \( Var(J_{t+1}) \), which
is not data-dependent, dominates the derivative of \( E(J_{t+1}) \) with probability approaching
one. This is because the former is increasing at the second order of \( z \), while the latter is
increasing only at the first order of \( z \), and so robustness entails \( \hat{\sigma}_{v,t}^2 > \sigma_v^2 \) with probability
approaching one.

4.4. Simulations

Corollary 4.3 shows that if the interest rate differential process \( y_t \) is close to a random
walk (i.e., \( \alpha \) is close to one), then with probability approaching one, robust agents distort
upwards the observational variance \( \hat{\sigma}_{v,t}^2 \) for any news history \( \{ y_t - a\hat{x}_{t-1}^\pi \} \). Here, present
simulations showing that this variance distortion tends to generate under-reaction to
news, which in turn gives rise to the anomalies.

Four parameters in our model determine the sign of the Fama coefficient: the per-
sistence parameter \( \alpha \), the noise-to signal ratio \( \sigma_v^2/\sigma_w^2 \), absolute risk-aversion \( (\gamma) \) and
uncertainty-aversion \( (1/\lambda) \). The first two characterize the data generating process (3.1),
while the last two parameters characterize the utility function (3.4). We use empirical
estimates found in the literature for \( (\gamma, \sigma_v^2/\sigma_w^2, \alpha) \). We then consider the values of \( \lambda \)
that satisfy condition (8.13) and that generate a negative slope coefficient in the Fama
regression.

Gourinchas and Tornell (2004) consider an interest rate differential process similar
to the one in this paper. Using data on interest rate differentials and their forecasts for
the G7 countries, they find that the drift of the differential \( \alpha \) is close to one both in
the actual data and the forecasts, but the noise-to signal ratio \( \sigma_v^2/\sigma_w^2 \) of the actual data
is much lower than that implied by the forecasts. Their estimates of \( \alpha \) range over the interval \((0.95, 1)\) and their estimates of \( \sigma_v^2 / \sigma_w^2 \) are near zero. Using the aggregate value of various risky and risk-free assets in the US., Bodie et. al. (2008) find that a coefficient of absolute risk-aversion \( \gamma \) of 2.6. According to Friend and Blume (1975) and Grossman and Shiller (1981) \( \gamma \) takes values on the interval \([2, 4]\). In our baseline simulations we set \( \gamma = 2.6, \alpha = 0.99, \sigma_v^2 = 1 \) and \( \sigma_w^2 = 100 \). As in the data, the resulting noise-to-signal ratio of our data generating process is close to zero: \( \sigma_v^2 / \sigma_w^2 = 0.01 \).

For each set of parameter values \((\gamma, \sigma_v^2, \sigma_w^2, \alpha, \lambda)\), we generate 100,000 shock sequences \( \{w_t, v_t\} \) and interest rate differential sequences \( \{y_t\} \) of length 100 by setting to zero the initial value of the state \((x_0 = 0)\). We obtain values of the distorted variance \( \tilde{\sigma}_{v,t}^2 \) by solving equation (8.14) in Proposition 4.6 recursively, starting from time \( t = 0 \). We then compute the Fama coefficient \( \beta^{\text{Fama}} \) from the linear regression, \( \Delta e_t = \beta_0 + \beta^{\text{Fama}} y_t + u_t \), where \( \Delta e_t \) is the exchange rate return computed from the equilibrium exchange rate function (4.4).

Figures 1 through 6 show the results from simulations where the degree of persistence \( \alpha \) varies over the range \((0.90, 0.99)\) and the uncertainty aversion coefficient \( \lambda \) is set to \(10^9\). Figure 1 illustrates the main result of Corollary 4.3 by showing the distributions of the distorted observational variance \( \tilde{\sigma}_{v,t}^2 \) for different values of \( \alpha \). As we can see, the larger \( \alpha \), the more the distorted variance \( \tilde{\sigma}_{v,t}^2 \) exhibits a systematic skew to values larger than the baseline \( \sigma_v^2 = 0.01 \).

Figures 2-4 show that observational uncertainty may account for the FPP if \( \alpha \) is close to one. Figure 2 exhibits the distributions of \( \beta^{\text{Fama}} \) over 100,000 regressions. As we can see, the larger \( \alpha \), the more the distribution of \( \beta^{\text{Fama}} \) is skewed to the left. Figure 3 shows the average value of \( \beta^{\text{Fama}} \) for different values of \( \alpha \).\(^{19}\) As we can see, a negative average \( \beta^{\text{Fama}} \) coefficient arises only for large values of \( \alpha \). Figure 4 shows the region in the space \((\alpha, 1/\lambda)\) over which average \( \beta^{\text{Fama}} \) is negative (blue area). As we can see, negative average \( \beta^{\text{Fama}} \) tends to arise when \( \alpha \) is large.

As we will explain below, a negative Fama coefficient is associated with a hump-shaped exchange rate response to a one-time interest rate differential shock of unknown

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\(^{19}\)The average Fama coefficient \( \bar{\beta}^{\text{Fama}} \) is defined as \( \bar{\beta}^{\text{Fama}} = \frac{\beta_i^{\text{Fama}}}{N}, i = 1, ..., N \), where \( N \) is total number of simulations and \( \beta_i^{\text{Fama}} \) is the Fama coefficient estimate from each simulation. According to the law of large numbers, \( \bar{\beta}^{\text{Fama}} \) converges to \( \beta^{\text{Fama}} \) in probability. See page 265-267, Green(2011).
duration, i.e., delayed overshooting (DO). Figure 5 exhibits an example \((a = 0.99\) and \(1/\lambda = 10^9\)) of impulse responses (IRFs) to a transitory shock and to a persistent shock, as well as the average IRF to a shock of unknown duration. As expected, the IRF to a transitory shock doesn’t exhibit DO, but the IRF to a persistent shock does. More importantly, the average IRF exhibits DO (momentum): the exchange rate continues appreciating over 13 periods following a shock to the interest rate differential of unknown duration. Lastly, Figure 6 shows the region in the space \((a, 1/\lambda)\) over which DO occurs (blue area).

4.5. Intuition for the Foreign Exchange Market Anomalies

Here, we provide intuition for the simulation results by deriving closed-form expressions for the IRF and the regression coefficient in the Fama regression. Unfortunately, the randomness of the observational variance distortion \(\sigma_{\nu t}^2\) makes such effort technically intractable. To attain technical tractability, we consider the case where the persistence coefficient \(a\) is close to one. As Corollary 4.3 shows, in this case the effect of news \(y_t - ax_{t-1}^\theta\) on the robust distortion of the variance \(\sigma_{\nu t}^2\) is dominated by the effect of the return variance \(Var_{t}^\theta (J_{t+1})\). That is, as \(a\) is close one, \(\left(\frac{\partial^2 Var_{t}^\theta (J_{t+1})}{\partial \sigma_{\nu t}^2}\right) >> -\gamma b_t^\theta \left(y_t - ax_{t-1}^\theta\right)\) and \(\frac{1}{2} (\gamma b_t^\theta)^2 Var_{t}^\theta (J_{t+1}) >> -\gamma b_t^\theta E_{t}^\theta (J_{t+1})\). This implies that a good approximation to FOC (4.6) can be obtained by omitting the terms

\[\frac{(\gamma b_t^\theta)^2 \partial Var_{t}^\theta (J_{t+1})}{\partial \sigma_{\nu t}^2}\]

To see this, we compute the following ratio:

\[
\lim_{a \to 1} \left( \frac{\gamma b_t^\theta}{2} \left( y_t - ax_{t-1}^\theta \right) - \frac{1}{a} \frac{dk_{t+1}^\theta}{a \partial \sigma_{\nu t}^2} \right) = \lim_{a \to 1} \left( \frac{\gamma b_t^\theta}{2} \left( k_{t+1}^\theta a + 1 - a \right) a \partial \sigma_{\nu t}^2 \right) = \lim_{a \to 1} \left( \frac{\gamma b_t^\theta}{2} \left( k_{t+1}^\theta a + 1 - a \right) a \partial \sigma_{\nu t}^2 \right) = +\infty
\]

The last equality follows because \(\lim_{a \to 1} \left| \frac{1}{a} \right| = +\infty\), \(|k_{t+1}^\theta|\) is bounded by \(1\), and \(\left| y_t - ax_{t-1}^\theta \right|\) is the realized value of the news which, by its nature,
related to the expected return, where the randomness of $\tilde{\sigma}_{v_t}^{2*}$ originates. The resulting approximating FOC for $\tilde{\sigma}_{v_t}^{2*}$ is

$$\left[\frac{1}{\sigma_v^2} - \frac{1}{\tilde{\sigma}_{v_t}^{2*}}\right] \frac{\lambda}{2} = \left[\frac{(\gamma b_t^s)^2}{2} \frac{\partial \text{Var}_{t}^{b_t^s}(J_{t+1})}{\partial \tilde{\sigma}_{v,t}^2}\right] \exp\left(\frac{1}{2} (\gamma b_t^s)^2 \text{Var}_{t}^{b_t^s}(J_{t+1})\right)$$

(4.7)

The attractive aspect of approximating FOC (4.7) is that it provides technical tractability, while preserving the main result of this Section: because $\frac{\partial \text{Var}_{t}^{b_t^s}(J_{t+1})}{\partial \tilde{\sigma}_{v,t}^2}$ is positive, the solution $\tilde{\sigma}_{v,t}^{2*}$ to (4.7) is unambiguously greater than $\sigma_v^2$.

To concentrate on the forecast-distortion mechanism we consider the case where the gain $k_t^{\theta_t}$ in filter (4.5) has converged. To do so we use the well known fact in the control literature that the gain $k_t^{\theta_t}$ converges rather fast if the underlying coefficients $(a, \sigma_w^2, \sigma_v^2)$ are constant. Lemma 8.1 in the appendix shows that if we set the supply of domestic bonds equal to a constant up to some time $T$, where $T$ can be very large, then on $t \in [0, T]$ the equilibrium exchange rate (4.4) has a deterministic $\alpha_t^*$, and the filter $\tilde{\alpha}_t^*$ has a constant gain, given by

$$k_t^{\theta_t} = \frac{a^2 \xi^* + \sigma_w^2}{a^2 \xi^* + \sigma_w^2 + \tilde{\sigma}_{v,t}^{2*}}, \quad \text{where} \quad \xi^* = \frac{- (\sigma_w^2 + \sigma_v^2 - a^2 \sigma_v^2) + \sqrt{(\sigma_w^2 + \sigma_v^2 - a^2 \sigma_v^2)^2 + 4 a^2 \sigma_w^2 \sigma_v^2}}{2 a^2}.$$

(4.8)

The distorted observational variance $\tilde{\sigma}_{v,t}^{2*}$ in (4.8) is the solution to (4.7) setting $b_t^s$ equal cannot be infinite. Similarly, the ratio $\frac{\frac{1}{2} (\gamma b_t^s)^2 \text{Var}_{t}^{b_t^s}(J_{t+1})}{-\gamma b_t^s \text{Var}_{t}^{b_t^s}(J_{t+1})}$ is infinite as $a \uparrow 1$ too. To see this, we compute

$$\lim_{a \downarrow 1} \frac{\frac{1}{2} (\gamma b_t^s)^2 \text{Var}_{t}^{b_t^s}(J_{t+1})}{-\gamma b_t^s \text{Var}_{t}^{b_t^s}(J_{t+1})} = \lim_{a \downarrow 1} \frac{\frac{1}{2} (\gamma b_t^s)^2 \left(\frac{a^2 \sigma_w^2}{a^2 \sigma_w^2 + \tilde{\sigma}_{v,t}^{2*}} + \sigma_w^2 + \tilde{\sigma}_{v,t}^{2*}\right)}{\left|y_t - \frac{a \tilde{\alpha}_t^{b_t^s}}{1-a} - \alpha_{t+1} + e_t\right|}$$

$$= \lim_{a \downarrow 1} \frac{\frac{1}{2} (\gamma b_t^s)^2 \left(\frac{a^2 \sigma_w^2}{a^2 \sigma_w^2 + \tilde{\sigma}_{v,t}^{2*}} + \sigma_w^2 + \tilde{\sigma}_{v,t}^{2*}\right)}{\left|1-a\right| \left(y_t - \alpha_{t+1} + e_t\right) - ax_t^{b_t^s}}$$

$$= +\infty$$
to a constant value $\bar{b}$.\textsuperscript{21} We can see that the higher uncertainty aversion (lower $\lambda$), the higher the distorted observational variance $\tilde{\sigma}_v^2$ relative to the baseline $\sigma_v^2$, and the lower the robust filter’s gain relative to the baseline gain.

The optimality condition for the holdings of the bond implies that the following robust interest parity condition holds in equilibrium.

$$E_t^{\theta_t}(e_{t+1}) - e_t = (i_t - i_t^f) - \zeta_t,$$  \hspace{1cm} (4.9)

Namely, the depreciation rate expected by robust agents equals the interest rate differential $(i_t - i_t^f)$ minus an uncertainty premium on domestic assets ($\zeta_t$). We can see from (4.9) that there are two channels through which the FPP and hump-shaped dynamics can arise in equilibrium: forecast-distortions and time-varying risk premia. Here, we focus on the first channel and we shut off the latter channel. We find that when there is no uncertainty aversion $(1/\lambda = 0)$, forecasts are generated by Bayes law under the baseline model, the Fama coefficient is one and there is no DO. In contrast, if uncertainty aversion is high, then under approximating FOC (4.7), DO and a negative Fama coefficient arise under observation uncertainty, provided the interest rate differential is highly persistent.

\subsection*{4.5.1. Delayed Overshooting (Conditional Momentum)}

As a preliminary step in explaining the FPP, we investigate the conditions under which the exchange rate exhibits a hump-shaped response conditional on the occurrence of a once-and-for-all shock to the interest rate differential, i.e., the forward premium.\textsuperscript{22} That is, a positive forward premium shock generates an initial appreciation of the domestic currency, which is followed by further appreciation for several periods afterwards before reverting to a depreciating path. This “delayed overshooting” pattern (DO), or conditional momentum, is consistent with the FPP because there is a period during which an

\textsuperscript{21}We set $b_t = \begin{cases} \bar{b}, & t \leq T \\ -\bar{b}\exp(T-t)\eta_t, & t \in (T, \infty), \hspace{0.5cm} \eta_t \equiv \left(V_t^{\theta_t}(J_{t+1})\right)^{-1}. \end{cases}$ For further details regarding the assumption on $b_t$, see Lemma 8.1 in the appendix.

\textsuperscript{22}The covered interest parity condition implies that the interest rate differential equals the forward premium: $f_t - e_t = i_t - i_t^f$, where $f_t$ is the log forward exchange rate one period ahead and $e_t$ is the log spot exchange rate.
exchange rate appreciation coexists with a positive interest rate differential.\(^{23}\)

To compute the impulse response to a random forward premium shock, suppose that at time \(t = 1\) the representative investor observes a forward premium realization \(y_1 = 1\) and no shocks occur afterwards. She knows that the forward premium shock is generated by a combination of a transitory shock \(v_1 = \kappa\) and a persistent shock \(w_0 = \varepsilon\) such that \(y_1 = \varepsilon + \kappa = 1\), but she does not observe the particular values of \(\varepsilon\) and \(\kappa\).

Importantly, to account for delayed overshooting we must consider the ‘average impulse response’ and not just a response to a persistent shock. This is because both investors and the econometrician cannot condition on whether the shocks are transitory or persistent. Since the data is generated by baseline model \(\theta'\), the average impulse response at time \(t\) to an initial \(y\)-shock is given by:

\[
\hat{\varepsilon}^\alpha_t \equiv E^{\theta'} (\hat{\varepsilon}_t (\varepsilon, \kappa)) \big| y_1 = \varepsilon + \kappa = 1, \quad \text{with} \quad \hat{\varepsilon}_t (\varepsilon, \kappa) \equiv e_t (\varepsilon, \kappa) - e_t (0, 0). \tag{4.10}
\]

The expression \(\hat{\varepsilon}_t (\varepsilon, \kappa)\) is the response of the log exchange rate at time \(t\) to an initial persistent shock \(\varepsilon\) and an initial transitory shock \(\kappa\). That is, the time \(t\) response to shock sequences \(w^s = \{\varepsilon, 0, \ldots, 0\}_{1 \times t}\) and \(v^s = \{\kappa, 0, \ldots, 0\}_{1 \times t}\). By definition, we have \(\hat{\varepsilon}_0 (\varepsilon, \kappa) = 0\).

Since in equilibrium the exchange rate and the forecasts are linear in the initial shocks \(\varepsilon\) and \(\kappa\), we can express the average impulse response (4.10) as a weighted average of the responses to a persistent shock \(\hat{\varepsilon}_t (\varepsilon, 0)\) and to a transitory shock \(\hat{\varepsilon}_t (0, \kappa)\):

\[
\hat{\varepsilon}^\alpha_t = q^{\theta'} \cdot \hat{\varepsilon}_t (\varepsilon, 0) + [1 - q^{\theta'}] \cdot \hat{\varepsilon}_t (0, \kappa), \quad q^{\theta'} \equiv \frac{\sigma^2_v}{\sigma^2_v + \sigma^2_w}. \tag{4.11}
\]

The weight \(q^{\theta'}\) is the expected value, under the data generating model \(\theta'\), of the persistent shock \(\varepsilon\) conditional on \(y_1 = 1\). The greater \(q^{\theta'}\), the greater the share of persistent shocks in the data. As we can see, \(q^{\theta'}\) is decreasing in \(\frac{\sigma^2_v}{\sigma^2_w}\), the noise-to-signal ratio of the data generating model.

The next Proposition states the conditions under which the average impulse response

\(^{23}\)This pattern stands in contrast to the overshooting pattern of Dornbusch (1976), and has been found in the data by Eichenbaum and Evans (1995). Empirically, it is more difficult to find delayed overshooting than the FPP. To identify delayed overshooting, it is necessary to determine when a shock to the interest rate differential occurs, an event that is hard to single out in the data.
exhibits hump-shaped dynamics. That is, following an increase in the interest rate differential, whether the average exchange rate appreciates at time \( t = 1 \), and continues appreciating until some time \( \tau \) after which it reverts back to its long run level. In other words, whether there exists an integer \( \tau \geq 2 \), such that \( \hat{e}_{t+1} - \hat{e}_{t} < 0 \) for all integers \( t \in (1, \tau) \) and \( \hat{e}_{t+1} - \hat{e}_{t} > 0 \) for all \( t \geq \tau + 1 \).

**Proposition 4.4 (Delayed Overshooting).** If the agent sets the robust observational variance based on the approximating FOC (4.7), then in a robust linear equilibrium, the average impulse response of the exchange rate to an interest rate differential shock satisfies: \( \hat{e}_{1} = -\frac{1-a(1-k^{\theta})}{1-a} \) and for \( t \geq 1 \):

\[
\hat{e}_{t+1} - \hat{e}_{t} = \frac{a^{t-1}}{1-a} \left[ \left( k^{\theta} - q^{\theta} \right) a \left( 1 - a \left( 1 - k^{\theta*} \right) \right) \left( 1 - k^{\theta*} \right)^{t-1} + (1-a) q^{\theta} \right] \tag{4.12}
\]

where \( k^{\theta*} \) is the robust gain and \( q^{\theta} \) is the share of persistent shocks in the data.

i. There is delayed overshooting with probability approaching one, if the unobservable component of the interest rate differential is highly persistent (i.e., \( a \) is sufficiently close to one) and uncertainty aversion satisfies \( 1/\lambda \in (1/\lambda^DO, 1/\lambda^v) \). The thresholds \( 1/\lambda^DO \) and \( 1/\lambda^v \) are defined by (8.24) and (8.30).

ii. Delayed overshooting occurs up to a time \( \tau \), after which mean-reversion takes place. Time \( \tau \) is the smallest integer that satisfies:

\[
\tau \geq 1 + \frac{\log \left( q^{\theta} \left[ \frac{1}{a} - 1 \right] \right) - \log \left( \left[ 1 - a \left( 1 - k^{\theta*} \right) \right] [q^{\theta} - k^{\theta*}] \right)}{\log \left( 1 - k^{\theta*} \right)} \tag{4.13}
\]

To see the intuition recall that the exchange rate is a function of the estimate of the hidden trend \( \hat{x}_t \), and that the response of \( \hat{x}_t \) to forward premium news is determined by the gain \( k^{\theta*} \) because \( \hat{x}_t = k^{\theta*} y_t + (1 - k^{\theta*})a \hat{x}_{t-1} \). When a positive forward premium shock occurs, the agent’s robust estimate initially reacts on average less to the news than what

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\( ^{24} \)Time \( \tau \) converges to positive infinity when \( a \) goes to 1. To see this, take the limit of the second term on the right hand side of (4.13): \( \lim_{a \rightarrow 1} \frac{\log \left( q^{\theta} \left[ \frac{1}{a} - 1 \right] \right) - \log \left( \left[ 1 - a \left( 1 - k^{\theta*} \right) \right] [q^{\theta} - k^{\theta*}] \right)}{\log \left( 1 - k^{\theta*} \right)} = -\infty \). The last equality follows because the denominator is negative and \( \lim_{a \rightarrow 1} k^{\theta*} = \frac{\xi^{*} + \sigma_{\alpha}^{2}}{\xi^{*} + \sigma_{\alpha}^{2} + \sigma_{\theta}^{2}} \) is finite; \( \xi^{*} = \frac{-\sigma_{\alpha}^{2} + \sqrt{(\sigma_{\alpha}^{2})^{2} + 4\sigma_{\theta}^{2}\sigma_{\alpha}^{2}}}{2} \).
it should under the data generating process (because $k^{0*} < k^{0}$), and so the exchange rate does not appreciate at $t = 1$ as much as it should. As a result, forecasts will later on have to catch up. This catch-up will be strong enough to generate momentum in the exchange rate if persistent shocks are long lasting ($a$ is high) and sensitivity to news $k^{0*}$ is lower than $q^{0'}$, but strictly positive. If instead $k^{0*}$ were greater than $q^{0'}$, most of the reaction would occur initially, and so subsequent forecast revisions would be very small and would be dominated by the inherent mean reversion of the forward premium. It is necessary that $k^{0*} < q^{0'}$ because momentum cannot be generated from transitory shocks, and so the initial forecast reaction to the shock $(k^{0*})$ must be less than the expected value of the persistent shock $(q^{0'})$. If $a$ were small, the results in Corollary 4.3 would not apply, and so observation uncertainty would not necessarily induce $k^{0*} < k^{0}$. Moreover, with a small $a$, even persistent shocks would disappear fast, so the initial small reaction would not lead to momentum.

Robustness against observation uncertainty generates a lower gain $k^{0*}$ by inducing agents to distort upwards the variance of observation shocks relative to the data generating process $\theta'$ (i.e., $\tilde{v}_{2}^{2} > \sigma_{v}^{2}$). This distorted variance is in turn determined by the degree of uncertainty aversion $1/\lambda$: the greater $1/\lambda$, the greater $\tilde{v}_{2}^{2}$ and so the smaller $k^{0*}$. In contrast, if there is no uncertainty aversion, $1/\lambda = 0$, there is no distortion as $\tilde{v}_{2}^{2}$ must be equal to $\sigma_{v}^{2}$ and the robust gain $k^{0*}$ must equal the baseline Bayes gain $k^{0'}$. In this case there cannot be momentum because $k^{0'}$ is necessarily greater than $q^{0'}$.

Lastly, notice that the approximating average IRF characterized in Proposition 4.4, generates a hump-shaped path that resembles that in the simulations of Figure 5: after

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25 The response to a purely transitory shock is $\hat{e}_{t}(0, \kappa) = -\kappa - \frac{a}{1-a} k^{0*} \kappa$ for $t = 1$ and $\hat{e}_{t}(0, \kappa) = -\frac{a k^{0*} (1-k^{0*})^{t-1}}{1-a} \kappa$ for $t \geq 2$. This response does not exhibit delayed overshooting for any $a \in (0, 1)$. The response to a purely persistent shock is $\hat{e}_{t}(\varepsilon, 0) = -\frac{a^{t-1}}{1-a} \left(1-a (1-k^{0*})^{t}\right) \varepsilon$ for $t \geq 1$. It follows that for any $k^{0*} \in (0, 1)$ there exists a sufficiently large $a$ such that this response exhibits delayed overshooting. For instance, $e_{2}(\varepsilon, 0) = e_{1}(\varepsilon, 0) < e_{0}$ provided $a \geq \frac{4}{3}$ and $k^{0*} \in \left(\frac{2a-1-\sqrt{4a-3}}{2a}, \frac{2a+1+\sqrt{4a-3}}{2a}\right)$. Since this interval converges to $(0, 1)$ as $a \to 1$, it follows that for any $k^{0*} \in (0, 1)$ there exists a sufficiently large $a$ such that $e_{2}(\varepsilon, 0) < e_{1}(\varepsilon, 0)$.

26 If $k^{0*}$ were zero, DO would not arise: $\lim_{k^{0*} \to 0} (e^{av}_{t+1} - e^{av}_{t}) = \frac{a^{t-1}}{1-a} (1-a)^{2} q^{0'}$, which implies that there is no integer $\tau \geq 1$ such that $e^{av}_{\tau+1} < e^{av}_{\tau}$. Gourinchas and Tornell (2004) find similar results: in their paper DO exists if agents use a small gain, but not if the gain is zero. Notice that in our equilibrium $k^{0*}$ is positive because $\tilde{v}_{2}^{2}$ is bounded.

27 A simple computation shows that $k^{0*} = \frac{\sigma^{2} \xi* + \sigma^{2}_{2}}{\sigma^{2}_{1} + \sigma^{2}_{2} + \sigma^{2}_{e}} > \frac{\sigma^{2}_{2}}{\sigma^{2}_{1} + \sigma^{2}_{2}} = q^{0'}$ for any $\xi* > 0$. 

25
the initial jump, the exchange rate continues appreciating for a long time period before reverting.

4.5.2. The Forward Premium Puzzle (FPP)

Consider the "Fama regression" $e_{t+1} - e_t = \alpha + \beta_{Fama} (i_t - i_t^f) + u_t$. Under the null of uncovered interest parity and rational expectations the slope (Fama) coefficient $\beta_{Fama}$ is one. However, many empirical studies find that $\hat{\beta}_{Fama} < 1$ (weak FPP) and often $\hat{\beta}_{Fama} < 0$ (strong FPP). The next proposition states the conditions under which the robust forecast-distortion mechanism generates an approximating $\beta_{Fama}$ that converges to a negative number with probability approaching one.

Proposition 4.5 (Forward Premium Puzzle). If the agent sets the robust observational variance based on the approximating FOC (4.7), then under observational uncertainty the Fama regression coefficient converges in plim to

$$\beta_{Fama} = 1 - \frac{(k^{\theta'} - k^{\theta^*}) a (a (1 + a) k^{\theta^*} + (1 - a^2))\left(\frac{1}{(1-a^2)(1-k^{\theta'})}\right) + \frac{\sigma^2_w}{\sigma^2_u}}{(1 - a^2)^2 + 1}$$

(4.14)

1. Weak FPP. $\beta_{Fama}$ is less than one with probability approaching one, if the interest rate differential is highly persistent (i.e., $a$ is close to one) and investors are both uncertainty-averse and risk-averse.

2. Strong FPP. $\beta_{Fama}$ is negative with probability approaching one, if in addition the degree of uncertainty aversion satisfies $1/\lambda \in (1/\lambda^F, 1/\lambda^v)$. The thresholds $1/\lambda^F$ and $1/\lambda^v$ are defined by (8.24) and (??).

This proposition is based on the result that if the interest rate differential is highly persistent and there is observational uncertainty, then investors under-react to news (Corollary 4.3): $\text{Pr}(k^{\theta^*} < k^{\theta'}) \uparrow 1$ as $a \uparrow 1$. Part 1 follows from (4.14) because $k^{\theta^*} < k^{\theta'}$ implies that the asymptotic value of the Fama coefficient is strictly smaller than one. Part 2 states that the strong form of the FPP (negative $\beta_{Fama}$) results if in addition uncertainty aversion is high enough—but not too high—so that under-reaction to news $(k^{\theta'} - k^{\theta^*})$ is large enough, but not too large. Finally, part 3 says that $\beta_{Fama}$ cannot be
negative if there is little under-reaction to news or interest rate differential shocks die out fast.\textsuperscript{28}

Before we describe the intuition, we note that the FPP implies predictable excess returns. To see this, notice that in our model “predictable excess returns under the data generating model $\theta''$ equal the forward premium minus the expected—under the baseline model—exchange rate depreciation

$$
\Lambda_t'' := y_t - E_t'' (e_{t+1} - e_t)
$$

where $y_t \equiv i_t - \hat{i}_t$. Using (4.9) to substitute for $y_t$, we show in the appendix that in the robust equilibrium

$$
\Lambda_t'' = \left[ E_t'' (x_{t+1}) - E_t''' (x_{t+1}) \right] \left[ 1 + \frac{ak''}{1-a} \right] + \zeta_t
$$

That is, under the data generating model $\theta''$, predictable excess returns equal the expectation distortion plus the uncertainty premium.\textsuperscript{29} Let us now analyze the regression coefficient

$$
\beta_{\text{Fama}} = \rho \lim_{\mathbb{E} \to \infty} \frac{\text{cov}^i (\Delta e_{t+1}, y_t)}{\text{var}^i (y_t)}
$$

Using (4.15) and (4.16) we obtain

$$
E_t'' \Delta e_{t+1} = y_t - \Lambda_t''
$$

The first term in (4.17) is the direct effect of the forward premium on average depreciation. The second term is the catch-up effect on average depreciation, and it is the source of the FPP. The third term captures the effect of past forecast errors. The last term is the uncertainty premium, which is constant in the equilibrium we are considering.

Consider an increase in $y_t$ and ignore the third term in (4.17) for a moment. Equation (4.17) shows that if forecasts are less sensitive to news than baseline forecasts (i.e., $k'' < k'''$), expected depreciation responds by less than the change in the forward pre-

\textsuperscript{28}When $k'' = 0$, $\beta_{\text{Fama}} = \frac{(1-a^2)^2}{(1-a^2)^2 + 1} \frac{(1-a)(1-k'') + (1+a)(1-k''')}{(1+a)(1-k''') - 1} > 0$. By continuity $\beta_{\text{Fama}}$ is positive for very small $k'''$.

\textsuperscript{29}In contrast, condition (4.9) implies that predictable excess returns under the agents’ robust model $\theta^*$ (i.e., $(i_t - \hat{i}_t) - E_t^* (e_{t+1} - e_t)$) equal simply the uncertainty premium $\zeta_t$. 

27
mium. The result is a Fama coefficient being less than one, as stated in part 1 of Proposition 4.5. To derive part 2 notice that if both the persistence parameter \( a \) and the undersensitivity to news \( k^{\theta'} - k^{\theta*} \) are large, the catch-up effect in (4.17) captured by \((1 + \frac{ak^{\theta*}}{1-a})(k^{\theta'} - k^{\theta*})\) can dominate the direct effect of one. As a result, the initial mispricing is so large that it requires the currency to appreciate further in the future.

When \( a \) is large, this future upward revision in beliefs will have a large effect on the exchange rate because agents will expect high domestic interest rates to persist far into the future. This mispricing results in an extreme scenario where a high domestic interest rate coexists with an appreciating currency. Hence, under the data generating model \( \psi_0 \), the forward premium and the expected depreciation tend to move in opposite directions on average, generating a negative regression coefficient \( \beta^{Fama} \).

Consider now the third term in (4.17). Since the gain differential \((k^{\theta'} - k^{\theta*})\) is a positive constant, the sign of the third term equals the sign of \( a \left( E^\theta_{t-1}(x_t) - E^{\theta'}_{t-1}(x_t) \right) \). Furthermore, since \( E^{\theta*}_{t-1}(x_t) \) reacts less to \( y_{t-1} \) than \( E^{\theta'}_{t-1}(x_t) \), and \( y_{t-1} \) is positively correlated with \( y_t \), it follows that on average \( E^{\theta*}_{t-1}(x_t) \) is smaller (larger) than \( E^{\theta'}_{t-1}(x_t) \) when \( y_{t-1} \) is positive (negative). Thus, \( E^{\theta*}_{t-1}(x_t) - E^{\theta'}_{t-1}(x_t) \) and \( y_t \) are negatively correlated. Therefore, the error in past forward premium estimates makes \( \beta^{Fama} \) smaller.

Finally, notice that in our robust equilibrium all deviations of \( \beta^{Fama} \) from one are generated by the forecast-distortion mechanism because the uncertainty premium \( \zeta_t \) is deterministic.31

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30 Notice that \( aE^{\theta*}_{t-1}(x_t) \) and \( aE^{\theta'}_{t-1}(x_t) \) are the estimates of \( y_t \) at time \( t-1 \).

31 To see this decompose the realized log exchange rate change \( (\Delta e_{t+1} := e_{t+1} - e_t) \) into its robust forecast \( E^\theta_t(\Delta e_{t+1}) \) and a forecast error \( v_{t+1} \):

\[
\Delta e_{t+1} = E^\theta_t(\Delta e_{t+1}) + v_{t+1}, \quad E^\theta_t(v_{t+1}) = 0;
\]

where we use the fact that the forecast error is zero under the robust measure \( \theta^* \). Using the above equation and (4.9) we have that

\[
\beta^{Fama} = \frac{\text{cov}^{\theta'}(\Delta e_{t+1}, y_t)}{\text{var}^{\theta'}(y_t)} = 1 + \frac{\text{cov}^{\theta'}(v_{t+1}, y_t)}{\text{var}^{\theta'}(y_t)} - \frac{\text{cov}^{\theta'}(\zeta_t, y_t)}{\text{var}^{\theta'}(y_t)}
\]

Since \( \zeta_t \) is deterministic, \( \text{cov}^{\theta'}(\zeta_t, y_t) = 0. \)
5. Other Types of Uncertainty

We have seen that if the interest rate differential is highly persistent, then structured observational uncertainty may generate the forward premium puzzle and delayed overshooting with probability approaching one. In this section, we consider two other types of uncertainty: structured uncertainty in the hidden-state equation and unstructured uncertainty, under which investors fear misspecification of the entire interest rate differential process but cannot pinpoint either its nature or location. We find that in both cases the forecast-distortion mechanism underlying the anomalies is not operative. In the first case, if a systematic distortion were to arise, it would go in the wrong direction as forecasts would be more sensitive to news than baseline forecasts, which would lead to $\beta^{Fama} > 1$. In the second case, the result is surprising: robust forecasts have the same sensitivity to news as Bayesian forecasts under the baseline model. Thus, the forecast-distortion mechanism underlying the anomalies is not operative.

5.1. Structured Uncertainty in the State Equation

To study state uncertainty we consider the following uncertainty set.32

$$\Theta^w = \left\{ \theta \in P(\Omega) : \frac{d\theta}{d\theta^0} = \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma^2_w} - \frac{1}{\tilde{\sigma}^2_w} \right) (x_t - a x_{t-1})^2 \right) \cdot \sqrt{\frac{\sigma^2_w}{\tilde{\sigma}^2_w}}, \sigma^2_w \in [\varepsilon, \infty], \omega > 0 \right\}$$

The representative investor solves Problem (3.8) with the same conjecture and prior as in Section 4. The following Lemma allows us to convert the optimization over probability measures into a parametric optimization.

Lemma 5.1 (Change of Measure II). If under the baseline probability measure $\theta'$ the random variables in (3.1) are distributed as $x_t | I_{t-1} \sim^{\theta'} N(a \hat{x}_{t-1}, \sigma^2_w)$, $y_t | x_t \sim^{\theta'} N(x_t, \sigma^2_v)$ and $x_{t-1} \sim^{\theta'} N(\hat{x}_{t-1}, \sigma^2_{t-1})$, then:

1. Under any probability measure $\theta \in \Theta^w$, $x_t | I_{t-1} \sim^{\theta} N(a \hat{x}_{t-1}, \tilde{\sigma}^2_w)$, while the distributions of $y_t | x_t$ and $x_{t-1}$ are preserved.

32 The condition $\varepsilon > 0$ ensures that set $\Theta^w$ is closed and convex.
2. The relative entropy of measure $\theta$ with respect to baseline measure $\theta'$ is
\[
R(\theta||\theta') = \frac{1}{2} \left( \tilde{\sigma}_w^2 - \log \frac{\tilde{\sigma}_w^2}{\sigma_w^2} - 1 \right) \text{ for any } \theta \in \Theta^w. \tag{5.1}
\]

This Lemma shows that under any probability measure in the set $\Theta^w$, the interest rate differential process is given by baseline model (3.1), except that the variance of persistent shocks has a distorted value $\tilde{\sigma}_w^2$ instead of the baseline $\sigma_w^2$. The next Proposition characterizes the robust linear equilibria under uncertainty set $\Theta^w$.

**Proposition 5.2 (Equilibrium Under Hidden-State Uncertainty).** Under uncertainty set $\Theta^w$ there exists a robust linear equilibrium if the degree of uncertainty aversion $1/\lambda$ is lower than a threshold $1/\lambda^w_t$ defined in (5.2).

1. The log exchange rate is
\[
e^*_t = -(i_t^* - i_t^*) - \frac{a}{1-a} \hat{\alpha}_t + \alpha^*_t, \quad \text{where } \alpha^*_t = \alpha^*_{t+1} + \gamma b^*_t \left( a k^*_{t+1} \hat{\alpha} + 1 \right)^2 \left( a \sigma_v^2 k^*_t + \sigma^2_v + \tilde{\sigma}^2_{w,t} \right). \tag{5.2}
\]

2. The robust forecast of the interest rate differential is given by (4.5) with gain
\[
k^*_t \equiv \frac{a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t}}{a^2 \sigma^2_{t-1} + \sigma^2_v + \tilde{\sigma}^2_{w,t}}, \quad \sigma^2_{t-1} = \frac{a^2 \sigma^2_{t-2} + \sigma^2_w}{a^2 \sigma^2_{t-2} + \sigma^2_w + \sigma^2_v} \tag{5.3}
\]

3. Under robust measure $\theta^*_t$, the distorted variance $\tilde{\sigma}_{w,t}^2$ of the persistent shock satisfies:
\[
\frac{\lambda}{2} \left( \frac{1}{\sigma^2_w} - \frac{1}{\tilde{\sigma}^2_{w,t}} \right) = \left( -\gamma b^*_t \left[ \frac{a}{1-a} \right] (y_t - a x^*_t) \right) \left( d_k^{\theta^*_t} \frac{d^2 \sigma^2_{w,t}}{d \tilde{\sigma}^2_{w,t}} + \frac{(\gamma b^*_t)^2}{2} \frac{\partial V a r^{\theta^*_t}_{t} (J_{t+1})}{\partial \tilde{\sigma}_{w,t}^2} \right) W^*_t \tag{5.4}
\]
where $\frac{d k_{t+1}^{\theta^*_t}}{d \tilde{\sigma}_{w,t}^2} = \frac{\sigma^2_t}{(a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma^2_v)} > 0, \frac{\partial V a r^{\theta^*_t}_{t} (J_{t+1})}{\partial \sigma^2_{w,t}} = \left( k_{t+1}^{\theta^*_t} \frac{a}{1-a} + 1 \right)^2 \left( \frac{a^2 \sigma^2_t}{(a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma^2_v)^2} + 1 \right) > 0$ and $W^*_t = \exp \left( -\gamma b^*_t E^{\theta^*_t}_{t} (J_{t+1}) + \frac{1}{2} (\gamma b^*_t)^2 V a r^{\theta^*_t}_{t} (J_{t+1}) \right) > 0$.

4. There is no forecast distortion (i.e., $\tilde{\sigma}^2_{w,t} = \sigma^2_w$) if the primitive utility function is risk neutral ($\gamma = 0$), there is no aversion to uncertainty ($1/\lambda = 0$) or the dollar bond is in zero net supply ($b^*_t = 0$).
This Proposition shows that the equilibrium exchange rate function has the same form as under observational uncertainty in (4.4). The only difference is that the robust filter distorts the variance of the persistent shock $\tilde{\sigma}_{w,t}^2$ rather than that of the observational shock $\bar{\sigma}_{w,t}^2$.

Since the anomalies arise in equilibrium only if there is under-reaction to news (i.e., $k_t^g < k_t^\theta$), we require $\tilde{\sigma}_{w,t}^2 < \sigma_w^2$. In general, however, the direction of the equilibrium variance distortion is data-dependent, as we can see in (5.4). Corollary 5.3 shows that this indeterminacy disappears in a probabilistic sense if the data-generating process for the interest rate differential is either highly persistent or close to white noise.

Corollary 5.3 (Overreaction to News). If there is structured uncertainty in the hidden-state equation and the data generating process for the interest rate differential $i_t - i_t^f$ is either highly persistent or close to white noise (i.e., the drift $a$ is close to either one or zero), then with probability approaching one the robust variance of the persistent shock $\tilde{\sigma}_{w,t}^2$ is distorted upwards and there is over-reaction to news in a robust linear equilibrium:

$$\Pr \left( k_t^g (\tilde{\sigma}_{w,t}^2) > k_t^\theta (\sigma_w^2) \right) \uparrow 1 \quad \text{as } a \uparrow 1 \text{ or } a \downarrow 0.$$  

The robust and Bayes gains are given, respectively, by $k_t^g (\tilde{\sigma}_{w,t}^2) = \frac{a^2 \sigma_{t-1}^2 + \tilde{\sigma}_{w,t}^2}{\sigma_{t-1}^2 + \tilde{\sigma}_{w,t}^2 + \sigma_e^2}$ and $k_t^\theta (\sigma_w^2) = \frac{\sigma_{t-1}^2 + \sigma_w^2}{\sigma_{t-1}^2 + \sigma_w^2 + \sigma_e^2}$.

This Corollary shows that in the two cases in which the indeterminacy in the direction of the forecast distortion disappears, robustness induces forecasts that are more sensitive to news than baseline forecasts. From the perspective of the anomalies the bad news is that we cannot account for the FPP nor delayed overshooting. To see this note that the equilibrium exchange rate (5.2) has the same form as that under observational uncertainty in (4.4). Since the robust gain is strictly greater than the baseline gain $\left( k_t^g (\tilde{\sigma}_{w,t}^2) > k_t^\theta (\sigma_w^2) \right)$ with probability approaching one, it follows from (4.14) that the Fama coefficient $\beta^{Fama}$ would be greater than one with probability approaching one if $\theta \in \Theta^w$. Similarly, the average impulse response function (4.12) implies that if $\theta \in \Theta^w$, there cannot be delayed overshooting with any likelihood. To see this note that (4.12) is strictly positive because $q^\theta$ is necessarily lower than $k^\theta$ for any $\theta \in \Theta^w$.  

31
Following a positive forward premium shock the exchange rate appreciates initially and immediately reverts to a depreciating path. That is, there is no delayed overshooting because with probability approaching one the investors’ filter takes persistent shocks to be more abundant than what they actually are in the data.

The proof shows that if the drift $\alpha$ tends to either zero or one in (5.4), then the effect of the variance of excess returns $\frac{\partial \text{Var}_{\theta t}^\gamma (J_{t+1})}{\partial \sigma_{z,t}}$ dominates that of expected returns $\frac{\partial E_{\theta t}^\gamma (J_{t+1})}{\partial \sigma_{z,t}}$. In these limiting cases, the costliest misspecification occurs when the investor’s model wrongly sets the variance of persistent shocks lower than what it actually is. Therefore, under state equation uncertainty, robustness entails distorting upwards the variance of persistent shocks, which in turn implies more sensitivity to news.

5.2. Unstructured Uncertainty

Under unstructured uncertainty the investor does not know the nature of the misspecification and does not know whether it is located in the observation equation, or in the hidden-state equation or in both. Here, we define the unstructured uncertainty set as the set of all probability measures on the measurable space $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(\Omega)$ is the Borel $\sigma$-algebra.

$$\Theta^u = P(\Omega) \equiv \{\theta : \mathcal{B}(\Omega) \rightarrow [0,1] \text{ and } \theta(\Omega) = 1\} \quad (5.5)$$

This set allows for a truly worst-case scenario. Optimizing over the set $\Theta^u$ seems a daunting task. Fortunately a result of the theory of large deviations—the Representation Lemma of Dupuis and Ellis (1997)—implies that the problem of the investor simplifies significantly.

**Lemma 5.4 (Representation Lemma).** Under unstructured uncertainty, the robust forecasting-portfolio problem (3.4) reduces to the following Bayesian problem under the unique baseline probability measure $\theta'$.

$$\Gamma_t = \max_{\theta_t} \inf_{\theta \in P(\Omega)} \left\{ E_t^\theta [u (W_{t+1}) + \lambda \cdot \mathcal{R}(\theta||\theta')] \right\} = \max_{\theta_t} \left\{ -\lambda \log \left( E_t^\theta \exp \left( -\frac{1}{\lambda} u (W_{t+1}) \right) \right) \right\} \quad (5.6)$$

---

33 The set of all probability measures on $\Omega$ is compact by Alaoglu’s theorem (Folland (2001), pp 169).
This Representation Lemma says that under unstructured uncertainty, the robust agent’s problem reduces to a familiar expected utility maximization problem under a unique probability measure, which corresponds to the baseline measure $\theta'$. Under the equivalent representation, the relative entropy disappears and $\inf_{\theta \in P(\Omega)} E_t^\theta u (W_{t+1})$ is replaced by the so called risk-sensitive utility function $-\lambda \log \left( E_t^{\theta'} \exp \left( -\frac{1}{\lambda} u (W_{t+1}) \right) \right)$. This function keeps the baseline probability measure unchanged and, because of the exponential function, captures the desire for robustness by putting more weight on the tails of the distribution. Hence a risk-sensitive agent is more concerned about tail events than a typical risk-averse agent.

From the perspective of accounting for the exchange rate anomalies the bad news is that in this problem the “separation principle” applies: Expectations can be computed independently of the portfolio strategy. The investor forms her expectations using Bayes law under the baseline probability measure $\theta'$ and based on these expectations she then chooses her portfolio. This separation implies that the gain that will appear in the equilibrium exchange rate function is the Bayesian gain $k^{\theta'}$. Thus, we will not get the lower sensitivity of forecasts to news ($k^{\theta^*} < k^{\theta'}$) that underpins the explanation for delayed overshooting and for the forward premium puzzle in Propositions 4.4 and 4.5.

In order to derive the equilibrium we replace $u(W_{t+1}) = -\exp(-\gamma W_{t+1})$ in (5.6) and reexpress problem $\Gamma_t$ as follows:

$$
\Gamma_t = \max_{b_t} \left\{ -\lambda \log \left( E_t^{\theta'} \exp \left( -\frac{1}{\lambda} \exp(-\gamma W_{t+1}) \right) \right) \right\} \Leftrightarrow \Gamma_t = \min_{b_t} \left\{ E_t^{\theta'} \exp \left( -\frac{1}{\lambda} \exp(-\gamma W_{t+1}) \right) \right\}.
$$

We show in the appendix that taking the first order condition with respect to $b_t$, using Stein’s Lemma and the exchange rate conjecture (3.7), it follows that there is an interior solution for $b_t$ only if returns satisfy the following condition:

$$
E_t^{\theta'}(e_{t+1}) - e_t = \left( i_t - i_t^I \right) + l_t.
$$

Condition (5.7) is the well known uncovered interest parity condition plus a time-varying uncertainty premium $l_t$, which is given by

$$
l_t \equiv \frac{VaR_t^{\theta'}(e_{t+1}) E_t^{\theta'} g(J_{t+1})}{E_t^{\theta'} g(J_{t+1})}, \quad \text{where } J_{t+1} \equiv \left( i_t - i_t^I \right) - (e_{t+1} - e_t)
$$

$$
g(J_{t+1}) \equiv -\exp(\gamma b_t^J J_{t+1}) \exp \left( -\frac{1}{\lambda} \exp(\gamma b_t^J e_{t+1}) \right), \quad g'(J_{t+1}) = \left[ 1 - \frac{1}{\lambda} \exp(\gamma b_t J_{t+1}) \right] \gamma b_t^J g'(J_{t+1})
$$
Notice that when the risk-aversion coefficient $\gamma$ is large, $g'(J_{t+1})$ is negative and thus the uncertainty premium is large and positive. In contrast, in the risk-neutral case ($\gamma = 0$), the uncertainty premium becomes zero (because $g'(J_{t+1}) = 0$). This result implies that risk-neutrality combined with a desire for robustness against unstructured uncertainty yields the same equilibrium as risk neutrality under no model uncertainty. Finally, when there is no aversion to model uncertainty (i.e., $1/\lambda$ goes to zero), the uncertainty premium $l_t$ equals $\gamma b_t^2 \text{Var} g_t (e_{t+1})$, which is the same as in a standard rational expectations equilibrium.\(^{34}\)

From the uncovered interest parity condition (5.7) we can find the equilibrium.

Proposition 5.5 (Equilibrium under unstructured uncertainty). Under uncertainty set $\Theta$, the log exchange rate in a robust linear equilibrium is

$$e^*_t = -\left( i_t - i^*_t \right) - \frac{a}{1-a} \hat{x}'_t + \alpha^*_t, \quad \alpha^*_{t+1} = \alpha^*_t + l^*_t,$$

where the estimate of the unobservable trend is given by the standard Kalman filter under the unique baseline probability measure $\theta'$

$$\hat{x}'_t = \left( 1 - k'_t \right) a \hat{x}'_{t-1} + k'_t \left( i_t - i^*_t \right), \quad k'_t = \frac{a^2 \sigma^2_t - \sigma^2_w + \sigma^2_v}{\sigma^2_t - a^2 \sigma^2_{t-1} + \sigma^2_w + \sigma^2_v}, \quad \sigma^2_t - a^2 \sigma^2_{t-1} + \sigma^2_w + \sigma^2_v,$$

and $l^*_t$ is given by (5.8) evaluated at $\beta_1 = -\frac{a}{1-a}, \beta_2 = -1$ and $b_t = b^*_t$.

Can the forecast-distortion mechanism generate the anomalies? The answer is no. Since $e^*_t$ has the same linear form as (4.4), we know from (4.12) and (4.14) in Propositions 4.4 and 4.5 that the first two terms in $e^*_t$: (i) do not generate momentum, and (ii) give rise to $\beta_{\text{Fama}} = 1$ because the robust gain equals the baseline gain under no misspecification $k'_t$. This implies that the forecast-distortion mechanism is not operational under unstructured uncertainty.

6. Literature Review

Following Fama (1984) the FPP has been documented for many currency pairs and time periods (Chinn (2006), Engel (1996) and Lewis (1995) survey the literature). Several

\[^{34}\text{This is because } \lim_{\lambda \to \infty} \frac{E_t^2 g'(J_{t+1})}{E_t g(J_{t+1})} = \gamma b^2_t\]
mechanisms have been proposed to account for the FPP. One group of papers concentrates on the risk premium (Alvarez, et. al. (2006), Bekaert (2006), Frankel and Engel (1984), Lusting and Verdelhan (2006), Nelson and Wu (1998)). A second group concentrates on expectational biases (Frankel and Froot (1989), Lewis (1989), Gourinchas and Tornell (2004)). A third group considers a peso-problem (Kaminski (2003), Farhi and Gabaix (2007)). A forth group looks at the microstructure of the trading mechanism (Bacchetta and Van Wincoop (2006), Burnside et. al. (2007), Evans and Lyons (2002), and Sarno, et. al. (2006)). Finally, Backus et. al. (2001) link the FPP to affine models of the term structure. On the empirical front, Frankel and Froot (1989) find, using exchange rate survey data, that expectational biases can better account for the FPP than risk premia. Using interest rate survey data Gourinchas and Tornell (2004) find that among G7 currencies forecasts of interest rate differentials overstate the relative importance of transitory shocks and that the resulting hump-shaped forecast pattern can account for the FPP. Our model has characterized conditions under which such a forecast-distortion mechanism is the outcome of utility optimization. A delayed response also arises in Bacchetta and Van Wincoop (2008) because transactions costs lead agents to make infrequent portfolio decisions.

Burnside et. al. (2008) construct a hedged carry trade portfolio that exploits the FPP and hedges against large adverse exchange rate changes by buying out-of-the-money options. They find that this strategy is profitable, and that neither a large peso-problem nor standard macroeconomic risk factors typically associated with risk premia can account for the main part its returns. These findings are consistent with our model. Although, in our setup robust agents can guard against many classes of misspecification, like a large peso-problem, in the equilibrium that generates the FPP, agents do not guard against such an extreme event. They guard only against misspecification in the link between observations and the underlying return process, and robustness involves only distorting second moments. In fact, when uncertainty aversion is extreme, so that the distortion reflects the worst-case scenario, robustness does not account for the FPP.

We have focused on the robust filtering problem, and have investigated when is it that it delivers the lower gain that is required for the hump-shaped dynamics that underlie the FPP. Unfortunately, under unstructured uncertainty, the robust filtering problem yields an estimator with the same gain as that of the baseline Bayes estimator, and the
$\mathcal{H}_\infty$ filtering problem yields a larger gain (Basar and Bernhard (1995)). These results lead us to consider instead different types of structured uncertainty, and determine conditions under which the robust filtering problem yields a gain lower than the baseline gain. Then we ask when is it that this property is carried over to the equilibrium exchange rate function in a setup where agents solve a joint portfolio-filtering problem.

This paper is linked to several papers that have used robust control to analyze macroeconomic issues. Brock, et. al. (2007), Cogley and Sargent (2005) and Svensson and Williams (2007) have addressed the issue of robust monetary policy design. Another group of papers investigates whether robustness against model misspecification can shed light on the fact that US monetary policy responses tend to be more cautious than those implied by optimal policies (Dennis (2007), Giannoni (2002), Kasa (2002), Onatski and Stock (2002), Tetlow and von zur Muehled (2006)). Like in this paper, whether robust policy entails cautiousness or aggressiveness depends on the structure of the uncertainty set. A third group studies asset pricing under aversion to model uncertainty (Caggetti, et. al. (2002), Epstein and Wang (1994), Hansen, et. al. (1999), Rigotti and Shannon (2005), and Tornell (2001)). Finally, several papers have analyzed the long-run horizon link between exchange rates and fundamentals as well as their expectations (e.g., Engel and West (2005) and Mark (1995)). Our forecast distortion mechanism works at higher frequencies—a few quarters—and so can be considered as complementary to these papers.

7. Conclusions

This paper analyzes the forward premium puzzle (FPP), a major anomaly in foreign exchange markets, by bringing together the robustness literature and the international finance literature. It characterizes conditions under which robustness against model misspecification tends to generate a negative correlation between interest rate differentials and exchange rate changes, i.e., a negative Fama coefficient. Specifically, we ask when is it that a desire for robustness leads optimizing agents—that hold no misperception—to distort the probability distribution of the data-generating process and to make forecasts of interest rate differentials that tend to under-react to news. For instance, when there is

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35 The $\mathcal{H}_\infty$ filtering problem minimizes the mean-squared error attenuated by the cumulative energy of the disturbances.
a positive shock to the interest rate differential, the currency does not appreciate enough and therefore needs to catch up subsequently before mean reverting. This under-reaction pattern generates a negative Fama coefficient because along the catch-up phase there is a gradual appreciation of the currency alongside a positive interest rate differential. Deriving the forecast distortion that underlies the FPP from an optimizing-robust framework is our contribution to the international finance literature.

We have found that the structure of the uncertainty set is a determining factor of the sensitivity to news of the filter. Robustness against the so-called unstructured uncertainty does not generate forecast under-reaction in the standard robust forecasting-portfolio problem solved by investors in our model. The search for an under-reaction pattern has lead us to consider several types of structured uncertainty. We have solved for the equilibrium with the aid of Girsanov-like change of measure techniques that translate sets of probability measures into parameter distortions and have allowed us to derive in closed-form the forward looking exchange rate and the Fama coefficient. To our knowledge, this constitutes a novel contribution to the robustness literature.

We have found that the forecast-distortion mechanism generates a Fama coefficient that converges in probability to a negative limit if there is uncertainty about the link between observations and the persistent component of the interest rate differential process, provided the differential is highly persistent and there is enough uncertainty aversion. In contrast, a negative Fama coefficient does not arise under unstructured uncertainty nor under uncertainty about the persistent component of the differential. These results are consistent with the finding in the empirical literature that the FPP is less prevalent in high inflation environments than in low inflation ones (e.g., Bansal and Dahlquist (2000)). In the former we should expect that the major source of misspecification is the inflation process, and that nominal interest rate differentials reflect mainly inflation differentials. If the inflation process is persistent and uncertain, we should expect that uncertainty about the persistent component of the nominal interest rate differential is a major source of model uncertainty. Our model would then predict that robust agents will use a higher gain in their forecasts, thereby generating a larger Fama coefficient in high inflation environments than in low inflation ones.

Finally, we would like to mention a few possible extensions. The under-reaction that underlies our explanation for the FPP can be used to account for the event-based
momentum anomaly found in the finance literature. Since our forecast distortion mechanism generates price changes in the same direction as the price change at the time of the event, and our equilibrium has the same linear form as the standard rational expectations equilibrium price functions, such a link seems relatively straightforward. Another extension would be to explain the existence of the carry trade, under which investors borrow in low yielding currencies and invest in high yielding currencies.

References


8. Appendix

Proof of Lemma 4.1. To prove part 1, first we show that if \( y_t \mid x_t \) follows a normal distribution \( N(x_t, \sigma_v^2) \) under the baseline model \( \theta' \), then under any model \( \theta \in \Theta^\nu \) we have that \( y_t \mid x_t \sim N(x_t, \sigma_v^2) \). Second, we show that \( x_t \mid x_{t-1} \) and \( x_{t-1} \) have the same distribution under \( \theta \) as under \( \theta' \). We start by computing the probability distribution of \( y_t \mid x_t \) under probability measure \( \theta \).

\[
P^\theta(y_t < z \mid x_t) = \int_{\{y_t < z\}} d\theta = \int_{\{y_t < z\}} d\theta' = \int_{\{y_t < z\}} \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) (y_t - x_t)^2 \right) \cdot \sqrt{\frac{\sigma_v^2}{\sigma_v^2}} d\theta'
\]

\[
= \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) (y_t - x_t)^2 \right) \cdot \sqrt{\frac{\sigma_v^2}{\sigma_v^2}} \sqrt{\frac{1}{2\pi\sigma_v^2}} \exp \left( -\frac{(y_t - x_t)^2}{2\sigma_v^2} \right) dy_t
\]

The last equality shows that, conditional on \( x_t \), \( y_t \) follows \( N(x_t, \sigma_v^2) \) under measure \( \theta \).

Next, we compute the distribution function of \( x_t \mid x_{t-1} \) under \( \theta \)

\[
P^\theta(x_t < z \mid x_{t-1}) = E^\theta 1_{\{x_t < z \mid x_{t-1}\}} = E^\theta' 1_{\{x_t < z \mid x_{t-1}\}} \frac{d\theta}{d\theta'}
\]

\[
= E^\theta' 1_{\{x_t < z \mid x_{t-1}\}} \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) v_t^2 \right) \cdot \sqrt{\frac{\sigma_v^2}{\sigma_v^2}}
\]

\[
= E^\theta' 1_{\{x_t < z \mid x_{t-1}\}} E^\theta' \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) v_t^2 \right) \frac{d\theta'}{\sqrt{2\pi\sigma_v^2}}
\]

The last equality follows because random variables \( x_t \) and \( v_t \) are independent under \( \theta' \).

Notice that the second expectation in the last equality is equal to one because

\[
E^\theta' \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) v_t^2 \right) \sqrt{\frac{\sigma_v^2}{\sigma_v^2}} = \int \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) v_t^2 \right) \frac{d\theta'}{\sqrt{2\pi\sigma_v^2}}
\]

\[
= \int \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) v_t^2 \right) \sqrt{\frac{\sigma_v^2}{\sigma_v^2}} \frac{1}{\sqrt{2\pi\sigma_v^2}} \exp \left( -\frac{1}{2} \frac{v_t^2}{\sigma_v^2} \right) dv_t = 1
\]

Therefore, we have that \( P^\theta(x_t < z \mid x_{t-1}) = E^\theta 1_{\{x_t < z \mid x_{t-1}\}} = E^\theta' 1_{\{x_t < z \mid x_{t-1}\}} = P^\theta'(x_t < z \mid x_{t-1}) \).

This shows that \( x_t \mid x_{t-1} \) has the same distribution under \( \theta \) as under \( \theta' \). Lastly, we use a
similar argument to show that \( x_{t-1} \) has the same distribution under \( \theta \) as under \( \theta' \).

\[
P^\theta(x_{t-1} < z) = E^\theta 1_{\{x_{t-1} < z\}} = E^{\theta'} 1_{\{x_{t-1} < z\}} \frac{d\theta}{d\theta'} = E^{\theta'} 1_{\{x_t < z\}} \exp \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) v_t^2 \right) \sqrt{\frac{\sigma_v^2}{\sigma_v^2}} = E^{\theta'} 1_{\{x_t < z\}}
\]

To prove part 2 we derive the relative entropy.

\[
R(\theta || \theta') = E^\theta \log \left( \frac{d\theta}{d\theta'} \right) = E^\theta \left( -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) (y_t - x_t)^2 + \frac{1}{2} \left( \log (\sigma_v^2) - \log (\sigma_v^2) \right) \right) = -\frac{1}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) \cdot E^\theta (v_t^2) + \frac{1}{2} \left( \log (\sigma_v^2) - \log (\sigma_v^2) \right) = \frac{1}{2} \left( \frac{\sigma_v^2}{\sigma_v^2} - \log \left( \frac{\sigma_v^2}{\sigma_v^2} \right) - 1 \right)
\]

**Proof of Proposition 4.2.** At time \( t \), the \( t \)-agent observes \( y_t \) and \( e_t \). Given probability measure \( \theta_t \) (equivalently, given the distorted variance \( \sigma_v^2 \)), she constructs estimates of \( x_t \) and \( x_{t+1} \) using Bayes law, given prior estimate \( \hat{x}_{t-1}^{\theta_t} \) and prior variance \( \sigma_{t-1}^2 \):

\[
\frac{d\theta_t}{d\theta_{t-1}} = \frac{1}{1 - k_t^{\theta_t}} a x_{t-1}^{\theta_t} + k_t^{\theta_t} y_t, \quad k_t^{\theta_t} = \frac{a^2 \sigma_{t-1}^2 + \sigma_v^2}{a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_v^2}, \quad \sigma_{t-1}^2 = \frac{(a^2 \sigma_v^2 + \sigma_v^2)^2}{a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_v^2}
\]

\[
x_t | I_t \sim N \left( \hat{x}_{t-1}^{\theta_t}, \frac{a^2 \sigma_{t-1}^2 + \sigma_v^2}{a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_v^2} \right), \quad x_{t+1} | I_t \sim N \left( \hat{x}_{t}^{\theta_t}, \frac{a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_v^2}{a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_v^2} \right)
\]

where \( I_t = \{ e_t, y_t, \hat{x}_{t-1}^{\theta_t} \} \). Using her knowledge of the equilibrium exchange rate function (3.7), the \( t \)-agent calculates the excess rate of return \( (J_{t+1} = y_t - e_{t+1} + e_t) \) as \( J_{t+1} = y_t - (\alpha_{t+1} + \beta_1 x_{t+1}^{\theta_t} + \beta_2 y_{t+1}) + e_t \). Furthermore, the \( t \)-agent knows that the \( t + 1 \)-agent will (i) use the same method to distort the probability measure \( \theta_{t+1} \) as the one used by the \( t \)-agent, and (ii) will estimate the hidden state \( x_{t+1} \) using Bayes law under this \( \theta_{t+1} \), with a given prior mean \( \hat{x}_{t}^{\theta_t} \) and variance \( \sigma_v^2 \). It follows that we can replace \( \hat{x}_{t+1}^{\theta_t} \) by \( (1 - k_{t+1}^{\theta_t}) a \hat{x}_{t}^{\theta_t} + k_{t+1}^{\theta_t} y_{t+1} \) and obtain \( J_{t+1} = y_t - \alpha_{t+1} - (1 - k_{t+1}^{\theta_t}) \beta_1 a \hat{x}_{t}^{\theta_t} - (k_{t+1}^{\theta_t} \beta_1 + \beta_2) y_{t+1} + e_t \). Since \( E_t^{\theta_t}(y_{t+1}) = a \hat{x}_{t}^{\theta_t} \), under probability measure \( \theta_t \in \Theta^v \), \( J_{t+1} \) is normally distributed with the following conditional mean and variance.

\[
E_t^{\theta_t}(J_{t+1}) = y_t - (\beta_1 + \beta_2) a \hat{x}_{t}^{\theta_t} - \alpha_{t+1} + e_t
\]

\[
V_t^{\theta_t}(J_{t+1}) = \left( k_{t+1}^{\theta_t} \beta_1 + \beta_2 \right)^2 \frac{a^2 \sigma_{t-1}^2 + \sigma_v^2}{a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_v^2}
\]

\[
(8.1)
\]

\[
(8.2)
\]
Next, note that problem (4.2) is equivalent to (3.8) because for any normally distributed random variable \( z \), \( E^\theta (\exp [-\gamma z]) = \exp \left(-\gamma E^\theta (z) + \frac{\gamma^2}{2} \text{Var}^\theta (z) \right) \). We solve problem (4.2) by considering the investor as a Stackelberg leader, that takes into account the strategy of nature: \( \sigma^2_{v,t} = \tilde{\sigma}^2_{v,t} (b_t) \). Nature then selects \( \tilde{\sigma}^2_{v,t} \) conditioning on the agent’s choice of \( b_t \). The FOC with respect to \( b_t \) is

\[
\frac{\partial \Gamma}{\partial b_t} = \left( \gamma E^\theta_t (J_{t+1}) - \gamma^2 b_t \text{Var}^\theta_t (J_{t+1}) \right) \frac{\exp \left( \frac{(\gamma b_t)^2}{2} \text{Var}^\theta_t (J_{t+1}) \right)}{\exp \left( \gamma b_t E^\theta_t (J_{t+1}) \right)} + \frac{\partial \Gamma}{\partial \tilde{\sigma}^2_{v,t}} \frac{d\tilde{\sigma}^2_{v,t}}{db_t} = 0 \quad (8.3)
\]

Since the FOC of nature’s problem implies \( \frac{\partial \Gamma}{\partial \tilde{\sigma}^2_{v,t}} = 0 \), the last term in (8.3) is zero, and so in an interior solution, \( \frac{\partial \Gamma}{\partial b_t} = 0 \) implies \( b_t = \frac{E^\theta_t (J_{t+1})}{\gamma \text{Var}^\theta (J_{t+1})} \), which is equation (4.3) in the text.

The SOC with respect to \( b_t \) is

\[
0 > \frac{\partial^2 \Gamma}{\partial b_t^2} = \Gamma_{b_t b_t} (b_t, \tilde{\sigma}^2_{v,t} (b_t)) + \Gamma_{b_t \tilde{\sigma}^2_{v,t} (b_t)} \left( \frac{d\tilde{\sigma}^2_{v,t} (b_t)}{db_t} \right)^2 + \Gamma_{\tilde{\sigma}^2_{v,t} \tilde{\sigma}^2_{v,t} (b_t)} \left( \frac{d\tilde{\sigma}^2_{v,t} (b_t)}{db_t} \right)^2 \left( \frac{\Gamma_{\tilde{\sigma}^2_{v,t} \tilde{\sigma}^2_{v,t} (b_t)}}{\Gamma_{\tilde{\sigma}^2_{v,t} \tilde{\sigma}^2_{v,t} (b_t)}} \right)^2 + \Gamma_{\tilde{\sigma}^2_{v,t} b_t} (b_t, \tilde{\sigma}^2_{v,t} (b_t)) \left( \frac{d\tilde{\sigma}^2_{v,t} (b_t)}{db_t} \right) = \Gamma_{b_t b_t} (b_t, \tilde{\sigma}^2_{v,t} (b_t)) < 0.
\]

This condition is unambiguously satisfied because

\[
0 > \Gamma_{b_t b_t} = -\left( \gamma E^\theta_t (J_{t+1}) + (\gamma b_t) \text{Var}^\theta_t (J_{t+1}) \right) \frac{\exp \left( \frac{(\gamma b_t)^2}{2} \text{Var}^\theta_t (J_{t+1}) \right)}{\exp \left( \gamma b_t E^\theta_t (J_{t+1}) \right)}.
\]

To derive the equilibrium, notice that the market demand for the bond can be expressed as \( b_t^* = \frac{y_t - a(\beta_1 + \beta_2) \hat{x}_t + \alpha_{t+1} + \alpha_{t+2}}{\gamma \text{Var}^\theta_t (J_{t+1})} \). Thus, the market-clearing condition \( b_t^* = b_t^* \) implies that the equilibrium exchange rate \( e_t^* = -y_t - a(\beta_1 + \beta_2) \hat{x}_t^* + \alpha_{t+1} + \gamma b_t^* \text{Var}^\theta_t (J_{t+1}) \). To derive the equilibrium values of \((\beta_1, \beta_2, \alpha)\) we equalize coefficients with conjecture (3.7), and obtain \( \alpha_t^* = \alpha_{t+1} + \gamma b_t^* \text{Var}^\theta_t (J_{t+1}) \), \( \beta_2 = -1 \), and \( \beta_1 = \alpha_1 + \alpha_2 = -\frac{a}{a} \).

This proves that, given \( \sigma^2_{v,t} \), the equilibrium exchange rate function \( e_t^* \) is (4.4). Next, we analyze \( \tilde{\sigma}^2_{v,t} \). The first order condition with respect to \( \tilde{\sigma}^2_{v,t} \) is

\[
\frac{\partial \Gamma}{\partial \tilde{\sigma}^2_{v,t}} = \left[ \gamma b_t \frac{\partial E^\theta_t (J_{t+1})}{\partial \tilde{\sigma}^2_{v,t}} - \frac{(\gamma b_t)^2}{2} \frac{\partial \text{Var}^\theta_t (J_{t+1})}{\partial \tilde{\sigma}^2_{v,t}} \right] \frac{\exp \left( \frac{(\gamma b_t)^2}{2} \text{Var}^\theta_t (J_{t+1}) \right)}{\exp \left( \gamma b_t E^\theta_t (J_{t+1}) \right)} + \frac{\lambda}{2} \left( \frac{1}{\sigma^2_v} - \frac{1}{\tilde{\sigma}^2_{v,t}} \right) = 0 \quad (8.4)
\]

Notice that expected excess returns \( E^\theta_t (J_{t+1}) \) depend on the estimate of the state \( \hat{x}_t^* \), that in turn depends on the gain \( k_t^* \), which is a function of \( \tilde{\sigma}^2_{v,t} \). From (8.1) we have that
given $\beta_1$ and $\beta_2$

$$\frac{\partial E_t^{\theta_t} (J_{t+1})}{\partial \sigma_{v,t}^2} = -a (\beta_1 + \beta_2) \frac{d\lambda_i^{\theta_t}}{d\lambda_{v,t}^2},$$

where

$$\frac{d\lambda_i^{\theta_t}}{d\lambda_{v,t}^2} = y_t - a \dot{\lambda}_{v,t-1}, \quad \frac{d\lambda_{v,t}^{\theta_t}}{d\sigma_{v,t}^2} = -\frac{a^2 \sigma_{v,t-1}^2 + \sigma_w^2}{(a^2 \sigma_{v,t-1}^2 + \sigma_w^2 + \sigma_{v,t}^2)^2} < 0. \quad (8.5)$$

From (8.2) we have that the derivative of the conditional variance is:

$$\frac{\partial V ar_t^{\theta_t} (J_{t+1})}{\partial \sigma_{v,t}^2} = \left( k_{t+1}^{\theta_t} \beta_1 + \beta_2 \right)^2 \left( a^2 \left( \frac{(a^2 \sigma_{v,t-1}^2 + \sigma_w^2)^2}{(a^2 \sigma_{v,t-1}^2 + \sigma_w^2 + \sigma_{v,t}^2)^2} + 1 \right) \right) > 0. \quad (8.6)$$

Replacing $\beta_1$ and $\beta_2$ by their equilibrium values $\beta_1^* = -\frac{a}{1-a}$ and $\beta_2^* = -1$, it follows that $\sigma_{v,t}^{2*}$ is determined by

$$\frac{\lambda}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_{v,t}^{2*}} \right) = \left[ -\gamma b_i^* \frac{a}{1-a} \left( y_t - a \dot{\lambda}_{v,t-1} \right) \frac{d\lambda_{v,t}^{\theta_t}}{d\sigma_{v,t}^2} + \frac{(\gamma b_i^*)^2}{2} \frac{\partial V ar_t^{\theta_t} (J_{t+1})}{\partial \sigma_{v,t}^2} \right] W_t^*, \quad (8.7)$$

where

$$\frac{d\lambda_{v,t}^{\theta_t}}{d\sigma_{v,t}^2} = -\frac{a^2 \sigma_{v,t-1}^2 + \sigma_w^2}{(a^2 \sigma_{v,t-1}^2 + \sigma_w^2 + \sigma_{v,t}^2)^2} < 0, \quad (8.8)$$

$$\frac{\partial V ar_t^{\theta_t}}{\partial \sigma_{v,t}^2} = \left( k_{t+1}^{\theta_t} \frac{a}{1-a} + 1 \right)^2 \left( a^2 \left( \frac{(a^2 \sigma_{v,t-1}^2 + \sigma_w^2)^2}{(a^2 \sigma_{v,t-1}^2 + \sigma_w^2 + \sigma_{v,t}^2)^2} + 1 \right) \right) > 0, \quad (8.9)$$

and $W_t^*$ is the optimized utility function:

$$W_t^* = \exp \left( -\gamma b_i^* E_t^{\theta_t} (J_{t+1}) + \frac{1}{2} (\gamma b_i^*)^2 V ar_t^{\theta_t} (J_{t+1}) \right) > 0. \quad (8.10)$$

Next, we verify the conditions for the existence of a positive and bounded distorted variance $\sigma_{v,t}^{2*}$. First, notice that (8.7) implies

$$\dot{\sigma}_{v,t}^{2*} = \left[ \frac{1}{\sigma_v^2} + \frac{2}{\lambda} \left[ \gamma b_i \frac{\partial E_t^{\theta_t} (J_{t+1})}{\partial \sigma_{v,t}^2} - \frac{(\gamma b_i)^2}{2} \frac{\partial V ar_t^{\theta_t} (J_{t+1})}{\partial \sigma_{v,t}^2} \right] \right] W_t^*$$

Thus, $\dot{\sigma}_{v,t}^{2*}$ is positive only if $\lambda$ satisfies the following condition

$$\lambda > \lambda_t^# \equiv \sigma_v^2 \left[ -\gamma b_i \frac{\partial E_t^{\theta_t} (J_{t+1})}{\partial \sigma_{v,t}^2} + \frac{(\gamma b_i)^2}{2} \frac{\partial V ar_t^{\theta_t} (J_{t+1})}{\partial \sigma_{v,t}^2} \right] W_t^* \quad (8.11)$$
Second, the SOC of nature’s problem in equilibrium is

\[
\frac{\partial^2 \Gamma}{\partial (\sigma_{v,t}^2)^2} = \left[ \frac{\gamma_b}{2} \frac{\partial^2 E_t^{g_t} (J_t+1) \partial E_t^{g_t} (J_t+1)}{\partial (\sigma_{v,t}^2)^2} - \frac{(\gamma_b)^2}{2} \frac{\partial^2 V a r_t^{g_t} (J_t+1) \partial V a r_t^{g_t} (J_t+1)}{\partial (\sigma_{v,t}^2)^2} \right] W_t^* \\
+ \left[ \frac{\gamma_b}{2} \frac{\partial E_t^{g_t} (J_t+1)}{\partial \sigma_{v,t}^2} - \frac{(\gamma_b)^2}{2} \frac{\partial V a r_t^{g_t} (J_t+1)}{\partial \sigma_{v,t}^2} \right] \frac{\partial W_t^*}{\partial \sigma_{v,t}^2} + \frac{\lambda}{2} \frac{1}{(\sigma_{v,t}^2)^2} > 0
\]

It holds if and only if \( \lambda > \lambda_t^* \), where \( \lambda_t^* \) is defined by

\[
\lambda_t^* \equiv 2 (\sigma_{v,t}^2)^2 \left[ \frac{(\gamma_b)^2}{2} \frac{\partial^2 V a r_t^{g_t} (J_t+1) \partial V a r_t^{g_t} (J_t+1)}{\partial (\sigma_{v,t}^2)^2} - \gamma_b \frac{\partial E_t^{g_t} (J_t+1) \partial E_t^{g_t} (J_t+1)}{\partial (\sigma_{v,t}^2)^2} \right] W_t^* \\
+ 2 (\sigma_{v,t}^2)^2 \left[ \frac{(\gamma_b)^2}{2} \frac{\partial V a r_t^{g_t} (J_t+1)}{\partial \sigma_{v,t}^2} - \gamma_b \frac{\partial E_t^{g_t} (J_t+1)}{\partial \sigma_{v,t}^2} \right] \frac{\partial W_t^*}{\partial \sigma_{v,t}^2}
\]

(8.12)

where \( \frac{\partial^2 V a r_t^{g_t} (J_t+1)}{\partial (\sigma_{v,t}^2)^2} = -2a^2 \left[ k_t^{\sigma_{t-1}} \beta_1 + \beta_2^2 \right] \frac{a^2 (a^2 \sigma_{t-1}^2 + \sigma_w^2)}{(a^2 \sigma_{t-1}^2 + \sigma_w^2 + \sigma_{v,t}^2)^2} < 0 \). Since \( \frac{1}{2} (\gamma_b)^2 \left( \frac{\partial V a r_t^{g_t} (J_t+1)}{\partial \sigma_{v,t}^2} \right)^2 > 0 \), \( \lambda_t^* \) has an ambiguous sign. Combining the positivity constraint and the SOC for \( \sigma_{v,t}^2 \), it follows that a linear robust equilibrium exists if \( \lambda > \lambda_t^* \) and \( \lambda > \lambda_t^\# \). That is,

\[
1/\lambda_t^\# = \min \{1/\lambda_t^*, 1/\lambda_t^\#\}
\]

(8.13)

This proves part 2. To prove part 3 note that if either \( 1/\lambda = 0 \) or \( \gamma = 0 \), or \( b_t^S = 0 \) then there is no distortion because (??) becomes \( \frac{1}{\sigma_t^v} - \frac{1}{\sigma_{v,t}^2} = 0 \Rightarrow \sigma_{v,t}^2 = \sigma_v^2 \).

**Proof of Corollary ??**. Notice that the RHS of (8.7) is a quadratic function of \( z \equiv \frac{1}{2} \gamma b_t^S \frac{a}{1-a} \):

\[
\lambda \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_{v,t}^2} \right) = \left[ A (a,t) z^2 + B (a,t) z + C (a,t) \right] W_t^* \equiv f (z) W_t^*, \quad z \equiv \frac{1}{2} \gamma b_t^S \frac{a}{1-a}.
\]

(8.14)

The coefficients \( A (a,t) \), \( B (a,t) \), \( C (a,t) \) are given by:

\[
A (a,t) \equiv \left( k_t^{\sigma_{t-1}} \right)^2 \left( a^2 \frac{(a^2 \sigma_{t-1}^2 + \sigma_w^2)^2}{(a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_{v,t}^2)^2} + 1 \right)
\]

(8.15)

\[
B (a,t) \equiv 2 \left( y_t - a \sigma_{t-1} \right) \frac{a^2 \sigma_{t-1}^2 + \sigma_w^2}{(a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_{v,t}^2)^2} + 2 \gamma b_t^S \left( a^2 \frac{(a^2 \sigma_{t-1}^2 + \sigma_w^2)^2}{(a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_{v,t}^2)^2} \right)
\]

(8.16)

\[
C (a,t) \equiv (\gamma b_t^S)^2 \left( a^2 \frac{(a^2 \sigma_{t-1}^2 + \sigma_w^2)^2}{(a^2 \sigma_{t-1}^2 + \sigma_v^2 + \sigma_{v,t}^2)^2} + 1 \right)
\]

(8.17)

We will show that if \( a \uparrow 1 \), then given any bond position \( b_t^S \) and any news surprise \( y_t - a \sigma_{t-1} \), the RHS of (8.14) is positive. This is because as \( a \uparrow 1 \), the roots of the
equation $f(z) = 0$ remain bounded, while $z$ goes to either positive or negative infinity. Since $f(z)$ is a convex parabola (because the coefficient $A$ is positive), it follows that $f(z)$ goes to positive infinity when $a \uparrow 1$.

**Step 1.** We show that as $a \uparrow 1$, the roots of the equation $f(z) = 0$ remain bounded by showing that the limits, as $a \uparrow 1$, of parameters $A(a,t)$, $B(a,t)$ and $C(a,t)$ exist. To see this, we derive the limits of $A(a,t)$, $B(a,t)$ and $C(a,t)$ as follows. First, by substituting $k_{t+1}^{\theta_t} = \frac{a^2 \sigma_t^2 + \sigma_w^2}{a^2 \sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}}$ in (8.15) we have:

$$A_t \equiv \lim_{a \uparrow 1} A(a,t)$$

$$= \lim_{a \uparrow 1} \left( \frac{a^2 \sigma_t^2 + \sigma_w^2}{a^2 \sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) \left( \frac{a^2}{a^2 \sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) + 1 \right) > 0,$$

where $\hat{\sigma}_t^2 = \lim_{a \uparrow 1} \sigma_t^2$. Recall that $\sigma_t^2 = \frac{a \sigma_t^2 + \sigma_w^2}{a \sigma_t^2 + \sigma_w^2 + \sigma_{t+1}^2}$, so we have $\sigma_t^2 = \frac{a \sigma_t^2 + \sigma_w^2}{a \sigma_t^2 + \sigma_w^2 + \sigma_{t+1}^2}$, with initial value $\sigma_0^2 = 0$. Since $\sigma_t^2$ is bounded above by $\sigma_w^2$, we can see that $\sigma_t^2$ is well defined and does not explode as $a \uparrow 1$. Second, substituting for $k_{t+1}^{\theta_t}$ and $\hat{x}_t^{\theta_t} = \left(1 - k_t \right) \hat{x}_{t-1}^{\theta_t} + k_t \hat{x}_t$ in (8.16) we have:

$$B_t \equiv \lim_{a \uparrow 1} B(a,t)$$

$$= \lim_{a \uparrow 1} \left( y_t - a \hat{x}_{t-1}^{\theta_t} \right) \left( \frac{a^2 \sigma_t^2 + \sigma_w^2}{a^2 \sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) + 2 \left( \gamma b_t^x \right) k_{t+1}^{\theta_t} \left( \frac{a^2}{a^2 \sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) + 1 \right)$$

$$= \left( y_t - a \hat{x}_{t-1}^{\theta_t} \right) \left( \frac{\sigma_t^2 + \sigma_w^2}{\sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) + 2 \left( \gamma b_t^x \right) \left( \frac{\sigma_t^2 + \sigma_w^2}{\sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) + 1 \right) + 1,$$

where $\hat{x}_{t-1}^{\theta_t} = \left(1 - \frac{\sigma_t^2 + \sigma_w^2}{\sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) \hat{x}_{t-2}^{\theta_t} + \frac{\sigma_t^2 + \sigma_w^2}{\sigma_t^2 + \sigma_w^2 + \sigma_{t+1}^2} y_{t-1}$ with initial values $\hat{x}_0^0 = x_0$ and $y_0 = 0$. Third, taking the limit of $C(a,t)$ we have

$$C_t \equiv \lim_{a \uparrow 1} C(a,t)$$

$$= \lim_{a \uparrow 1} \left( \gamma b_t^y \right)^2 \left( \frac{a^2}{a^2 \sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) + 1 \right)$$

$$= \left( \gamma b_t^y \right)^2 \left( \frac{\sigma_t^2 + \sigma_w^2}{\sigma_t^2 + \sigma_w^2 + \hat{\sigma}_{t+1}^{2*}} \right) + 1 \right).$$

Since $y_t$ and $b_t^x$ are given exogenously, $B_t$ and $C_t$ are well-defined if a robust linear equilibrium exists.

**Step 2.** The function $f(z)$ is a convex parabola because the coefficient $A(a,t)$ is
positive for any \( a \in (0,1] \). Let us consider the three possible cases. First, if \( B(a,t)^2 < 4A(a,t)C(a,t) \), the equation \( f(z) = 0 \) has no real roots, and so \( f(z) > 0 \) for all \( z \). Second, if \( B(a,t)^2 > 4A(a,t)C(a,t) \), the equation \( f(z) = 0 \) has two real roots, both of which are finite as \( a \) converges to one: \( \overline{z}_1 = \frac{-B_1+\sqrt{B_1^2-4A_1C_1}}{2A_1} \) and \( \overline{z}_2 = \frac{-B_1+\sqrt{B_1^2-4A_1C_1}}{2A_1} \).

But as \( a \) converges to one, \( z \) goes to either \(+\infty\) (if \( b_t^* > 0 \)) or \(-\infty\) (if \( b_t^* < 0 \)). Therefore, there exists an \( a \) close to one, such that either \( z > \overline{z}_2 \) or \( z < \overline{z}_1 \). In either case \( f(z) > 0 \).

Third, if \( B(a,t)^2 = 4A(a,t)C(a,t) \), the equation \( f(z) = 0 \) has only one real root. When \( a \) gets closer to one, the root converges to \( \overline{z} = -\frac{B_1}{2A_1} \), while \( z \) goes to either \(+\infty\) (if \( b_t^* > 0 \)) or \(-\infty\) (if \( b_t^* < 0 \)). Thus, as \( a \uparrow 1 \), we have that \( |z| > \overline{z} \), and so \( f(z) \) must be positive too.

Since in the three possible cases, \( f(z) > 0 \), it follows that there exists a small \( \varepsilon > 0 \), such that \( f(z) \) is large positive for any \( a \in (1-\varepsilon,1) \). Therefore, if \( a \uparrow 1 \), the RHS of (8.14) is positive for any data news \( y_t - ax^t_{t-1} \) and any bond position \( b_t^* \). Hence, we conclude that for any \( \delta > 0 \):

\[
\lim_{a \uparrow 1} \Pr \{ \text{RHS of (8.14)} > \delta \} = 1.
\]

It follows from (8.14) that if \( a \uparrow 1 \), then for any data news and any bond position, the observational variance is distorted upwards with probability approaching one:

\[
\Pr \{ \tilde{\sigma}_{v,t}^2 > \sigma_v^2 \} \uparrow 1, \quad \text{as } a \uparrow 1.
\]

Lastly, it follows from the gain equations \( k_t^{\theta_t} \equiv \frac{a^2\sigma_{t-1}^2 + \sigma_w^2}{a^2\sigma_{t-1}^2 + \sigma_w^2 + \sigma_{v,t}^2} \) and \( k_t^\theta \equiv \frac{a^2\sigma_{t-1}^2 + \sigma_w^2}{a^2\sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} \) that

\[
\Pr \{ k_t^{\theta_t} < k_t^\theta \} \uparrow 1, \quad \text{as } a \uparrow 1.
\]

This concludes the proof.

**Lemma 8.1 (Steady State).** If the supply of bonds follows

\[
b_t^* = \begin{cases} 
\bar{b}, & t \leq T \\
-\bar{b} \exp (T-t) \eta_t, & t \in (T,\infty),
\end{cases}
\quad \eta_t \equiv (V_t^{\theta_t} (J_{t+1}))^{-1},
\]

and initial time is in the infinite past, then on \( t \in [0,T] \): (i) the equilibrium exchange rate (4.4) has a deterministic \( \alpha_t^* \) (given by (8.23)); (ii) the filter has a constant gain given by (4.8); (iii) the distorted variance \( \tilde{\sigma}_v^2 \) is a constant given by (8.20) and \( \lambda^v \), the lower bound for \( \lambda \), is given by (8.24).

**Proof.** We use the standard result in the control literature that if time starts in the infinite past, then \( \sigma_t^2 \) has converged to its steady-state value \( \xi^* \) for any time \( t \in [0,T] \), where \( \xi^* \) solves the steady-state equation for \( \sigma_t^2 = \frac{(a^2\sigma_{t-1}^2 + \sigma_w^2)\sigma_v^2}{a^2\sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} \). Since the supply of bonds \( b_t^* \) is a constant \( \bar{b} \) on \([0,T]\), in equilibrium \( k^{\theta^*} \) and \( \tilde{\sigma}_v^{2*} \) are jointly determined by
the following system of equations.

\[ k^{\theta^*} = \frac{a^2 \xi^* + \sigma_w^2}{a^2 \xi^* + \sigma_w^2 + \sigma_v^2}, \quad \text{with} \quad \xi^* = \frac{-(\sigma_w^2 + \sigma_v^2 - a^2 \sigma_t^2) + \sqrt{(\sigma_w^2 + \sigma_v^2 - a^2 \sigma_t^2)^2 + 4a^2 \sigma_w^2 \sigma_v^2}}{2a^2} \]  

(8.19)

\[ \lambda \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_v^2} \right) = \left[ \gamma \bar{b} \left( \frac{a k^{\theta^*}}{1-a} + 1 \right) k^{\theta^*} \right]^2 \cdot \left( \frac{a^2}{a^2 \xi^* + \sigma_w^2 + \sigma_v^2} + \frac{1}{\sigma_v^2} \right) \cdot \exp \left( \frac{(\gamma \bar{b})^2}{2} \left[ \frac{a k^{\theta^*}}{1-a} + 1 \right]^2 \left( \frac{a^2 k^{\theta^*} \sigma_v^2 + \sigma_w^2 + \sigma_v^2}{\sigma_v^2} \right) \right). \]  

(8.20)

Therefore, \( Var_t^{\theta^*} (J_{t+1}) \) is also constant for \( t \in [0, T] \): \( Var_t^{\theta^*} (J_{t+1}) = \left[ \frac{a k^{\theta^*}}{1-a} + 1 \right]^2 \Phi^* \) with \( \Phi^* = \left( \frac{a^2 (a^2 \xi^* + \sigma_v^2)}{a^2 \xi^* + \sigma_w^2 + \sigma_v^2} + \sigma_w^2 + \sigma_v^2 \right). \) Next, we show that for any fixed \( T \) there exists a finite \( \alpha_t^* \) so that the exchange rate function does not explode. For \( t \in [0, T] \),

\[ \alpha_t = \alpha_{t+1} + \frac{1}{2} \gamma b^t Var_t^{\theta^*} (J_{t+1}) = \alpha_{T+1} + \frac{1}{2} \gamma \sum_{i=1}^{T-t} \tilde{b} \left[ \frac{a k^{\theta^*}}{1-a} + 1 \right]^2 \Phi^*. \]  

(8.21)

Notice that \( \alpha_{T+1} = \alpha_T + \frac{1}{2} \gamma \sum_{i=1}^{j} b^t Var_t^{\theta^*} (J_{T+i+1}) \). Letting \( j \) go to infinity, we have

\[ \alpha_{T+1} = \alpha_{\infty} + \frac{1}{2} \gamma \sum_{i=1}^{\infty} b_i^t Var_t^{\theta^*} (J_{T+i+1}) \]  

(8.22)

To find \( \alpha_{\infty} \) note that (8.21) implies \( \alpha_{\infty} = \alpha_{\infty} + \lim_{t \to \infty} \frac{1}{2} \gamma b^t Var_t^{\theta^*} (J_{t+1}) \), and that the \( b^t \) process in (8.18) implies that for \( t \in (T, \infty] \)

\[ \lim_{t \to \infty} \frac{1}{2} \gamma b^t Var_t^{\theta^*} (J_{t+1}) = \lim_{t \to \infty} \frac{1}{2} \gamma b \exp (- (t-T)) \eta_t Var_t^{\theta^*} (J_{t+1}) = \lim_{t \to \infty} \frac{1}{2} \gamma b \exp (- (t-T)) = 0. \]

Therefore, \( \alpha_{\infty} \) exists and is equal to a constant \( \alpha_{\infty} = c \in R \). Plugging back \( \alpha_{\infty} = c \) into equation (8.22), we have that

\[ \alpha_{T+1} = c + \frac{1}{2} \gamma \sum_{i=1}^{\infty} b_{T+i} Var_{T+i}^{\theta^*} (J_{T+i+1}) = c + \frac{1}{2} \gamma \sum_{i=1}^{\infty} \tilde{b} \exp (-i) = c + \frac{1}{2} \gamma \frac{\tilde{b}}{1 - \exp(-1)}. \]

This proves that \( \alpha_{T+1} = c + \frac{1}{2} \gamma \frac{\tilde{b}}{1 - \exp(-1)} \) is finite. Hence, for any fixed \( T \) and \( t \in [0, T] \),
\( \alpha_t^* \) can be expressed as

\[
\alpha_t^* = c + \gamma \frac{\bar{b}}{2(1 - \exp(-1))} + \frac{T - t}{2} \gamma \bar{b} \left[ \frac{ak^\theta^*}{1 - a} + 1 \right]^2 \Phi^* < \infty, \quad \text{with} \quad (8.23)
\]

\[
\Phi^* = \left( \frac{a^2 (a^2 + \sigma_w^2) \sigma_v^2}{a^2 + \sigma_w^2 + \sigma_v^2} \right) \quad \text{and} \quad c \in R.
\]

Furthermore, the lower bound for \( \lambda \) is

\[
\lambda^v = Max\{\lambda^*, \lambda^\#\} \quad (8.24)
\]

where \( \lambda^* \) is \( \lambda_t^* \) in equation (8.13) and \( \lambda^\# \) is \( \lambda_t^\# \) in equation (8.11) evaluated at steady-state values.

**Proof of Proposition 4.4.** Let \( y_0 = x_0 = \hat{x}_0 = 0 \) and recall that under the data generating process \( \theta', \upsilon_t \sim N(0, \sigma_w^2) \) and \( \upsilon_t \sim N(0, \sigma_w^2) \) for \( t = 1, 2, \ldots \). Since in equilibrium the exchange rate and the forecasts are linear in the initial shocks \( \epsilon \) and \( \kappa \), the average impulse response (4.10) can be expressed as

\[
\hat{c}_t^{aw} = E^{\theta'}(\epsilon|y_1 = 1) \cdot e_t(\epsilon, 0) + E^{\theta'}(\kappa|y_1 = 1) \cdot e_t(0, \kappa) - e_t(0, 0)
\]

where \( e_t(0, 0) = \alpha_t^* \). The result \( E(\epsilon|y_1 = 1) = q^{\theta'} \) and \( E(\kappa|y_1 = 1) = 1 - q^{\theta'} \) is derived from Bayes Law. To see this, consider the prior distribution \( \epsilon \sim \theta' \sim N(0, \sigma_w^2) \) and the observation \( y_1|x \sim \theta' \sim N(\epsilon, \sigma_w^2) \). Bayes law implies that \( E^{\theta'}(\epsilon|y_1 = 1) = \frac{\sigma_w^2}{\sigma_w^2 + \sigma_w^2} y_1 = \frac{1}{1 + \sigma_w^2/\sigma_w^2} = q^{\theta'} \).

Next, we compute the impulse responses to a transitory shock and to persistent shock separately. Then we compute the impulse response to an ‘average’ shock \( y_1 = 1 \).

**IRF to a transitory shock.** If we condition on a transitory shock at time \( t = 1 \), then \( \upsilon_1 = \kappa \) and all other shocks are zero. It follows that the data generated are \( x_t = 0 \) for all \( t \geq 1 \), \( y_t = \kappa \), and \( y_t = 0 \) for all \( t \geq 2 \). Plugging these data in filter (4.8), it follows that the estimate of the unobservable trend at time \( t \) is \( \hat{x}_t = a^{t-1} \Pi_{i=1}^{t-1} (1 - k^{\theta_{i+1}}) k^{\theta_{i+1}} \kappa \). Plugging \( \hat{x}_t \) and \( y_t \) into the exchange rate function, we have that the IRF to a transitory shock is 0 for \( t = 0 \) and

\[
e_t(0, \kappa) = \begin{cases} 
\alpha_t^* - \left( 1 + \frac{ak^\theta}{1-a} \right) \kappa & t = 1 \\
\alpha_t^* - \frac{a^{t-1} (1 - k^{\theta_{i+1}}) k^{\theta_{i+1}}}{1-a} \kappa & t \geq 2
\end{cases}
\quad (8.26)
\]

**IRF to a persistent shock.** If we condition on a persistent shock at time \( t = 1 \), then \( \upsilon_0 = \epsilon \) and all other shocks are zero. It follows that the data generated are \( x_t = a^{t-1} \epsilon \) and \( y_t = a^{t-1} \epsilon \) for all \( t \geq 1 \). Plugging these data in filter (4.8), it follows that the estimate of the unobservable trend at time \( t \) is \( \hat{x}_t = a^{t-1} \left( 1 - (1 - k^{\theta})^t \right) \epsilon \). Plugging \( \hat{x}_t \)
and \( y_t \) into the exchange rate function, we have that the IRF to a persistent shock is

\[
e_t(\varepsilon, 0) = \alpha_t^* - \frac{a^{t-1}}{1-a} (1 - a (1 - k^{\theta^*})^t) \varepsilon, \quad t \geq 1
\]

(8.27)

**IRF to an average shock.** Substituting (8.26) and (8.27) in (8.25) we have that

\[
\hat{e}_{1t}^{av} = \left[ \alpha_t^* - \left(1 + \frac{a k^{\theta^*}}{1-a}\right) \right] (1 - q^{\theta'}) + \left[ \alpha_t^* - \frac{1}{1-a} (1-a (1 - k^{\theta^*})) \right] q^{\theta'} - \alpha_t^*
\]

\[
= - \left(1 + \frac{a k^{\theta^*}}{1-a} \right) + \left[1 + \frac{a k^{\theta^*}}{1-a} \right] \left[\frac{1}{1-a} (1-a (1 - k^{\theta^*})) \right] q^{\theta'} = \left(1 + \frac{a k^{\theta^*}}{1-a} \right) q^{\theta'}
\]

For \( t \geq 2 \), we have that

\[
\hat{e}_{tt}^{av} = \left[ \alpha_t^* - \frac{a^t k^{\theta^*} (1 - k^{\theta^*})^{t-1}}{1-a} \right] (1 - q^{\theta'}) + \left[ \alpha_t^* - \frac{a^{t-1}}{1-a} (1-a (1 - k^{\theta^*})^t) \right] q^{\theta'} - \alpha_t^*
\]

\[
= - \frac{a^t k^{\theta^*} (1 - k^{\theta^*})^{t-1}}{1-a} - \frac{a^{t-1} (1-a (1 - k^{\theta^*})^t) - a^t k^{\theta^*} (1 - k^{\theta^*})^{t-1}}{1-a} q^{\theta'}
\]

\[
= - \frac{a^t k^{\theta^*} (1 - k^{\theta^*})^{t-1}}{1-a} - \frac{a^{t-1} (1-a (1 - k^{\theta^*})^t) - a^t k^{\theta^*} (1 - k^{\theta^*})^{t-1}}{1-a} q^{\theta'} = \frac{a^{t-1}}{1-a} \left[q^{\theta'} + a (1 - k^{\theta^*}) (k^{\theta^*} - q^{\theta'}) \right]
\]

Therefore, the IRF to an interest rate differential shock \( y_1 = 1 \) equals 0 for \( t = 0 \) and

\[
\hat{e}_{tt}^{av} = \begin{cases} 
- \frac{1-a (1-k^{\theta^*})}{1-a} & t = 0 \\
- \frac{a^{t-1}}{1-a} [q^{\theta'} + a (1 - k^{\theta^*}) (k^{\theta^*} - q^{\theta'})] & t \geq 2
\end{cases}
\]

We obtain the exchange rate return (4.12) in the Proposition by taking the first difference of this equation. It follows directly from (4.12) that if \( \hat{e}_{t+1}^{av} - \hat{e}_{tt}^{av} < 0 \) then \( k^{\theta^*} < q^{\theta'} \) is necessary. To prove part (i) notice that as \( a \uparrow 1 \), (4.12) implies that \( \hat{e}_{tt}^{av} - \hat{e}_{tt}^{av} \) goes to either \(-\infty\) or \(+\infty\). The sign equals the sign of the difference \( k^{\theta^*} (1) - q^{\theta'} \), where \( k^{\theta^*} (1) \) is \( k^{\theta^*} \) evaluated at \( a = 1 \). Thus, as \( a \uparrow 1 \), there is DO with probability approaching one if and only if \( k^{\theta^*} (1) < q^{\theta'} \). Replacing \( \xi^* = \frac{-\sigma^2_v + \sqrt{\sigma^4_v + 4\sigma^2_v \sigma^2_w}}{2} \) in the formula for \( k^{\theta^*} \), we have that \( k^{\theta^*} (1) < q^{\theta'} \) is equivalent to

\[
\frac{1+\sqrt{1+4\frac{\sigma^2_v}{\sigma^2_w}}}{2} \quad \frac{1}{1+\sqrt{1+4\frac{\sigma^2_v}{\sigma^2_w}}} < \frac{1}{1+\frac{\sigma^2_v}{\sigma^2_w}} \Leftrightarrow \frac{\sigma^2_v}{\sigma^2_w} > \frac{1+\sqrt{1+4\frac{\sigma^2_v}{\sigma^2_w}}}{2}
\]

(8.28)

Next, we determine values of \( \lambda \) such that (8.28) holds. In steady state, the approximating
FOC 4.7 implies that
\[
\frac{\tilde{\sigma}_v^{2*}}{\sigma_v^2} = 1 + \frac{\tilde{\sigma}_v^{2*}}{\lambda} \left[ \frac{(\gamma \tilde{b})^2}{2} \frac{\partial Var^{\theta^*}(J)}{\partial \tilde{\sigma}_{v,t}^2} \right] \exp \left( \frac{1}{2} (\gamma \tilde{b})^2 Var^{\theta^*}(J) \right)
\] (8.29)

Combining (8.28) and (8.29) it follows that the condition for DO as \( a \uparrow 1 \) is
\[
\lambda < \lambda^{DO} \equiv \tilde{\sigma}_v^{2*} \left( \frac{-1 + \sqrt{1 + 4 \frac{\sigma_v^2}{\sigma_w^2}}}{2} \right)^{-1} \left[ \frac{(\gamma \tilde{b})^2}{2} \frac{\partial Var^{\theta^*}(J)}{\partial \tilde{\sigma}_{v,t}^2} \right] \exp \left( \frac{1}{2} (\gamma \tilde{b})^2 Var^{\theta^*}(J) \right)
\] (8.30)

Since an equilibrium exists provided \( \lambda > \lambda^v \), DO arises with probability approaching one in equilibrium as \( a \uparrow 1 \) if and only if \( \lambda \in (\lambda^v, \lambda^{DO}) \). Since (4.12) is continuous in the drift \( a \), there exist a high enough \( a \in (0, 1) \) such that there is DO for \( \lambda \in (\lambda^v, \lambda^{DO}) \).

Part (ii) follows because there exists a unique time \( \tau \), such that mean-reversion occurs after \( \tau \):
\[
e_{t+1} - e_t \geq 0, \text{ for } t \geq \tau \iff [1 - k^{\theta^*}]^{\tau - 1}[1 - a(1 - k^{\theta^*})][q^{\theta^*} - k^{\theta^*}] \leq q^{\theta^*} \left[ \frac{1}{a} - 1 \right].
\]

\( \tau \) is the smallest integer greater or equal to \( \tau^* \)
\[
\tau^* = 1 + \left[ \log \left( q^{\theta^*} \left[ \frac{1}{a} - 1 \right] \right) - \log \left( [1 - a(1 - k^{\theta^*})][q^{\theta^*} - k^{\theta^*}] \right) \right] \left[ \log \left( 1 - k^{\theta^*} \right) \right]^{-1}
\]

**Derivation of (4.16).** Given the exchange rate function \( e_{t+1} = \alpha_{t+1} - \frac{a}{1-a} \hat{x}_{t+1}^\theta + y_{t+1} \) and updating formula \( \hat{x}_{t+1}^\theta = (1 - k^{\theta^*}) a \hat{x}_{t}^\theta + k^{\theta^*} y_{t+1} \), it follows that \( e_{t+1} = \alpha_{t+1} - \frac{a^2}{1-a} (1 - k^{\theta^*}) \hat{x}_{t}^\theta - (k^{\theta^*} \frac{a}{1-a} + 1) y_{t+1} \). Conditioning on information \( I_t \), and taking expectations under the robust measure \( \theta^* \), as well as under the baseline \( \theta \), we have
\[
E_t^{\theta^*}(e_{t+1}) = \alpha_{t+1} - \frac{a^2}{1-a} (1 - k^{\theta^*}) \hat{x}_{t}^\theta - \left( k^{\theta^*} \frac{a}{1-a} + 1 \right) E_t^{\theta^*}(y_{t+1})
\]
\[
E_t^{\theta}(e_{t+1}) = \alpha_{t+1} - \frac{a^2}{1-a} (1 - k^{\theta^*}) \hat{x}_{t}^\theta - \left( k^{\theta^*} \frac{a}{1-a} + 1 \right) E_t^{\theta}(y_{t+1})
\]

Equation (4.16) follows from \( E_t^{\theta^*}(y_{t+1}) = E_t^{\theta^*}(x_{t+1}) \) and \( E_t^{\theta^*}(y_{t+1}) = E_t^{\theta^*}(x_{t+1}) \)
\[
\Lambda_t = E_t^{\theta^*}(e_{t+1}) - e_t + \zeta_t - E_t^{\theta}(e_{t+1} - e_t) = \left[ E_t^{\theta^*}(x_{t+1}) - E_t^{\theta^*}(x_{t+1}) \right] \left[ 1 + \frac{ak^{\theta^*}}{1-a} \right] + \zeta_t
\]
\[
= \left[ E_t^{\theta^*}(x_{t+1}) - E_t^{\theta^*}(x_{t+1}) \right] \left[ 1 + \frac{ak^{\theta^*}}{1-a} \right] + \zeta_t
\]

**Proof of Proposition 4.5.** From the robust uncovered interest parity condition (4.9) it follows that \( E_t^{\theta^*}(\Delta e_{t+1}) = y_t - \zeta_t \), where \( y_t \equiv i_t - i_t^f \). If we define the forecast error
as \( v_{t+1} \equiv \Delta e_{t+1} - E_t^{\varphi} (\Delta e_{t+1}) \), it follows that

\[
\text{cov}^\varphi (\Delta e_{t+1}, y_t) = \text{cov}^\varphi (E_t^{\varphi} (\Delta e_{t+1}) + v_{t+1}, y_t) = \text{cov}^\varphi ((y_t - \zeta_t) + v_{t+1}, y_t) = \text{var}^\varphi (y_t) + \text{cov}^\varphi (v_{t+1}, y_t)
\]

(8.31)

The last step follows because the bond’s supply process (8.18) implies that the premium \( \zeta_t \) is deterministic (by Lemma 8.1), so \( \text{cov}^\varphi (\zeta_t, y_t) = 0 \). Let’s develop \( \text{cov}^\varphi (v_{t+1}, y_t) \).

\[
\text{cov}^\varphi (v_{t+1}, y_t) = \text{cov}^\varphi (\Delta e_{t+1} - E_t^{\varphi} (\Delta e_{t+1}), y_t) = \text{cov}^\varphi (\zeta_t - \Lambda_t, y_t) = -\text{cov}^\varphi (\Lambda_t, y_t)
\]

(8.32)

The second equality follows from (4.16): \( \Lambda_t = [E_t^{\varphi} (\Delta e_{t+1}) - E_t^{\varphi} (\Delta e_{t+1})] + \zeta_t \). The third equality follows because the premium \( \zeta_t \) is deterministic. Replacing (8.32) in (8.31) we get \( \beta^{Fama} = 1 - \lim_{t \to \infty} \frac{\text{cov}^\varphi (\Lambda_t, y_t)}{\text{var}^\varphi (y_t)} \). Notice that we can express \( \text{cov}^\varphi (\Lambda_t, y_t) \) as

\[
\text{cov}^\varphi (\Lambda_t, y_t) = \xi \cdot \text{cov}^\varphi (E_t^{\varphi} (x_{t+1}) - E_t^{\varphi} (x_{t+1}), y_t), \quad \xi := \left( \frac{k^{\varphi} a}{1 - a} + 1 \right)
\]

\[
= \xi \cdot \text{cov}^\varphi (a (\hat{x}^{\varphi} - \hat{x}^{\varphi*}), y_t) = a \xi \cdot \text{cov}^\varphi (\hat{x}^{\varphi} - \hat{x}^{\varphi*}, y_t)
\]

Here, we have used the fact that \( E_t^{\varphi} (x_{t+1}) = a \hat{x}^{\varphi} \) and \( E_t^{\varphi*} (x_{t+1}) = a \hat{x}^{\varphi*} \) because \( E_t^{\varphi} (w_t) = 0 \) and \( E_t^{\varphi*} (w_t) = 0 \). Next, we compute \( \text{cov}^\varphi (\hat{x}^{\varphi*}, y_t) \).

\[
\text{cov}^\varphi (\hat{x}^{\varphi*}, y_t) = \text{cov}^\varphi (\hat{x}^{\varphi*}, \tau_t) + k^{\varphi*} \sigma_v^2 = \text{cov}^\varphi (a (1 - k^{\varphi}) \hat{x}_{t-1}^{\varphi*} + k^{\varphi} y_t, x_t) + k^{\varphi} \sigma_v^2
\]

\[
= a^2 (1 - k^{\varphi}) \text{cov}^\varphi (\hat{x}_{t-1}^{\varphi*}, x_{t-1}) + k^{\varphi} \text{Var}^\varphi (x_t) + k^{\varphi*} \sigma_v^2.
\]

Using the stationarity of \( \text{cov}^\varphi (\hat{x}_t, y_t) \) and the stationary value of \( \text{Var}^\varphi (x) = \frac{\sigma_x^2}{1 - a^2} \)

\[
\text{cov}^\varphi (\hat{x}, y) \equiv \lim_{t \to \infty} \text{cov}^\varphi (\hat{x}_t, y_t) = \frac{k^{\varphi}}{1 - a^2} \frac{1}{1 - a^2 (1 - k^{\varphi})} \sigma_w^2 + k^{\varphi} \sigma_v^2
\]

\[
\text{cov}^\varphi (\hat{x}^{\varphi*}, y) \equiv \lim_{t \to \infty} \text{cov}^\varphi (\hat{x}^{\varphi*}_t, y_t) = \frac{k^{\varphi}}{1 - a^2} \frac{1}{1 - a^2 (1 - k^{\varphi})} \sigma_w^2 + k^{\varphi*} \sigma_v^2
\]

Finally, by substituting back in \( \beta^{Fama} \) we obtain

\[
\beta^{Fama} = 1 - \lim_{t \to \infty} \frac{a \xi \cdot \text{cov}^\varphi (\hat{x}^{\varphi} - \hat{x}^{\varphi*}, y_t)}{\text{var}^\varphi (y_t)} = 1 - a \xi \frac{\text{cov}^\varphi (\hat{x}^{\varphi}, y) - \text{cov}^\varphi (\hat{x}, y)}{\lim_{t \to \infty} \text{var}^\varphi (y_t)}.
\]
Therefore, $\beta^{Fama} = 1 - \frac{K_1}{K_2}$, where $K_2 := \frac{\sigma^2_w + \sigma^2_v}{1 - a^2}$, and

$$K_1 = a \left( \frac{ak^\theta}{1 - a} + 1 \right) \left[ \frac{k'\theta}{1 - a^2} \frac{1}{1 - a^2 (1 - k'^\theta)} \sigma^2_w - \frac{k^\theta}{1 - a^2 (1 - k^\theta)} \sigma^2_v + \left( k'\theta - k^\theta \right) \sigma^2_v \right]$$

$$= a \left( \frac{ak^\theta}{1 - a} + 1 \right) \left[ \frac{1}{1 - a^2} \left( \frac{k'\theta}{1 - a^2 (1 - k'^\theta)} - \frac{k^\theta}{1 - a^2 (1 - k^\theta)} \right) \sigma^2_w + \left( k'\theta - k^\theta \right) \sigma^2_v \right]$$

$$= a \left( \frac{ak^\theta}{1 - a} + 1 \right) \left( \frac{1}{(1 - a^2 (1 - k'^\theta))(1 - a^2 (1 - k^\theta))} \sigma^2_w + \left( k'\theta - k^\theta \right) \sigma^2_v \right)$$

It follows that

$$\frac{K_1}{K_2} = a \left( \frac{ak^\theta}{1 - a} + 1 \right) \frac{\left( 1 - a^2 (1 - k'^\theta)(1 - a^2 (1 - k^\theta)) \right) \sigma^2_v + \left( k'\theta - k^\theta \right) \sigma^2_v}{\frac{a^2}{\sigma^2_w} (1 - a^2) + 1}$$

To prove part 1 of Proposition 4.5. Note that the fact that the sign of $k'\theta - k^\theta$ equals the sign of $\bar{k}' - \bar{k}$, because the rest of terms in $\frac{K_1}{K_2}$ is always positive for any $a \in (0, 1)$. As $a$ is close to one, Corollary (4.3) implies that $k'\theta - k^\theta > 0$ with probability approaching one, therefore $\beta^{Fama} < 1$. To prove part 2, we need to show that $K_1/K_2 > 1$ for some large $a$:

$$\lim_{a \to 1} \frac{K_1}{K_2} = 2\bar{k} (\bar{k}' - \bar{k}) \left( \frac{1}{kk'} + \frac{\sigma^2_v}{\sigma^2_w} \right) = 2 \left[ 1 - \frac{\bar{k}}{k'} + \frac{\bar{k}\bar{k}'}{\sigma^2_v} - \frac{\bar{k}^2}{\sigma^2_w} \right]$$

where $\bar{k} = k'^\theta (1) = \frac{\xi' + \sigma^2_w}{\xi' + \sigma^2_w + \sigma^2_v}$, $\bar{k}' = k'^\theta (1) = \frac{\xi' + \sigma^2_w}{\xi' + \sigma^2_w + \sigma^2_v}$ and $\xi' = -\frac{\sigma^2_w + 2 \sigma^2 w \sigma^2 v}{2} \sigma^2_w \sigma^2_v$ denote $k'^\theta$, $k'^\theta$ and $\sigma^2$ evaluated at $a = 1$, respectively. If we use the notation $\tilde{k} = c\bar{k}'$, with $c \in (0, 1)$, then we have

$$\lim_{a \to 1} \frac{K_1}{K_2} = 2 \left( (1 - c) + c (1 - c) \frac{\sigma^2_v}{\sigma^2_w} \right) > 1$$

if and only if $c \in (0, c_0)$, where $c_0 = \sqrt{\frac{2 \sigma^2_w \sigma^2_v}{\sigma^2_v}} - 1$. This is equivalent to

$$\frac{\sigma^2_v}{\sigma^2_w} > c_1 \equiv \left( c_0^{-1} \left( \sigma^2_v - 2 \left( \sigma^2_w + \sqrt{\sigma^4_w + 4 \sigma^2_w \sigma^2_v} \right)^{-1} \right) - \sigma^2_v \right) \frac{\sigma^2_w + \sqrt{\sigma^4_w + 4 \sigma^2_w \sigma^2_v}}{2}$$

(8.33)
The approximating FOC (4.7) implies that \( \tilde{\sigma}_{v}^{2*} = \frac{2}{\lambda} \left[ \frac{\gamma b}{2} \frac{\partial V_{ar^{\theta}}(J)}{\partial \tilde{\sigma}_{v}^{2}} \right] \exp \left( \frac{1}{2} (\gamma b)^{2} V_{ar^{\theta}}(J) \right) \) + \( \tilde{\sigma}_{v}^{2*} \), it follows that \( \lim_{a_{11} \rightarrow \infty} \frac{K_{1}}{K_{2}} > 1 \) if and only if

\[
0 < \lambda < \lambda^F \equiv (c_{1} - 1)^{-1} \tilde{\sigma}_{v}^{2*} \left[ \frac{\gamma b}{2} \frac{\partial V_{ar^{\theta}}(J)}{\partial \tilde{\sigma}_{v}^{2}} \right] \exp \left( \frac{1}{2} (\gamma b)^{2} V_{ar^{\theta}}(J) \right) \]

(8.34)

By the continuity of \( \frac{K_{1}}{K_{2}} \) in \( a \), we conclude that in equilibrium there exist a large enough \( a \in (0, 1) \) such that \( \beta_{Fama} < 0 \) if and only if \( \lambda \in (\lambda^v, \lambda^F) \), where \( \lambda^v \) is defined by 8.13. This proves part 2.

**Proof of Lemma 5.1.** This proof follows the same steps at those in the proof of Lemma 4.1 and can be found in the extended appendix in our websites.

**Proof of Proposition 5.2** For a given probability measure \( \theta_{t} \) (or for a given distorted variance \( \tilde{\sigma}_{w,t}^{2} \)), the investor constructs her estimates of \( x_{t} \) and \( x_{t+1} \) using Bayes law, given prior estimate \( \hat{x}_{t-1} \) and prior variance \( \sigma_{t-1}^{2} \). These estimates are

\[
\hat{x}_{t|t}^{\theta_{t}} (x_{t}) = (1 - k_{t}^{\theta_{t}}) \tilde{a}_{t}^{\theta_{t}^{t-1}} + k_{t}^{\theta_{t}} y_{t}, \quad k_{t}^{\theta_{t}} = \frac{a^{2} \sigma_{t-1}^{2} + \tilde{\sigma}_{w,t}^{2}}{a^{2} \sigma_{t-1}^{2} + \sigma_{w,t}^{2} + \sigma_{v}^{2}}, \quad \sigma_{t-1}^{2} = \frac{(a^{2} \sigma_{t-1}^{2} + \tilde{\sigma}_{w,t}^{2}) \tilde{\sigma}_{v}^{2}}{a^{2} \sigma_{t-1}^{2} + \sigma_{w,t}^{2} + \sigma_{v}^{2}}
\]

\[
x_{t} I_{t} \sim N \left( \hat{x}_{t|t}, \frac{a^{2} \sigma_{t-1}^{2} + \tilde{\sigma}_{w,t}^{2}}{a^{2} \sigma_{t-1}^{2} + \sigma_{w,t}^{2} + \sigma_{v}^{2}} \right), \quad x_{t+1} I_{t} \sim N \left( \hat{x}_{t|t}, \frac{a^{2} \sigma_{t-1}^{2} + \tilde{\sigma}_{w,t}^{2}}{a^{2} \sigma_{t-1}^{2} + \sigma_{w,t}^{2} + \sigma_{v}^{2}} \right), \]

where \( I_{t} = \{ e_{t}, y_{t}, \hat{x}_{t-1} \} \). The \( t \)-agent computes the excess rate of return \( J_{t+1} = y_{t} - e_{t+1} + e_{t} \) based on her observations of \( y_{t} \) and \( e_{t} \) and her knowledge of the equilibrium exchange rate function (3.7): \( J_{t+1} = y_{t} - \left( \alpha_{t+1} + \beta_{1} \hat{x}_{t+1} + \beta_{2} y_{t+1} \right) + e_{t} \). Furthermore, the \( t \)-agent knows that the \( t+1 \)-agent will (i) use the same method to distort the probability measure \( \theta_{t+1} \) as the one used by the \( t \)-agent, and (ii) will estimate the hidden state \( x_{t+1} \) using Bayes law under this \( \theta_{t+1} \), with a given prior mean \( \hat{x}_{t+1} \) and variance \( \sigma_{t}^{2} = \frac{(a^{2} \sigma_{t-1}^{2} + \tilde{\sigma}_{w,t}^{2}) \tilde{\sigma}_{v}^{2}}{a^{2} \sigma_{t-1}^{2} + \sigma_{w,t}^{2} + \sigma_{v}^{2}} \). It follows that we can replace \( \hat{x}_{t+1} \) by \( (1 - k_{t+1}^{\theta_{t+1}}) \tilde{a}_{t}^{\theta_{t}^{t+1}} + k_{t+1}^{\theta_{t+1}} y_{t+1} \) and obtain \( J_{t+1} = y_{t} - \alpha_{t+1} - \left( 1 - k_{t+1}^{\theta_{t+1}} \right) \beta_{1} \tilde{a}_{t}^{\theta_{t}} - \left( k_{t+1}^{\theta_{t+1}} \beta_{1} + \beta_{2} \right) y_{t+1} + e_{t} \). Since \( E_{t}^{\theta_{t}} (y_{t+1}) = \tilde{a}_{t}^{\theta_{t}} \), under probability measure \( \theta_{t} \in \Theta^{w} \), we have that \( J_{t+1} \) is normally distributed with conditional mean and variance

\[
E_{t}^{\theta_{t}} (J_{t+1}) = y_{t} - \alpha_{t+1} - \beta_{1} \left( (1 - k_{t+1}^{\theta_{t+1}}) \tilde{a}_{t}^{\theta_{t}^{t+1}} + k_{t+1}^{\theta_{t+1}} \tilde{a}_{t}^{\theta_{t}^{t}} \right) - a_{\theta_{2}} \tilde{a}_{t}^{\theta_{t}} + e_{t} \quad (8.35)
\]

\[
V_{t}^{\theta_{t}} (J_{t+1}) = \left( k_{t+1}^{\theta_{t+1}} \beta_{1} + \beta_{2} \right)^{2} \frac{a^{2} \tilde{\sigma}_{t}^{2} \tilde{\sigma}_{v}^{2}}{a^{2} \tilde{\sigma}_{t}^{2} + \tilde{\sigma}_{w,t}^{2} + \tilde{\sigma}_{v}^{2} + \tilde{\sigma}_{w,t}^{2}} + \tilde{\sigma}_{v}^{2} \]

Note that problem (4.2) is equivalent to (3.8) because for any normally distributed random variable \( z \), \( E^{\theta} (\exp [-\gamma z]) = \exp \left( -\gamma E^{\theta} (z) + \frac{\gamma^{2}}{2} V_{ar^{\theta}} (z) \right) \). We solve problem

55
(4.2) by considering the investor as a Stackelberg leader, that takes into account the strategy of nature: \( \hat{\sigma}_{w,t}^2 = \sigma_{w,t}^2(b_t) \). Nature then selects \( \hat{\sigma}_{w,t}^2 \) conditioning on the agent’s choice of \( b_t \). The first-order-condition with respect to \( b_t \) is

\[
\frac{\partial \Gamma}{\partial b_t} = -\left(-\gamma E_t^{\theta_t}(J_{t+1}) + \gamma^2 b_t Var_t^{\theta_t}(J_{t+1})\right) \frac{\exp\left(\frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1})\right)}{\exp\left(\gamma b_t E_t^{\theta_t}(J_{t+1})\right)} + \frac{\partial \Gamma}{\partial \sigma_{w,t}^2} \frac{d\sigma_{w,t}^2}{db_t} = 0
\] (8.36)

Since the FOC of nature’s problem implies \( \frac{\partial \Gamma}{\partial \sigma_{w,t}^2} = 0 \), the last term in (8.36) is zero, and so in an interior solution, \( \frac{\partial \Gamma}{\partial b_t} = 0 \) implies \( b_t = \frac{E_t^{\theta_t}(J_{t+1})}{\gamma Var_t^{\theta_t}(J_{t+1})} \), which is equation (4.3) in the text.

The SOC with respect to \( b_t \) is

\[
\frac{\partial^2 \Gamma}{\partial b_t^2} = \Gamma_{b_t b_t}(b_t, \sigma_{w,t}^2(b_t)) + \Gamma_{b_t \sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b_t)) \cdot \left(\frac{\partial \sigma_{w,t}^2(b_t)}{\partial b_t}\right)^2 + \Gamma_{\sigma_{w,t}^2 \sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b_t)) \cdot \frac{\partial^2 \sigma_{w,t}^2(b_t)}{\partial b_t^2}
\]

Notice that the total derivative of nature’s first order condition \( \Gamma_{\sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b)) = 0 \) is \( \Gamma_{\sigma_{w,t}^2} \frac{db_t}{\sigma_{w,t}^2} + \frac{d\sigma_{w,t}^2}{db_t} = 0 \).

Thus, \( \frac{\partial^2 \sigma_{w,t}^2(b_t)}{\partial b_t^2} = -\frac{\Gamma_{\sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b))}{\Gamma_{\sigma_{w,t}^2 \sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b))} \). Combining this equation with \( \Gamma_{b_t \sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b_t)) = \sigma_{w,t}^2(b_t, \sigma_{w,t}^2(b_t)) \), the investor’s second order condition becomes

\[
\frac{\partial^2 \Gamma}{\partial b_t^2} = \Gamma_{b_t b_t}(b_t, \sigma_{w,t}^2(b_t)) + \Gamma_{b_t \sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b_t)) \cdot \left(\frac{\partial \sigma_{w,t}^2(b_t)}{\partial b_t}\right)^2 + \Gamma_{\sigma_{w,t}^2 \sigma_{w,t}^2}(b_t, \sigma_{w,t}^2(b_t)) \cdot \frac{\partial^2 \sigma_{w,t}^2(b_t)}{\partial b_t^2} = \Gamma_{b_t b_t}(b_t, \sigma_{w,t}^2(b_t)) < 0.
\]

This condition is unambiguously satisfied because \( 0 > \Gamma_{b_t b_t} = -\left(-\gamma E_t^{\theta_t}(J_{t+1}) + \gamma^2 b_t Var_t^{\theta_t}(J_{t+1})\right) \cdot \frac{\exp\left(\frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1})\right)}{\exp\left(\gamma b_t E_t^{\theta_t}(J_{t+1})\right)}. \\
To derive the equilibrium note that the market demand for the dollar bond can be expressed as \( b_t^* = \frac{y_t - a(\beta_1 + \beta_2) \sigma_{w,t}^2 - \alpha_1 + \epsilon_t}{\gamma Var_t^{\theta_t}(J_{t+1})} \). Thus, the market-clearing condition \( b_t^* = b_t^* \) implies \( \epsilon_t = -y_t + a (\beta_1^* + \beta_2^*) \hat{\sigma}_{w,t}^2 + \alpha_1^* + \gamma b_t^* Var_t^{\theta_t}(J_{t+1}) \). To derive the equilibrium values of \( (\beta_1, \beta_2, \alpha_1) \) we equalize coefficients with conjecture (3.7), and obtain \( \alpha_1 = \alpha_1^* + \gamma b_t^* Var_t^{\theta_t}(J_{t+1}) \), \( \beta_2 = -1 \), and \( \beta_1 = a (\beta_1^* + \beta_2^*) = -\frac{a}{1-a} \). This proves that, given a solution \( \hat{\sigma}_{w,t}^2 \), the equilibrium \( \epsilon_t \) is (4.4).

The first order condition with respect to \( \hat{\sigma}_{w,t}^2 \) is

\[
\frac{\partial \Gamma}{\partial \hat{\sigma}_{w,t}^2} = \left[ \gamma b_t \frac{\partial E_t^{\theta_t}(J_{t+1})}{\partial \hat{\sigma}_{w,t}^2} - \frac{(\gamma b_t)^2}{2} \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \hat{\sigma}_{w,t}^2} \right] \frac{\exp\left(\frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1})\right)}{\exp\left(\gamma b_t E_t^{\theta_t}(J_{t+1})\right)} + \frac{\lambda}{2} \left( \frac{1}{\sigma_{w,t}^2} - \frac{1}{\hat{\sigma}_{w,t}^2} \right) = 0
\] (8.37)

Notice that the expected excess return \( E_t^{\theta_t}(J_{t+1}) \) depends on the estimate of the state.
\[ \hat{x}_t^{\theta_t}, \text{ that in turn depends on the gain } k_t^{\theta_t}, \text{ which is a function of } \sigma^2_{w,t}. \text{ Thus,} \]

\[
\frac{\partial E_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} = -a (\beta_1 + \beta_2) \frac{d\hat{x}_t^{\theta_t}}{dk_t^{\theta_t}} \frac{dk_t^{\theta_t}}{d\sigma^2_{w,t}},
\]

where \( \frac{d\hat{x}_t^{\theta_t}}{dk_t^{\theta_t}} = y_t - ax_{t-1}^{\theta_t} \) and \( \frac{dk_t^{\theta_t}}{d\sigma^2_{w,t}} = \frac{\sigma^2_w}{(a^2\sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma^2_v)^2} > 0 \)

The derivative of the returns variance is

\[
\frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} = \left( (k_t^{\theta_t+1}\beta_1 + \beta_2) \right)^2 \left( \frac{a^2\sigma^4_v}{(a^2\sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma^2_v)^2} + 1 \right)
\]

It thus follows that

\[
\frac{\partial \Gamma}{\partial \sigma^2_{w,t}} = -\gamma b_t [a (\beta_1 + \beta_2)] (y_t - ax_{t-1}^{\theta_t}) \frac{dk_t^{\theta_t}}{d\sigma^2_{w,t}} \exp \left( \frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1}) \right) \exp(\gamma b_t E_t^{\theta_t}(J_{t+1}))
\]

\[
- \frac{(\gamma b_t)^2}{2} \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} \frac{\exp \left( \frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1}) \right)}{\exp(\gamma b_t E_t^{\theta_t}(J_{t+1}))} + \lambda \left( 1 - \frac{1}{\sigma^2_w} \right)
\]

Replacing the equilibrium coefficients \( \beta_1^* = -\frac{a}{1-a} \) and \( \beta_2^* = -1 \), the FOC becomes

\[
\frac{\lambda}{2} \left( \frac{1}{\sigma^2_w} - \frac{1}{\sigma^2_{w,t}} \right) = \left( -\gamma b_t^* \left[ \frac{a}{1-a} \right] (y_t - ax_{t-1}^{\theta_t}) \frac{dk_t^{\theta_t}}{d\sigma^2_{w,t}} + \frac{(\gamma b_t)^2}{2} \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} \right) W^*_t,
\]

where \( \frac{dk_t^{\theta_t}}{d\sigma^2_{w,t}} = \frac{\sigma^2_w}{(a^2\sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma^2_v)^2} > 0 \), \( \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} = \left( k_t^{\theta_t+1}\frac{a}{1-a} + 1 \right) \left( \frac{a^2\sigma^4_v}{(a^2\sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma^2_v)^2} + 1 \right) > 0 \) and \( W^*_t = \exp \left( -\gamma b_t^* E_t^{\theta_t}(J_{t+1}) + \frac{1}{2} (\gamma b_t)^2 Var_t^{\theta_t}(J_{t+1}) \right) > 0 \) is the optimal utility function.

Next, we derive conditions for the existence of a positive and bounded distorted variance \( \sigma^2_{w,t} \). First, notice that (8.38) implies

\[
\sigma^2_{w,t} = \left[ \frac{1}{\lambda} + \frac{2}{\lambda} \left[ \frac{\gamma b_t^* E_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} - \frac{(\gamma b_t)^2}{2} \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} \right] W^*_t \right]^{-1}.
\]

Thus, \( \sigma^2_{w,t} \) is positive only if \( \lambda \) satisfies the following condition

\[
\lambda > \lambda^*_t \equiv \sigma^2_w \left[ -\gamma b_t^* \frac{\partial E_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} + \frac{(\gamma b_t^*)^2}{2} \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \sigma^2_{w,t}} \right] W^*_t.
\]

(8.39)
Second, the SOC of nature’s problem in equilibrium is

$$\frac{\partial^{2} \Gamma}{\partial (\hat{\sigma}_{w,t}^{2})^{2}} = \left[ \gamma b_{t} \frac{\partial^{2} E_{t}^{\gamma t} (J_{t+1})}{\partial (\hat{\sigma}_{w,t}^{2})^{2}} \frac{\partial E_{t}^{\gamma_{t}} (J_{t+1})}{\partial \hat{\sigma}_{w,t}^{2}} - \frac{\gamma b_{t}^{2}}{2} \frac{\partial^{2} Var_{t}^{\gamma t} (J_{t+1})}{\partial \hat{\sigma}_{w,t}^{2}} \right] W_{t}^{*}$$

It holds if and only if $$\lambda > \lambda_{t}^{*}$$, where $$\lambda_{t}^{*}$$ is defined by

$$\lambda_{t}^{*} = 2 \left( \hat{\sigma}_{w,t}^{2} \right) \left[ \frac{\gamma b_{t}^{2}}{2} \frac{\partial^{2} Var_{t}^{\gamma_{t}} (J_{t+1})}{\partial \hat{\sigma}_{w,t}^{2}} - \frac{\gamma b_{t}}{\partial \hat{\sigma}_{w,t}^{2}} \right] W_{t}^{*}$$

Combining the positivity constraint and the SOC for $$\hat{\sigma}_{w,t}^{2}$$, it follows that a linear robust equilibrium exists if $$\lambda > \lambda_{t}^{*}$$ and $$\lambda > \lambda_{t}^{\#}$$. That is,

$$\frac{1}{\lambda_{t}^{v}} = \min \{ 1/\lambda_{t}^{*}, 1/\lambda_{t}^{\#} \}$$

(8.41)

To prove part 4 note that if either $$1/\lambda = 0$$, $$\gamma = 0$$ or $$b_{t}^{S} = 0$$, then there is no distortion because (8.38) becomes

$$\frac{1}{\sigma_{w}^{2}} - \frac{1}{\hat{\sigma}_{w,t}^{2}} = 0 \Rightarrow \hat{\sigma}_{w,t}^{2} = \sigma_{w}^{2}$$

**Proof of Corollary 5.3.** The sign of $$\sigma_{w}^{2} - \hat{\sigma}_{w,t}^{2}$$ is determined by the sign of the RHS of (8.38), which depends on the sign of the news $$\left( y_{t} - a_{x,t-1} \right)$$ and the sign of the dollar bond position $$b_{t}^{S}$$. We will show that the RHS of (8.38) is unambiguously positive in two limiting cases: $$a \downarrow 0$$ and $$a \uparrow 1$$. Expanding the RHS of (8.38) we have

$$RHS(8.38) = -\gamma b_{t} \left[ \frac{a}{1-a} \right] \left( y_{t} - a_{x,t-1} \right) \frac{\sigma_{v}^{2}}{a^{2} \sigma_{t-1}^{2} + \hat{\sigma}_{w,t}^{2} + \sigma_{v}^{2}} W_{t}^{*}$$

$$+ \frac{\gamma b_{t}^{2}}{2} \left( \frac{\sigma_{t+1}^{2}}{a} \frac{1}{1-a} + 1 \right) \left( \frac{a^{2} \sigma_{v}^{4}}{a^{2} \sigma_{t-1}^{2} + \hat{\sigma}_{w,t}^{2} + \sigma_{v}^{2}} + 1 \right) W_{t}^{*}$$

Thus, $$\lim_{a \rightarrow 0} RHS(8.38) = \frac{(\gamma b_{t}^{2})}{2} W_{t}^{**}$$, where $$W_{t}^{**} = \exp \left( -\gamma b_{t}^{\#} (y_{t} - \alpha_{t+1} + ct) + \frac{(\gamma b_{t}^{2})}{2} \left( \sigma_{w,t}^{2} + \sigma_{v}^{2} \right) \right) > 0$$, which is the value of $$W_{t}^{*}$$ at $$a = 0$$. It follows that if $$a$$ is close enough to zero, then for any $$\delta > 0$$

$$\lim_{a \rightarrow 0} Pr \{ RHS \ of \ (8.38) \ > \delta \} = 1.$$  

(8.42)

To see what happens when $$a \uparrow 1$$ , notice that the RHS of (8.38) is a quadratic function.
of $z \equiv \frac{1}{2} \gamma b_i^\theta \frac{a}{1-a}$:

$$
\lambda \left( \frac{1}{\sigma_w^2} - \frac{1}{\sigma_w^2} \right) = f(z) W^*_t
\equiv [A(a,t) z^2 + B(a,t) z + C(a,t)] W^*_t, \quad \text{with } z \equiv \frac{1}{2} \gamma b_i^\theta \frac{a}{1-a}.
$$

(8.43)

The coefficients $A(a,t), B(a,t), C(a,t)$ are given by:

$$
A(a,t) = \left( k_i^\theta t+1 \right)^2 \left( \frac{a^2 \sigma^4_v}{(a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma_v^2)^2} + 1 \right) > 0
$$

(8.44)

$$
B(a,t) = -2 \left( y_t - a\tilde{x}_t^{\theta t-1} \right) \frac{\sigma^2_v}{(a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma_v^2)^2} + 2 (\gamma b_i^\theta) k_i^\theta t+1 \left( \frac{a^2 \sigma^4_v}{(a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma_v^2)^2} \right)
$$

(8.45)

$$
C(a,t) = \frac{(\gamma b_i^\theta)^2}{2} \left( \frac{a^2 \sigma^4_v}{(a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma_v^2)^2} + 1 \right)
$$

(8.46)

We will prove that if $a$ is sufficiently close to one, then given any bond position $b_i^\theta$ and any news surprise $y_t - a\tilde{x}_t^{\theta t-1}$, the RHS of (8.14) is positive. We do so by showing that as $a \uparrow 1$, the roots of the equation $f(z) = 0$ remain bounded, while $z$ goes to either positive or negative infinity. Since $f(z)$ is a convex parabola (because the coefficient $A$ is positive), it follows that $f(z)$ goes to positive infinity when $a \uparrow 1$.

**Step 1.** We show that as $a \uparrow 1$, the roots of the equation $f(z) = 0$ remain bounded by showing that the limits, as $a \uparrow 1$, of parameters $A(a,t), B(a,t)$ and $C(a,t)$ exist. To see this, we derive limits of $A(a,t), B(a,t)$ and $C(a,t)$ as follows. First, by substituting $k_i^\theta t+1 = \frac{a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t+1}}{a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t+1} + \sigma_v^2}$ in (8.44) and taking the limit of $a \uparrow 1$, we have:

$$
A_t = \lim_{a \uparrow 1} A(a,t)
= \lim_{a \uparrow 1} \left( \frac{a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t+1}}{a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t+1} + \sigma_v^2} \right) \left( \frac{a^2 \sigma^4_v}{(a^2 \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma_v^2)^2} + 1 \right)
= \left( \frac{\sigma^2_{t-1} + \tilde{\sigma}^2_{w,t+1}}{\sigma^2_{t-1} + \tilde{\sigma}^2_{w,t+1} + \sigma_v^2} \right) \left( \frac{\sigma^4_v}{(\tilde{\sigma}_{w,t}^2 + \sigma_v^2)^2} + 1 \right) > 0.
$$

where $\sigma^2_t = \lim_{a \uparrow 1} \sigma^2_{t-1}$. Recall that $\sigma^2_t = \frac{(a \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t}) \sigma^2_v}{a \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma_v^2}$, so we have $\sigma^2_t = \frac{(a \sigma^2_{t-1} + \tilde{\sigma}^2_{w,t}) \sigma^2_v}{\sigma^2_{t-1} + \tilde{\sigma}^2_{w,t} + \sigma_v^2}$, with initial value $\sigma^2_0 = 0$. We can see that $\sigma^2_t$ remains bounded above by $\sigma^2_v$ even as $a \uparrow 1$.

Second, substituting for $k_i^\theta t+1$ and $\tilde{x}_t^{\theta t-1} = \left( 1 - k_i^\theta \right) a\tilde{x}_t^{\theta t-1} + k_i^\theta y_t$ in (8.16) and taking
the limit as \( a \uparrow 1 \), we have:

\[
B_t \equiv \lim_{a \uparrow 1} B(a,t)
\]

\[
= \lim_{a \uparrow 1} \frac{2}{a} \left( y_t - a \bar{x}_{t-1} \right) \left( \frac{\sigma_w^2}{\sigma_{t-1}^2 + \sigma_w^2} + 2\left( \gamma b_t^* \right) k_{t+1} \right) + 2\left( \gamma b_t^* \right) k_{t+1} \left( \frac{a^2 \sigma_v^2}{a^2 \sigma_{t-1}^2 + \sigma_w^2} + 1 \right)
\]

\[
= 2 \left( y_t - \bar{x}_{t-1} \right) \left( \frac{\sigma_w^2}{\sigma_{t-1}^2 + \sigma_w^2} + 2\left( \gamma b_t^* \right) \left( \frac{\sigma_{t-1}^2 + \sigma_w^2}{\sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} \right) \right) + 2\left( \gamma b_t^* \right) \left( \frac{\sigma_v^2}{\sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} + 1 \right)
\]

where \( \bar{x}_{t-1} = \left( 1 - \frac{\sigma_{t-1}^2 + \sigma_w^2}{\sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} \right) \bar{x}_{t-2} + \frac{\sigma_{t-1}^2 + \sigma_w^2}{\sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} y_{t-1} \) with initial values \( \bar{x}_0 = x_0 \) and \( y_0 = 0 \). Third, taking the limit of \( C(a,t) \) we have

\[
C_t \equiv \lim_{a \uparrow 1} C(a,t) = \lim_{a \uparrow 1} \frac{2}{a} \left( \frac{\sigma_w^2}{\sigma_{t-1}^2 + \sigma_w^2} + 2\left( \gamma b_t^* \right) \left( \frac{\sigma_v^2}{\sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} + 1 \right) \right)
\]

The coefficients \( B_t \) and \( C_t \), defined in equations (8.47) and (8.48), are bounded because \( y_t \) and \( b_t^* \) are given exogenously and assumed to be finite. In addition, the other terms in (8.47) and (8.48) are all bounded too.

**Step 2.** The function \( f(z) \) is a convex parabola because the coefficient \( A(a,t) \) is positive for any \( a \in (0,1] \). Let us consider the three possible cases. First, if \( B(a,t)^2 < 4A(a,t)C(a,t) \), the equation \( f(z) = 0 \) has no real roots, and so \( f(z) > 0 \) for all \( z \). Second, if \( B(a,t)^2 > 4A(a,t)C(a,t) \), the equation \( f(z) = 0 \) has two real roots, both of which are finite as \( a \) converges to one: \( \overline{z}_1 = \frac{-B_1 - \sqrt{B_1^2 - 4A_1C_1}}{2A_1} \) and \( \overline{z}_2 = \frac{-B_1 + \sqrt{B_1^2 - 4A_1C_1}}{2A_1} \).

But as \( a \) converges to one, \( z \) goes to either \( +\infty \) (if \( b_t^* > 0 \)) or \( -\infty \) (if \( b_t^* < 0 \)). Therefore, there exists an \( a \) close to one, such that either \( z > \overline{z}_2 \) or \( z < \overline{z}_1 \). In either case \( f(z) > 0 \). Third, if \( B(a,t)^2 = 4A(a,t)C(a,t) \), the equation \( f(z) = 0 \) has only one real root. When \( a \) gets closer to one, the root converges to \( \overline{z} = -\frac{B_1}{2A_1} \), while \( z \) goes to either \( +\infty \) (if \( b_t^* > 0 \)) or \( -\infty \) (if \( b_t^* < 0 \)). Thus, as \( a \uparrow 1 \), we have that \( |z| > \overline{z} \), and so \( f(z) \) must be positive too.

From these three cases, it follows that there exists a small \( \varepsilon > 0 \), such that \( f(z) \) is large positive for any \( a \in (1 - \varepsilon, 1) \). Therefore, if \( a \uparrow 1 \), the RHS of (8.43) is positive for any data news \( y_t - ax_{t-1} \) and any bond position \( b_t^* \). We conclude that for any \( \delta > 0 \):

\[
\lim_{a \uparrow 1} \Pr \{ \text{RHS of (8.43)} > \delta \} = 1.
\]

It follows from (8.42), (8.49) and the FOC (8.43) that if either \( a \uparrow 1 \) or \( a \downarrow 0 \), then for any data news and any bond position the variance of the shock in the hidden-state equation is distorted upwards with probability approaching one:

\[
\Pr \{ \sigma_{w,t}^2 > \sigma_w^2 \} \uparrow 1, \quad \text{as } a \uparrow 1 \text{ or } a \downarrow 0.
\]
Lastly, it follows from the gain equations \( k_t^{\theta_t} = \frac{a^2 \sigma_{t-1}^2 + \sigma_{w}^2}{a^2 \sigma_{t-1}^2 + \sigma_{w}^2 + \sigma_{\epsilon}^2} \) and \( k_t^{\theta_t} = \frac{a^2 \sigma_{t-1} + \sigma_{w}^2}{a^2 \sigma_{t-1} + \sigma_{w}^2 + \sigma_{\epsilon}^2} \) that

\[
\Pr \left\{ k_t^{\theta_t} > k_t^{\theta_t} \right\} \uparrow 1, \quad \text{as } a \uparrow 1 \text{ or } a \downarrow 0.
\]

This concludes the proof.

**Proof of Lemma 5.4.** We use a representation theorem in Dupuis and Ellis (1997) to show that if \( \theta' \) is the baseline model and \( \theta \in \Theta^u \), then the agent’s problem simplifies to the RHS in (5.6).

**Lemma 8.2 (Variational Representation).** Let \((\Omega, \mathcal{F})\) be a measurable space, and let \(\mathcal{P}(\Omega)\) denote the set of all probability measures defined on it. If \(f\) is a bounded measurable function mapping from \(\Omega\) into \(\mathbb{R}\), and \(\theta \in \mathcal{P}(\Omega)\), then; (a) the following variational formula holds:

\[
- \log \int_{\Omega} e^{-f(\omega)} \theta(d\omega) = \inf_{\theta' \in \mathcal{P}(\Omega)} \left\{ R(\theta||\theta') + \int_{\Omega} f(\omega) \theta(d\omega) \right\}
\]

(b) The infimum in (8.50) is uniquely achieved by a probability measure \(\theta^*\) which is absolutely continuous with respect to \(\theta\) and has the Radon-Nikodym derivative

\[
\frac{d\theta^*}{d\theta} = \frac{1}{\int_{\Omega} e^{-f(\omega)} \theta'(d\omega)}
\]

Applying the above Lemma to our robust utility function, we have that

\[
\Gamma = \max_{b_t} \inf_{\theta \in \Theta^u} \left\{ E^\theta_t \left[ u(W_{t+1}) + \lambda \cdot R(\theta||\theta') \right] \right\} = \max_{b_t} \left\{ -\lambda \log \left( E^\theta_t \exp \left( -\frac{1}{\lambda} u(W_{t+1}) \right) \right) \right\}.
\]

**Proof of (5.7).** From the first order condition for \(b_t\) we have

\[
E_t^{\theta_t} \left[ -J_{t+1} \exp \left( \gamma b_t(J_{t+1}) \right) \exp \left( -\frac{1}{\lambda} \exp \left( \gamma b_t(J_{t+1}) \right) \right) \right] = 0
\]

where \(J_{t+1} \equiv (i_t - i_t^2) - (e_{t+1} - e_t)\). Next, we use Stein’s Lemma (see for example, Casella and Berger [14] page 187), which states that for a normally distributed random variable \(X \sim N(\mu, \sigma^2)\), and a function \(g(x)\) with \(E|g'(X)| < \infty\), we have \(E[g(X)(X - \mu)] = \sigma^2 Eg'(X)\). Starting with the first order condition for \(b_t\), the conjecture implies that \(J_{t+1}\) has a conditional normal distribution with mean \(E_t^{\theta_t} \left( (i_t - i_t^2) - (e_{t+1} - e_t) \right)\) and variance \(Var_t^{\theta_t} (e_{t+1})\). Let

\[
g(J_{t+1}) = -\exp (\gamma b_t J_{t+1}) \exp \left( -\frac{1}{\lambda} \exp (\gamma b_t J_{t+1}) \right)
\]

The derivative of \(g(J_{t+1})\) is \(g'(J_{t+1}) = [1 - \frac{1}{\lambda} \exp (\gamma b_t J_{t+1})] \gamma b_t g(J_{t+1})\). It follows that
the right hand side of the first order condition can be written as

$$E_t^{o'}(J_{t+1}g(J_{t+1})) = E_t^{o'}\left( J_{t+1} - E_t^{o'}(J_{t+1}) + E_t^{o'}(J_{t+1}) \right) g(J_{t+1}) = E_t^{o'}\left( J_{t+1} - E_t^{o'}(J_{t+1}) \right) g(J_{t+1}) + E_t^{o'}(J_{t+1}) E_t^{o'} g(J_{t+1})$$

Applying Stein’s Lemma to the left side of the first order condition (8.51), we have that

$$0 = \text{Var}_t^{o'}(e_{t+1}) E g'(J_{t+1}) + E_t^{o'}(J_{t+1}) E_t^{o'} g(J_{t+1})$$

Therefore, $E_t^{o'}(-J_{t+1}) = \frac{\text{Var}_t^{o'}(e_{t+1}) E g'(J_{t+1})}{E_t^{o'} g(J_{t+1})}$.

Hence, $E_t^{o'}(-J_{t+1}) - l_t = 0$ or

$$E_t^{o'}\left( e_{t+1} - e_t + i_{t+1}^f - i_t \right) - l_t = 0$$

where $l_t = \frac{\text{Var}_t^{o'}(e_{t+1}) E_t^{o'} g(J_{t+1})}{E_t^{o'} g(J_{t+1})}$.

Substituting for $g(J_{t+1})$ and $g'(J_{t+1})$ in $l_t$, we write $l_t$ explicitly as

$$l_t = \frac{\text{Var}_t^{o'}(e_{t+1}) E_t^{o'} [(1 - \frac{1}{\chi} \exp(-\gamma b_t J_{t+1})) \gamma b_t \exp(-\gamma b_t (e_{t+1} - e_t + i_{t+1}^f - i_t)) \exp(-\frac{1}{\chi} \exp(-\gamma b_t (e_{t+1} - e_t + i_{t+1}^f - i_t)))]}{E_t^{o'} \exp(-\gamma b_t (e_{t+1} - e_t + i_{t+1}^f - i_t)) \exp(-\frac{1}{\chi} \exp(-\gamma b_t (e_{t+1} - e_t + i_{t+1}^f - i_t)))}}$$

**Proof of Proposition 5.5.** From the conjecture (3.7) $e_t = \alpha_t + \beta_1 \hat{x}_t + \beta_2 \left( i_t - i_t^* \right)$

$$E_t^{o'}(e_{t+1}) = \alpha_{t+1} + \beta_1 E_t^{o'} \hat{x}_{t+1} + \beta_2 E_t^{o'} \left( i_t - i_t^* \right) = \alpha_{t+1} + (\beta_1 + \beta_2) E_t^{o'} \hat{x}_{t+1} = \alpha_{t+1} + (\beta_1 + \beta_2) E_t^{o'} \hat{x}_{t+1}$$

In equilibrium, $b_t = b_t^*$, and the exchange rate function must satisfy $E_t^{o'}(e_{t+1} - e_t + i_{t+1}^f - i_t) - l_t = 0$. Plugging $e_t$ and $E_t^{o'}(e_{t+1})$ into the exchange rate function, we have $0 = \alpha_{t+1} - \alpha_t - l_t + [a (\beta_1 + \beta_2) - \beta_1] \hat{x}_t - (\beta_2 + 1) (i_t - i_t^*)$. Since the equation must hold for any $\hat{x}_t$ and $i_t - i_t^*$, it follows that $a (\beta_1 + \beta_2) - \beta_1 = 0$, $\beta_2 + 1 = 0$, $\alpha_{t+1} = \alpha_t + l_t = 0$. Hence, the coefficients of the equilibrium exchange rate function are $\beta_1^* = -\frac{a}{1-a}$, $\beta_2^* = -1$, and $\alpha_{t+1}^* = \alpha_t^* + l_t^*$, where $l_t^*$ is $l_t$ evaluated at $(\beta_1^*, \beta_2^*, b_t^*)$. 

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Here, we present some proofs as well as some auxiliary results.

**Proof of Lemma 5.1.** We prove part 1 in three steps. First, we find the distribution of random variable $x_t|x_{t-1}$ under any probability measure $\theta \in \Theta^w$, given that under the baseline measure $\theta'$ the random variable $x_t|x_{t-1}$ is normally distributed as $N (a x_{t-1}, \sigma^2_w)$. Second, we show that $y_t|x_t$ and $x_{t-1}$ have the same distribution under measure $\theta$ as under measure $\theta'$.

$$P^\theta(x_t < z|x_{t-1}) \equiv \int_{\{x_t < z|x_{t-1}\}} d\theta$$

$$= \int_{\{x_t < z|x_{t-1}\}} \frac{d\theta}{d\theta'} d\theta'$$

$$= \int_{\{x_t < z|x_{t-1}\}} \exp \left( -\left( \frac{1}{2\sigma^2_w} - \frac{1}{2\sigma^2_w} \right) (x_t - a x_{t-1})^2 \right) \cdot \sqrt{\frac{\sigma^2_w}{\sigma^2_w}} d\theta'$$

$$= \int_{-\infty}^z \exp \left( -\left( \frac{1}{2\sigma^2_w} - \frac{1}{2\sigma^2_w} \right) (x - a x_{t-1})^2 \right) \cdot \sqrt{\frac{\sigma^2_w}{2\pi\sigma^2_w}} \exp \left[ -\frac{(x - a x_{t-1})^2}{2\sigma^2_w} \right] dx$$

$$= \int_{-\infty}^z \frac{1}{\sqrt{2\pi}\sigma_w} \exp \left[ -\frac{1}{2\sigma^2_w} (x - a x_{t-1})^2 \right] dx$$

The RHS in the last equation is the PDF of a Normal distribution $N (a x_{t-1}, \sigma^2_w)$. This shows that $x \sim N (a x_{t-1}, \sigma^2_w)$. Second, we show that $y_t|x_t$ has the same distribution under measure $\theta$ as under measure $\theta'$.

$$P^\theta(y_t < z|x_t) = E^\theta 1_{\{y_t < z|x_t\}} = E^\theta 1_{\{y_t < z|x_t\}} \frac{d\theta}{d\theta'}$$

$$= E^\theta 1_{\{y_t < z|x_t\}} \exp \left( -\left( \frac{1}{2\sigma^2_w} - \frac{1}{2\sigma^2_w} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma^2_w}{\sigma^2_w}}$$

$$= E^\theta \left\{ 1_{\{y_t < z|x_t\}} \exp \left( -\left( \frac{1}{2\sigma^2_w} - \frac{1}{2\sigma^2_w} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma^2_w}{\sigma^2_w}} \right\}$$

$$= E^\theta \left\{ E^\theta \left\{ 1_{\{y_t < z|x_t\}} \exp \left( -\left( \frac{1}{2\sigma^2_w} - \frac{1}{2\sigma^2_w} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma^2_w}{\sigma^2_w}} \right\} | x_t \right\}$$

$$= E^\theta \left\{ E^\theta \left\{ 1_{\{y_t < z\}} E^\theta \left\{ \exp \left( -\left( \frac{1}{2\sigma^2_w} - \frac{1}{2\sigma^2_w} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma^2_w}{\sigma^2_w}} \right\} | x_t \right\} \right\}$$

The last inequality follows because $x_{t-1}$ and $w_{t-1}$ are independent with each other under
Notice that the third expectation in the last inequality equals one because

\[
E^{\theta'} \left\{ \exp \left( -\left( \frac{1}{2 \hat{\sigma}_w^2} - \frac{1}{2 \hat{\sigma}_w^2} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma_w^2}{\sigma_w^2}} \right\} \\
= \int \exp \left( -\left( \frac{1}{2 \hat{\sigma}_w^2} - \frac{1}{2 \hat{\sigma}_w^2} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma_w^2}{\sigma_w^2}} d\theta' \\
= \int \exp \left( -\left( \frac{1}{2 \hat{\sigma}_w^2} - \frac{1}{2 \hat{\sigma}_w^2} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma_w^2}{\sigma_w^2}} \exp \left( -\frac{1}{2} \frac{w_{t-1}^2}{\hat{\sigma}_w^2} \right) d\theta' \\
= \int \frac{1}{\sqrt{2\pi \hat{\sigma}_w^2}} \exp \left( -\frac{1}{2} \frac{w_{t-1}^2}{\hat{\sigma}_w^2} \right) d\theta' = 1
\]

Therefore, we have that \( P^{\theta} (y_t < z | x_t) = E^{\theta} \mathbb{1}_{\{y_t < z\}} = E^{\theta'} \mathbb{1}_{\{y_t < z\}} \). This shows that conditional on \( x_t, y_t \) has the same distribution under \( \theta \) as under \( \theta' \). Third, the same argument can be used to show that \( x_{t-1} \) has the same distribution under \( \theta \) as under \( \theta' \). To prove part 2 we compute the relative entropy

\[
R(\theta || \theta') = E^\theta \log \left( \frac{d\theta}{d\theta'} \right) = E^\theta \left( -\left( \frac{1}{2 \hat{\sigma}_w^2} - \frac{1}{2 \hat{\sigma}_w^2} \right) x^2 - \frac{1}{2} \log (\hat{\sigma}_w^2) \right) \\
= -\left( \frac{1}{2 \hat{\sigma}_w^2} - \frac{1}{2 \hat{\sigma}_w^2} \right) E^\theta (x^2) - \frac{1}{2} \log (\hat{\sigma}_w^2) = \frac{1}{2} \left( \frac{\sigma_w^2}{\hat{\sigma}_w^2} - \log \left( \frac{\sigma_w^2}{\hat{\sigma}_w^2} \right) - 1 \right)
\]

Auxiliary Results

**Lemma 8.3 (Convexity of the Uncertainty Sets).** The sets of probability measures \( \Theta^v, \Theta^w \) and \( \Theta^a \) are closed and convex.

**Proof.** The set \( \Theta^v \) is the image of the mapping \( g : [\varepsilon, \infty) \rightarrow P(\Omega) \), where \( g \) is defined by \( \theta (A) = g_A (z) = \int_A \exp \left( -\frac{1}{2} \left( \frac{1}{\hat{\sigma}_w^2} - \frac{1}{\sigma_w^2} \right) (y_t - x_t)^2 \right) \cdot \sqrt{\frac{\sigma_w^2}{\hat{\sigma}_w^2}} d\theta', \) \( z \in [\varepsilon, \infty), \varepsilon > 0 \), for a fixed arbitrary set \( A \in \mathcal{A} \). The function \( g \) is continuous since for any sequence \( z_n \rightarrow z \in R \) and a fixed arbitrary set \( A \in \mathcal{A} \), we have that \( \theta_n (A) = g_A (z_n) \rightarrow \theta (A) = g_A (z) \). Since \( [\varepsilon, \infty) \) is a closed set and \( g_A \) is a continuous function, the image set \( \Theta^v \) is also closed as a continuous function preserves topological properties. To prove that \( \Theta^v \) is convex, notice that the Intermediate Value Theorem implies that for any \( z_1, z_2 \in [\varepsilon, \infty) \) and \( \phi \in [0, 1] \) there always exists a \( z \in [\varepsilon, \infty) \) such that \( \phi f (z_1) + (1 - \phi) f (z_2) = f (z) \), where \( f (z) = \exp \left( -\frac{1}{2} \left( \frac{1}{\hat{\sigma}_w^2} - \frac{1}{\sigma_w^2} \right) (y_t - x_t)^2 \right) \cdot \sqrt{\frac{\sigma_w^2}{\hat{\sigma}_w^2}} \) is the Radon-Nikodym density function. This holds because \( f (z) \) is a continuous function. Hence, \( \phi \int_A f (z_1) + (1 - \phi) f (z_2) = \int_A f (z) \) for any fixed \( A \in \mathcal{A} \). This shows that \( \Theta^v \) is convex.

The same argument shows that \( \Theta^w \) and \( \Theta^a \) are closed and convex. These sets are generated by the density functions \( f^w (z) = \exp \left( -\frac{1}{2} \left( \frac{1}{\hat{\sigma}_w^2} - \frac{1}{\sigma_w^2} \right) (x_t - a x_{t-1})^2 \right) \cdot \sqrt{\frac{\sigma_w^2}{\hat{\sigma}_w^2}} \) over the closed set \( [\varepsilon, \infty), \varepsilon > 0 \), and \( f^a (z) = \exp \left[ -\frac{(x_t - z)^2 - 2x_{t-1} z(x_t - a x_{t-1})}{2\sigma_w^2} \right] \) over the closed set \([-1, 1]\) respectively.
Derivation of Equation (3.5). Let $Z_{t+1} = \left[ \exp(i_t) - \frac{E_{t+1}}{E_t} \exp(i_t^f) \right] b_t$ and $e_t := \log(E_t)$. The Taylor expansion around zero is given by

$$
Z_{t+1} = b_t \left[ \exp(i_t) - \exp(e_{t+1} - e_t + i_t^f) \right] = b_t \left[ (1 + i_t + o_2(2)) - (1 + e_{t+1} - e_t + i_t^f + o_1(2)) \right] = b_t \left[ -(e_{t+1} - e_t) + i_t - i_t^f + o(2) \right],
$$

where $o(2) = o_1(2) + o_2(2)$ and

$$
o_1(2) = -\frac{1}{2} \exp(\xi_1) \left( e_{t+1} - e_t + i_t^f \right)^2, \quad o_2(2) = \frac{1}{2} \exp(\xi_2) i_t^2,
\xi_1 \in (0, e_{t+1} - e_t + i_t^f), \quad \xi_2 \in (0, i_t)
$$

Clearly, $\lim_{x_i \to 0} \frac{o_i(x_i)}{x_i} = 0$, for $i = 1, 2$ where $x_1 = e_{t+1} - e_t + i_t^f$ and $x_2 = i_t$. Thus the terms $o_1(2)$ and $o_2(2)$ are approximately zero if $e_{t+1} - e_t + i_t^f$ and $i_t$ are small, which is the case for monthly and quarterly data across the major currency pairs considered in the literature.

We have defined the utility function in terms of the log excess return $W_{t+1} = b_t \left[ (i_t - i_t^f) - (e_{t+1} - e_t) \right]$. The following Lemma shows that for a given domestic interest rate $i_t$, $u_1 = E^\theta_t \left[ - \exp(-\gamma W_{t+1}) \right]$ is a monotonic transformation of $u_2 = E^\theta_t \left[ - \exp \left( -\gamma b_t \left( \exp(i_t) - \frac{E_{t+1}}{E_t} \exp(i_t^f) \right) \right) \right]$.

**Lemma 8.4.** For a fixed $i_t$, $u_1 = E^\theta_t \left\{ - \exp \left[ -\gamma b_t \left( \exp(i_t) - \frac{E_{t+1}}{E_t} \exp(i_t^f) \right) \right] \right\}$ is a monotonic transformation of $u_2 = E^\theta_t \left\{ - \exp \left[ -\gamma b_t \left( i_t - i_t^f - e_{t+1} + e_t \right) \right] \right\}$.

**Proof.** Suppose that under utility function $u_1$

$$
(b_{t,1}, E_{t+1,1}, E_{t,1}, i_{t,1}^f) \geq (b_{t,2}, E_{t+1,2}, E_{t,2}, i_{t,2}^f)
$$

$$
\Leftrightarrow
u_1 = E^\theta_t \left\{ - \exp \left[ -\gamma b_t \left( \exp(i_t) - \frac{E_{t+1}}{E_t} \exp(i_{t,1}^f) \right) \right] \right\}
\geq u_2 = E^\theta_t \left\{ - \exp \left[ -\gamma b_t \left( \exp(i_t) - \frac{E_{t+1}}{E_{t,2}} \exp(i_{t,2}^f) \right) \right] \right\}
$$

for any measure $\theta$.

$$
\Leftrightarrow \exp(i_t) - \frac{E_{t+1,1}}{E_{t,1}} \exp(i_{t,1}^f) \geq \exp(i_t) - \frac{E_{t+1,2}}{E_{t,2}} \exp(i_{t,2}^f)
$$
\[ 1 - \frac{E_{t+1,1}}{E_{t,1}} \exp \left( i'_{t,1} - i_t \right) \geq 1 - \frac{E_{t+1,2}}{E_{t,2}} \exp \left( i'_{t,2} - i_t \right) \]

\[ \exp \left( i_t - i'_{t,1} - e_{t+1,1} + e_{t,1} \right) \geq \exp \left( i_t - i'_{t,2} - e_{t+1,2} + e_{t,2} \right) \]

\[ i_t - i'_{t,1} - e_{t+1,1} + e_{t,1} \geq i_t - i'_{t,2} - e_{t+1,2} + e_{t,2} \]

\[ - \exp \left[ -\gamma b_t \left( i_t - i'_{t,1} - e_{t+1,1} + e_{t,1} \right) \right] \]

\[ \geq - \exp \left[ -\gamma b_t \left( i_t - i'_{t,2} - e_{t+1,2} + e_{t,2} \right) \right] \]

\[ \bar{u}_1 \equiv E^\theta \left\{ - \exp \left[ -\gamma b_t \left( i_t - i'_{t,1} - e_{t+1,1} + e_{t,1} \right) \right] \right\} \]

\[ \geq \bar{u}_2 \equiv E^\theta \left\{ - \exp \left[ -\gamma b_t \left( i_t - i'_{t,2} - e_{t+1,2} + e_{t,2} \right) \right] \right\} \]

Therefore, \( \left( E_{t+1,1}, E_{t,1}, i'_{t,1} \right) \equiv \left( E_{t+1,2}, E_{t,2}, i'_{t,2} \right) \).

**Equilibrium Under Baseline Priors**  Here we derive the equilibrium in the case where the prior of a \( t \)-agent equals the Bayesian prior estimate \( \hat{x}_t^\theta = a \hat{x}_{t-1}^\theta \) under the baseline model. As we shall see the key characteristics of the equilibrium are qualitatively the same as those in Proposition 4.2: forecasts are less sensitive to news \( y_t \) than Bayesian forecasts, and this under-reaction carries over to the equilibrium \( e_t^* \). In this case we need a new conjecture with two states: the Bayesian prior \( \hat{x}_t^\theta \) and the \( t \)-agent’s robust estimate of the state

\[ e_t = \alpha_t + \beta_1 \hat{x}_t^\theta + \beta_2 \hat{x}_t^\theta + \beta_3 y_t \]

It follows that expected excess returns are

\[ E_t^\theta J_{t+1} = y_t - E_t^\theta (e_{t+1}) + e_t \]

\[ = \left[ -a\beta_1 (1 - k_{t+1}^\theta) - a\beta_2 (1 - k_{t+1}^\theta) + \beta_2 \right] \hat{x}_t^\theta + \left[ -a\beta_1 k_{t+1}^\theta + a\beta_2 k_{t+1}^\theta - a\beta_3 + \beta_1 \right] \hat{x}_t^\theta + (\beta_3 + 1) y_t + \alpha_t - \alpha_{t+1} \]

and

\[ V_t^\theta (J_{t+1}) = \left( \beta_1 k_{t+1}^\theta + \beta_2 k_{t+1}^\theta + \beta_3 \right)^2 V_t^\theta (y_{t+1}) \]

\[ = \left( \beta_1 k_{t+1}^\theta + \beta_2 k_{t+1}^\theta + \beta_3 \right)^2 \left( \frac{a^2 (a^2 \sigma_{t-1}^2 + \sigma_w^2) \tilde{\sigma}_{v,t}^2 + \sigma_w^2 + \sigma_{v,t}^2}{a^2 \sigma_{t-1}^2 + \sigma_w^2 + \sigma_{v,t}^2} \right) \]
The first derivative for \( b_t \) is

\[
\frac{\partial \Gamma}{\partial b_t} = -(-\gamma E_t^{\theta_t} (J_{t+1}) + \gamma^2 b_t V a r_t^{\theta_t} (J_{t+1})) \exp\left(\frac{(\gamma b_t)^2}{2} V a r_t^{\theta_t} (J_{t+1})\right) \exp\left(\gamma b_t E_t^{\theta_t} (J_{t+1})\right) + \frac{\partial \Gamma}{\partial \sigma_{2,v,t}^2} \frac{d \sigma_{2,v,t}^2}{d b_t}
\]

The first order condition for \( b_t \) implies that for any linear equilibrium we will have the equality for coefficients.

\[-a \beta_1 k^{\theta_{t+1}} + a \beta_2 k^{\theta_t} - a \beta_3 + \beta_1 = 0\]

The exchange rate that clears the market is

\[e_t = \left[ -a \beta_1 \left( 1 - k^{\theta_{t+1}} \right) - a \beta_2 \left( 1 - k^{\theta_t} \right) + \beta_2 \right] \tilde{\alpha}_t + \left[ -a \beta_1 k^{\theta_{t+1}} + a \beta_2 k^{\theta_t} - a \beta_3 + \beta_1 \right] \tilde{\beta}_t + y_t + \gamma b^{\theta} V a r_t^{\theta_t} (J_{t+1})\]

If we equalize coefficients with the conjecture, we get that in equilibrium

\[\beta_3^* = -1\]

\[\beta_1^* = \left[ a \beta_1 k^{\theta_{t+1}} + a \beta_2 k^{\theta_t} - a \right] \left[ a \beta_1^* (1 - k^{\theta_{t+1}}) + a \beta_2^* (1 - k^{\theta_t}) \right] \alpha_t^* = \gamma b^{\theta} V a r_t^{\theta_t} (J_{t+1}) + \alpha_t^*\]

Some algebra reveals that

\[\beta_1^* = \frac{\beta_2^* k^{\theta_{t+1}} - 1}{1 - a k^{\theta_{t+1}}} a\]

Notice that if \( k^{\theta_{t+1}} = k^{\theta_t} \), then we get \( \beta_1^* + \beta_2^* = -\frac{a}{1-a} \), as in the simpler case considered in Proposition 4.2. The key to the result is that the equation of \( \beta_1^* \) and \( \beta_2^* \) imply that \( \frac{\partial E_t^{\theta_t} (J_{t+1})}{\partial \sigma_{2,v,t}^2} = 0 \) in the first derivative of \( \frac{d \Gamma}{d \sigma_{2,v,t}^2} \) because the FOC for \( \sigma_{2,v,t}^2 \) is

\[
\frac{\partial \Gamma}{\partial \sigma_{2,v,t}^2} = \left[ \gamma b_t E_t^{\theta_t} (J_{t+1}) - \frac{(\gamma b_t)^2}{2} \frac{\partial V a r_t^{\theta_t} (J_{t+1})}{\partial \sigma_{2,v,t}^2} \right] \exp\left(\frac{(\gamma b_t)^2}{2} V a r_t^{\theta_t} (J_{t+1})\right) \exp\left(\gamma b_t E_t^{\theta_t} (J_{t+1})\right) + \frac{\lambda}{2} \left( \frac{1}{\sigma_v^2} - \frac{1}{\sigma_{2,v,t}^2} \right)
\]

where

\[
\frac{\partial E_t^{\theta_t} (J_{t+1})}{\partial \sigma_{2,v,t}^2} = \left[ -a \beta_1 k^{\theta_{t+1}} + a \beta_2 k^{\theta_t} - a \beta_3 + \beta_1 \right] \frac{d \sigma_{2,v,t}^2}{d \sigma_{2,v,t}^2}
\]

is zero when evaluated at \( \beta_1^* \) and \( \beta_2^* \). This result in turn implies an upward distorted
variance $\sigma^2_{\text{v,t}} \geq \sigma^2_{\text{v}}$ in equilibrium, as the first order condition for $\sigma^2_{\text{v,t}}$ then becomes

$$\frac{1}{\sigma^2_{\text{v}}} \cdot \frac{1}{\sigma^2_{\text{v,t}}} = \frac{(\gamma b_t)^2}{\lambda} \cdot \frac{\partial \text{Var}\left(\theta^\tau_{t+1}\right)}{\partial \sigma^2_{\text{v,t}}} \cdot \frac{\exp\left(\frac{(\gamma b_t)^2}{2} \text{Var}\left(\theta^\tau_{t+1}\right)\right)}{\exp(\gamma b_t E^\tau_t (J_{t+1}))} > 0,$$

where $\frac{\partial \text{Var}\left(\theta^\tau_{t+1}\right)}{\partial \sigma^2_{\text{v,t}}} = \left(\beta_1^* k_{t+1}^{\theta^\tau_{t+1}} + \beta_2^* k_{t+1}^{\theta^\tau_{t+1}} + \beta_3^*\right)^2 \left(a^2 \cdot \frac{(\alpha^2 \sigma^2_{t-1} + \sigma^2_{\text{v,t}})^2}{(a^2 \sigma^2_{t-1} + \sigma^2_{\text{v,t}} + \sigma^2_{\text{v,t}})^2} + 1\right) \geq 0.$
Figure 2

Distributions of $\beta$
Figure 3

![Graph showing the average of β versus persistence α.]
Figure 5

For $a=0.99$:

- Transitory shock
- Persistent shock
- Average shock