
Here, we present some proofs as well as some auxiliary results.

**Proof of Lemma 6.1.** We prove part 1 in three steps. First, we find the distribution of random variable $x_t|x_{t-1}$ under any probability measure $\theta \in \Theta^w$, given that under the baseline measure $\theta'$ the random variable $x_t|x_{t-1}$ is normally distributed as $N(ax_{t-1}, \sigma_{w}^2)$. Second, we show that $y_t|x_t$ and $x_{t-1}$ have the same distribution under measure $\theta$ as under measure $\theta'$.

\[
P^\theta (x_t < z | x_{t-1}) \equiv \int_{\{x_t < z|x_{t-1}\}} d\theta
\]

\[
= \int_{\{x_t < z| x_{t-1}\}} \frac{d\theta}{d\theta'} d\theta'
\]

\[
= \int_{\{x_t < z| x_{t-1}\}} \exp \left( - \left( \frac{1}{2\sigma_{w}^2} - \frac{1}{2\sigma_{w}^2} \right) (x_t - ax_{t-1})^2 \right) \cdot \sqrt{\frac{\sigma_{w}^2}{\sigma_{w}^2}} d\theta'
\]

\[
= \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_{w}^2}} \exp \left[ - \frac{1}{2\sigma_{w}^2} (x - ax_{t-1})^2 \right] d\theta
\]

\[
= \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_{w}^2}} \exp \left[ - \frac{1}{2\sigma_{w}^2} (x - ax_{t-1})^2 \right] dx
\]

The RHS in the last equation is the PDF of a Normal distribution $N(ax_{t-1}, \sigma_{w}^2)$. This shows that $x \sim N(ax_{t-1}, \sigma_{w}^2)$. Second, we show that $y_t|x_t$ has the same distribution under measure $\theta$ as under measure $\theta'$.

\[
P^\theta (y_t < z | x_t) = E^\theta 1_{\{y_t < z| x_t\}} = E^\theta 1_{\{y_t < z| x_t\}} \frac{d\theta}{d\theta'}
\]

\[
= E^\theta 1_{\{y_t < z| x_t\}} \exp \left( - \left( \frac{1}{2\sigma_{w}^2} - \frac{1}{2\sigma_{w}^2} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma_{w}^2}{\sigma_{w}^2}}
\]

\[
= E^\theta \left\{ 1_{\{y_t < z| x_t\}} \exp \left( - \left( \frac{1}{2\sigma_{w}^2} - \frac{1}{2\sigma_{w}^2} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma_{w}^2}{\sigma_{w}^2}} | x_t \right\}
\]

\[
= E^\theta \left\{ E^\theta \left\{ 1_{\{y_t < z| x_t\}} \exp \left( - \left( \frac{1}{2\sigma_{w}^2} - \frac{1}{2\sigma_{w}^2} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma_{w}^2}{\sigma_{w}^2}} | x_{t-1} \right\} | x_t \right\}
\]

\[
= E^\theta \left\{ E^\theta \left\{ 1_{\{y_t < z\}} | x_{t-1} \right\} E^\theta \left\{ \exp \left( - \left( \frac{1}{2\sigma_{w}^2} - \frac{1}{2\sigma_{w}^2} \right) w_{t-1}^2 \right) \cdot \sqrt{\frac{\sigma_{w}^2}{\sigma_{w}^2}} | x_t \right\} \right\}
\]

The last inequality follows because $x_{t-1}$ and $w_{t-1}$ are independent with each other under
\( \theta' \). Notice that the third expectation in the last inequality equals one because

\[
E^{\theta'} \left\{ \exp \left( -\left( \frac{1}{2\sqrt{\hat{\sigma}^2_w}} - \frac{1}{2\sigma^2_w} \right) w^2_{t-1} \right) \cdot \sqrt{\frac{\sigma^2_w}{\hat{\sigma}^2_w}} \right\} \\
= \int \exp \left( -\left( \frac{1}{2\sqrt{\hat{\sigma}^2_w}} - \frac{1}{2\sigma^2_w} \right) w^2_{t-1} \right) \cdot \sqrt{\frac{\sigma^2_w}{\hat{\sigma}^2_w}} d\theta' \\
= \int \exp \left( -\left( \frac{1}{2\sqrt{\hat{\sigma}^2_w}} - \frac{1}{2\sigma^2_w} \right) w^2_{t-1} \right) \cdot \sqrt{\frac{\sigma^2_w}{\hat{\sigma}^2_w}} \sqrt{2\pi \sigma^2_w} \exp \left( -\frac{1}{2} \frac{w^2_{t-1}}{\sigma^2_w} \right) dw_{t-1} \\
= \int \frac{1}{\sqrt{2\pi \hat{\sigma}^2_w}} \exp \left( -\frac{1}{2} \frac{w^2_{t-1}}{\sigma^2_w} \right) dw_{t-1} = 1
\]

Therefore, we have that \( P^\theta (y_t < z | x_t) = E^{\theta'} 1_{\{y_t < z\}} = E^{\theta} 1_{\{y_t < z\}} \). This shows that conditional on \( x_t, y_t \) has the same distribution under \( \theta \) as under \( \theta' \). Third, the same argument can be used to show that \( x_{t-1} \) has the same distribution under \( \theta \) as under \( \theta' \). To prove part 2 we compute the relative entropy

\[
R(\theta || \theta') = E^\theta \log \left( \frac{d\theta}{d\theta'} \right) = E^\theta \left( -\left( \frac{1}{2\sqrt{\hat{\sigma}^2_w}} - \frac{1}{2\sigma^2_w} \right) x^2 - \frac{1}{2} \log \left( \frac{\sigma^2_w}{\hat{\sigma}^2_w} \right) \right) \\
= -\left( \frac{1}{2\sqrt{\hat{\sigma}^2_w}} - \frac{1}{2\sigma^2_w} \right) E^\theta (x^2) - \frac{1}{2} \log \left( \frac{\sigma^2_w}{\hat{\sigma}^2_w} \right) \\
= \frac{1}{2} \left( \frac{\sigma^2_w}{\hat{\sigma}^2_w} - \log \left( \frac{\sigma^2_w}{\hat{\sigma}^2_w} \right) - 1 \right)
\]

**Proof of Proposition 6.2.** The solution to the agent’s problem has the same form as in the case of observation uncertainty. The first order condition for \( b_t \) is given by (8.3), while that for \( \hat{\sigma}^2_{w,t} \) is

\[
0 = \frac{\partial \Gamma}{\partial \hat{\sigma}^2_{w,t}} \\
= -\frac{1}{2} (\gamma b_t)^2 \frac{\partial \text{Var}^\theta_t (J_{t+1})}{\partial \hat{\sigma}^2_{w,t}} \exp \left( -\gamma b_t E^\theta_t (J_{t+1}) + \frac{1}{2} (\gamma b_t)^2 \text{Var}^\theta_t (J_{t+1}) \right) + \frac{\lambda}{2} \left( \frac{1}{\sigma^2_w} - \frac{1}{\hat{\sigma}^2_{w,t}} \right)
\]

where

\[
\frac{\partial \text{Var}^\theta_t (J_{t+1})}{\partial \hat{\sigma}^2_{w,t}} = \left[ \left( \kappa^{\theta_t}_{t+1} + \beta_1 + \beta_2 \right) \right]^2 \left( \frac{\sigma^2_v}{a^2 \sigma^2_{t+1} + \sigma^2_v + \hat{\sigma}^2_{w,t}} \right)^2 + 1 > 0
\]

Analogous to the argument for \( \hat{\sigma}^2_{\theta,t} \), the first order conditions imply that a robust agent distorts upwards the variance of the trend shock: \( \hat{\sigma}^2_{w,t} > \sigma^2_w \). The second order condition
for $b_t$ is satisfied because
\[
\frac{\partial^2 \Gamma}{\partial b_t} = - \left( - \gamma E_t^\theta (J_{t+1}) + (\gamma^2 b_t) \text{Var}_t^\theta (J_{t+1}) \right)^2 + \gamma^2 \text{Var}_t^\theta (J_{t+1}) \cdot \exp \left( - (\gamma b_t) E_t^\theta (J_{t+1}) + \frac{1}{2} (\gamma b_t)^2 \text{Var}_t^\theta (J_{t+1}) \right) < 0
\]

The second derivative of $\tilde{\sigma}_{w,t}^2$ is
\[
\frac{\partial^2 \Gamma}{\partial (\tilde{\sigma}_{w,t}^2)^2} = - \frac{(\gamma b_t)^2}{2} \left[ \frac{1}{2} (\gamma b_t)^2 \left( \frac{\partial \text{Var}_t^\theta (J_{t+1})}{\partial \tilde{\sigma}_{w,t}^2} \right)^2 + \frac{\partial^2 \text{Var}_t^\theta (J_{t+1})}{\partial (\tilde{\sigma}_{w,t}^2)^2} \right] \cdot \exp \left( - (\gamma b_t) E_t^\theta (J_{t+1}) + \frac{(\gamma b_t)^2}{2} \text{Var}_t^\theta (J_{t+1}) \right) + \frac{\lambda}{2} \frac{1}{(\tilde{\sigma}_{w,t}^2)^2} \geq 0
\]
where
\[
\frac{\partial^2 \text{Var}_t^\theta (J_{t+1})}{\partial (\tilde{\sigma}_{w,t}^2)^2} = - \left( \left( k_{t+1}^\theta \beta_1 + \beta_2 \right)^2 - \frac{a^2 \sigma_t^2}{\left( a^2 \sigma_{t-1}^2 + \sigma_t^2 + \tilde{\sigma}_{w,t}^2 \right)} \right).
\]
Hence the second order condition for $\tilde{\sigma}_{w,t}^2$ holds if and only if $\lambda \geq \lambda_t^w$, where $\lambda_t^w$ is defined by (8.18).

**Proof of Lemma 6.3.** We follow the same steps as in the proof of Lemma 6.1. First, we find the distribution of random variable $x_t|x_{t-1}$ under any probability measure $\theta \in \Theta^w$, given that under the baseline measure $\theta'$ the random variable $x_t|x_{t-1}$ has a normal distribution $N(ax_{t-1}, \sigma_w^2)$. Then we prove that $y_t|x_t$ and $x_{t-1}$ have the same distributions under $\theta$ as under $\theta'$.

\[
P^\theta (x_t < z|x_{t-1}) = \int_{\{x_t < z|x_{t-1}\}} d\theta = \int_{\{x_t < z|x_{t-1}\}} \frac{d\theta}{d\theta'} d\theta' = \int_{\{x_t < z|x_{t-1}\}} \exp \left[ - \frac{(x_{t-1} - \delta)^2 - 2x_{t-1} \delta (x_t - ax_{t-1})}{2\sigma_w^2} \right] d\theta' = \int_{-\infty}^z \exp \left[ - \frac{(x_{t-1} - \delta)^2 - 2x_{t-1} \delta (x - ax_{t-1})}{2\sigma_w^2} \right] \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{(x - ax_{t-1})^2}{2\sigma_w^2} \right] dx = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp \left[ - \frac{1}{2\sigma_w^2} (x - (a + \delta) x_{t-1})^2 \right] dx
\]

The RHS in the last equation is the PDF of a Normal distribution $N((a + \delta) x_{t-1}, \sigma_w^2)$. This shows that $x_t \sim N((a + \delta) x_{t-1}, \sigma_w^2)$. The same argument as the one in the proof of Lemma 6.1 can be applied to prove that $y_t|x_t$ and $x_{t-1}$ have the same distributions.
under \( \theta \) as under \( \theta' \). To prove part 2, we compute the relative entropy
\[
R(\theta||\theta') = E^\theta \log \left( \frac{d\theta}{d\theta'} \right) = E^\theta \left( \frac{(x_{t-1} \delta)^2 - 2 x_{t-1} \delta (x_t - a x_{t-1})}{2 \sigma_w^2} \right) = \frac{E^\theta (x_t \delta)^2}{2 \sigma_w^2}.
\]

To derive the last equality we use the fact that \( E^\theta (x_t|x_{t-1}) = (a + \delta) x_{t-1} \).

**Proof of Proposition 6.4.** Since the prior of the young \( t \) agent is \( x_{t-1} \sim \theta' \cdot N \left( x_{t-1}^{\theta'_{t-1}}, \sigma_{t-1}^2 \right) \), where \( \sigma_{t-1}^2 = \frac{(a^2 \sigma_{t-2}^2 + \sigma_w^2)}{a^2 \sigma_{t-2}^2 + \sigma_w^2 + \sigma_v^2} \), it follows that under the robust model \( \theta_t \), the agent’s posterior estimate of the state \( x_t \) is
\[
\hat{x}_t^{\theta_t} = E_t^{\theta_t} (x_t|I_t) = \left( 1 - k_t^{\theta_t} \right) (a + \delta_t) \hat{x}_{t-1}^{\theta_{t-1}} + k_t^{\theta_t} y_t
\]
\[
x_t|I_t \sim \theta_t \cdot N \left( \hat{x}_t^{\theta_t}, \left( \frac{((a + \delta_t)^2 \sigma_{t-1}^2 + \sigma_w^2)}{(a + \delta_t)^2 \sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} \right) \right),
\]
\[
x_{t+1}|I_t \sim \theta_t \cdot N \left( (a + \delta_t) \hat{x}_t, (a + \delta_t)^2 \frac{((a + \delta_t)^2 \sigma_{t-1}^2 + \sigma_w^2)}{(a + \delta_t)^2 \sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} \right)
\]

In this case the gain of the filter is \( k_t^{\theta_t} = \frac{(a + \delta_t)^2 \sigma_{t-1}^2 + \sigma_w^2}{(a + \delta_t)^2 \sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2} \). Note that the gain is increasing in the drift distortion because of the assumption \( \delta_t \geq -a \)
\[
\frac{dk_t}{d\delta_t} = \frac{2 (a + \delta_t) \sigma_{t-1}^2}{((a + \delta_t)^2 \sigma_{t-1}^2 + \sigma_w^2 + \sigma_v^2)^2} > 0
\]

Under probability measure \( \theta_t \) the log excess return is
\[
J_{t+1} \equiv (i_t - i_{t-1}^f) - (e_{t+1} - e_t)
\]
\[
= -\alpha_{t+1} - \beta_1 \hat{x}_{t+1}^{\theta_{t+1}} - \beta_2 \left( i_t - i_{t+1}^f \right) + e_t + (i_t - i_{t+1}^f)
\]
\[
= -\alpha_{t+1} - \left( 1 - k_{t+1}^{\theta_{t+1}} \right) \beta_1 (a + \delta_{t+1}) \hat{x}_t^{\theta_t} - \left( k_{t+1}^{\theta_{t+1}} \beta_1 + \beta_2 \right) \left( i_t - i_{t+1}^f \right) + e_t + (i_t - i_{t+1}^f)
\]

To compute the distribution of excess returns, under probability measure \( \theta_t \), the \( t \) agent uses the conjecture (3.7) to forecast next period’s exchange rate: \( e_{t+1}^{\text{conj}} = \alpha_{t+1} + \beta_1 \hat{x}_{t+1}^{\theta_{t+1}} + \beta_2 \left( i_{t+1} - i_{t+1}^f \right) \). To obtain \( E_t^{\theta_t}(\hat{x}_{t+1}^{\theta_{t+1}}; I_t) \) note that the \( t \) agent knows the problem that will be solved by \( t + 1 \) agents. Thus, the \( t \) agent knows the method that \( t + 1 \) agents will use to derive the robust probability measure \( \theta_{t+1} \), and that they will make forecasts using Bayes law under \( \theta_{t+1} \). Taking this into account, the \( t \) agent knows that \( t + 1 \) agents will use the updating formula \( \hat{x}_{t+1}^{\theta_{t+1}} = \left( 1 - k_{t+1}^{\theta_{t+1}} \right) (a + \delta_{t+1}) \hat{x}_t^{\theta_t} + k_{t+1}^{\theta_{t+1}} \left( i_{t+1} - i_{t+1}^f \right) \).

Therefore, the \( t \) agent sets \( E_t^{\theta_t}(\hat{x}_{t+1}^{\theta_{t+1}}) = (a + \delta_{t+1}) \hat{x}_t^{\theta_t} \). Replacing this formula in the
conjecture, it follows that under probability measure $\theta_t$, the log excess return $J_{t+1}$

$$E^\theta_t (J_{t+1}) = -\alpha_{t+1} - (1 - k^\theta_{t+1}) \beta_1 (a + \delta_{t+1}) \tilde{x}^\theta_t - (k^\theta_{t+1} \beta_1 + \beta_2) (a + \delta_t) \tilde{x}^\theta_t + e_t + (i_t - i^*_t)$$

$$V^\theta_t (J_{t+1}) = (k^\theta_{t+1} \beta_1 + \beta_2)^2 \left[ (a + \delta_t)^2 \left( \frac{(a + \delta_t)^2 \sigma^2_t + \sigma^2_e}{\sigma^2_t - 1 + \sigma^2_e} \right) + \lambda E^\theta_t \tilde{x}^2_t \delta_t \right]$$

We solve problem (3.4) by considering the investor as a Stackelberg leader that takes into account the strategy of nature: $\delta_t = s(b_t, e_t)$. Nature then selects $\delta_t$ conditioning on the agent’s choice of $b_t$. Notice that $\delta_t$ affects the investor’s payoff through its effect on $E^\theta_t (J_{t+1})$ and $Var^\theta_t (J_{t+1})$. The first order condition for $\delta_t$ is:

$$\frac{\partial \Gamma}{\partial \delta_t} = - \left[ \gamma b_t \frac{\partial E^\theta_t (J_{t+1})}{\partial \delta_t} + \frac{(\gamma b_t)^2}{2} \frac{\partial Var^\theta_t (J_{t+1})}{\partial \delta_t} \right] \exp \left( -\gamma b_t E^\theta_t (J_{t+1}) + \frac{(\gamma b_t)^2}{2} Var^\theta_t (J_{t+1}) + \lambda E^\theta_t \tilde{x}^2_t \delta_t \right)$$

The first order condition for $b_t$ is:

$$\frac{\partial \Gamma}{\partial b_t} = - \left( -\gamma E^\theta_t (J_{t+1}) + \gamma^2 b_t Var^\theta_t (J_{t+1}) \right) \exp \left( -\gamma b_t E^\theta_t (J_{t+1}) + \frac{(\gamma b_t)^2}{2} Var^\theta_t (J_{t+1}) \right)$$

In the second equality we have used the first order condition for nature: $\frac{\partial \Gamma}{\partial \delta_t} = 0$. The second order condition for the investor’s problem is

$$0 > \frac{\partial^2 \Gamma}{\partial b_t^2} = \Gamma_{bb_t} \left( b_t, \delta^*_t (b_t) \right) + \Gamma_{b\delta_t} \left( b_t, \delta^*_t (b_t) \right) \frac{d\delta^*_t (b_t)}{db_t}$$

$$+ \Gamma_{\delta_t \delta_t} \left( b_t, \delta^*_t (b_t) \right) \frac{d^2 \delta^*_t (b_t)}{db_t^2}$$

Notice that the total derivative of nature’s FOC $\Gamma_{\delta_t} (b_t, \delta^*_t (b_t)) = 0$ is $\Gamma_{\delta_t \delta_t} db_t + \Gamma_{\delta_t} d\delta_t = 0$. 

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Thus,
\[
\frac{d\delta_t^*(b_t)}{db_t} = -\frac{\Gamma_{\delta_t b_t}}{\Gamma_{\delta_t \delta_t}}
\]

Combining this equation with \(\Gamma_{b_t b_t} = \Gamma_{\delta_t b_t}\), the investor’s SOC becomes
\[
\frac{\partial^2 \Gamma}{\partial b_t^2} = \Gamma_{b_t b_t}(b_t, \delta_t^*(b_t)) + \Gamma_{b_t \delta_t}(b_t, \delta_t^*(b_t)) \cdot \left( \frac{\Gamma_{\delta_t b_t}(b_t, \delta_t^*(b_t))}{\Gamma_{\delta_t \delta_t}(b_t, \delta_t^*(b_t))} \right)^2 + \Gamma_{\delta_t \delta_t}(b_t, \delta_t^*(b_t)) \cdot \left( \frac{\Gamma_{\delta_t b_t}(b_t, \delta_t^*(b_t))}{\Gamma_{\delta_t \delta_t}(b_t, \delta_t^*(b_t))} \right)^2 = \Gamma_{b_t b_t}(b_t, \delta_t^*(b_t)) \leq 0
\]

This condition is unambiguously satisfied because \(\Gamma_{b_t b_t} \leq 0\):
\[
\Gamma_{b_t b_t} = -\left[ -\gamma E_t^{\theta_t}(J_{t+1}) + (\gamma^2 b_t )Var_t^{\theta_t}(J_{t+1})^2 + \gamma^2 Var_t^{\theta_t}(J_{t+1}) \right] \cdot \exp \left( -(\gamma b_t) E_t^{\theta_t}(J_{t+1}) + \frac{1}{2}(\gamma b_t)^2 Var_t^{\theta_t}(J_{t+1}) \right).
\]

Nature’s second order condition is
\[
\frac{\partial^2 \Gamma}{\partial \delta_t^2} = -\left( \frac{^2}{2} \right) \left[ \frac{1}{2} (\gamma b_t)^2 \left( \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \delta_t} \right)^2 + \frac{\partial^2 Var_t^{\theta_t}(J_{t+1})}{\partial (\delta_t)^2} \right] \cdot \exp \left( -\gamma b_t E_t^{\theta_t}(J_{t+1}) + \frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1}) \right) + \lambda \frac{E_t^{\theta_t} x_t^2}{\sigma^2} \geq 0
\]

Hence, the second order condition for \(\delta_t\) holds if and only if \(\lambda \geq \lambda_t^a\), where \(\lambda_t^a\) is defined by
\[
\lambda_t^a = \max \{0, \lambda_t^{***}\}, \quad \text{with}
\]
\[
\lambda_t^{***} \equiv \left( \frac{^2}{(\delta_t^a)^2 + 1} \right) \left[ \frac{(\gamma b_t)^2}{2} \left( \frac{\partial Var_t^{\theta_t}(J_{t+1})}{\partial \delta_t} \right)^2 + \frac{\partial^2 Var_t^{\theta_t}(J_{t+1})}{\partial (\delta_t)^2} \right] \cdot \exp \left( -\gamma b_t E_t^{\theta_t}(J_{t+1}) + \frac{(\gamma b_t)^2}{2} Var_t^{\theta_t}(J_{t+1}) \right)
\]

Next, we characterize the equilibrium. The first order condition for \(b_t\) implies that
\[
-\left[ \alpha_{t+1} + \left( \left( 1 - k_{t+1}^{\theta_t} \right) \beta_1 (a + \delta_{t+1}) + k_{t+1}^{\theta_t} \beta_1 + \beta_2 \right) (a + \delta_t) \right] \hat{x}_t - e_t - \left( i_t - i_t^f \right)
\]
\[
= \gamma b_t Var_t^{\theta_t}(J_{t+1})
\]

In equilibrium, \(\beta_2^* = -1\), and
\[
\left( \left( 1 - k_{t+1}^{\theta_t} \right) \beta_1 (a + \delta_{t+1}) + k_{t+1}^{\theta_t} \beta_1 + \beta_2^* \right) (a + \delta_t) = \beta_1^*
\]

\[
\Rightarrow \beta_1^* = \frac{a + \delta_t}{1 - \left[ \left( 1 - k_{t+1}^{\theta_t} \right) (a + \delta_{t+1}) + k_{t+1}^{\theta_t} (a + \delta_t) \right]}
\]
Since $\beta_2^* = -1$ in equilibrium, the first order condition for $\delta_t$ becomes

\[
0 = \frac{\partial \Gamma}{\partial \delta_t} = - \left[ \gamma b_t^2 \text{Var}_{t}^{\theta_t} (J_{t+1}) + \frac{(\gamma b_t)^2}{2} \left( k_{t+1}^{\theta_{t+1}} \beta_1^* + \beta_2^* \right) \left( 2 (a + \delta_t) k_t + (a + \delta_t)^2 \frac{dk_t}{d\delta_t} \right) \right] \times \exp \left( -\gamma b_t \text{Var}_{t}^{\theta_t} (J_{t+1}) + \frac{(\gamma b_t)^2}{2} \text{Var}_{t}^{\theta_t} (J_{t+1}) \right) + \frac{\lambda E_t^{\theta_t} x_t^2}{\sigma_w^2} \delta_t.
\]

This condition implies that $\delta_t^* > 0$. Since the gain $k_t$ is an increasing function of $\delta_t$, we conclude that $k_t^{\theta_t} > k^{\theta'}$.

**Auxiliary Results**

1. **Derivation of Equation (3.5).** Let $Z_{t+1} = \left[ \exp(i_t) - \frac{E_{t+1}}{E_t} \exp(i_t^f) \right] b_t$ and $e_t := \log(E_t)$. The Taylor expansion around zero is given by

\[
Z_{t+1} = b_t \left[ \exp(i_t) - \exp(e_{t+1} - e_t + i_t^f) \right] = b_t \left[ (1 + i_t + o_2(2)) - (1 + e_{t+1} - e_t + i_t^f + o_t(2)) \right] = b_t \left[ -(e_{t+1} - e_t) + i_t - i_t^f + o(2) \right],
\]

where $o(2) = o_1(2) + o_2(2)$ and

\[
io_1(2) = \frac{1}{2} \exp(\xi_1) \left( e_{t+1} - e_t + i_t^f \right)^2, \quad o_2(2) = \frac{1}{2} \exp(\xi_2) i_t^2,
\]

\[
\xi_1 \in \left( 0, e_{t+1} - e_t + i_t^f \right), \quad \xi_2 \in (0, i_t)
\]

Clearly, $\lim_{x_i \to 0} \frac{o_i(x_i)}{x_i} = 0$, for $i = 1, 2$ where $x_1 = e_{t+1} - e_t + i_t^f$ and $x_2 = i_t$. Thus the terms $o_1(2)$ and $o_2(2)$ are approximately zero if $e_{t+1} - e_t + i_t^f$ and $i_t$ are small.

2. To show that if we let $\gamma$ go to zero, the primitive utility function becomes risk neutral consider the following monotonic transformation of the primitive utility function: $-\frac{1}{\gamma} E^{\theta'} \left( \exp(-\gamma W_{t+1}) - 1 \right)$.

**Lemma 8.3.** $\lim_{\gamma \to 0} -\frac{1}{\gamma} E^{\theta} \left( \exp(-\gamma W_{t+1}) - 1 \right) = E^{\theta}(W_{t+1})$.

**Proof.** Applying L’Hopital’s rule, we have

\[
\lim_{\gamma \to 0} E^{\theta} \left[ -\frac{1}{\gamma} \left( \exp(-\gamma W_{t+1}) - 1 \right) + \lambda \cdot \mathcal{R}(\theta||\theta') \right] = E^{\theta} \left[ \lim_{\gamma \to 0} W_{t+1} \exp(-\gamma W_{t+1}) + \lambda \cdot \mathcal{R}(\theta||\theta') \right] = E^{\theta} \left[ W_{t+1} + \lambda \cdot \mathcal{R}(\theta||\theta') \right].
\]
3. We have defined the utility function in terms of the log excess return \( W_{t+1} = b_t \left[ (i_t - i_t^f) - (e_{t+1} - e_t) \right] \). The following Lemma shows that for a given domestic interest rate \( i_t \), \( u_1 = E_t^\theta \left[ - \exp \left( -\gamma W_{t+1} \right) \right] \) is a monotonic transformation of \( u_2 = E_t^\theta \left[ - \exp \left( -\gamma b_t \left( \exp (i_t) - \frac{E_{t+1}}{E_t} \exp (i_t^f) \right) \right) \right] \).

**Lemma 8.4.** For a fixed \( i_t \), \( u_1 = E_t^\theta \left\{ - \exp \left[ -\gamma b_t \left( \exp (i_t) - \frac{E_{t+1}}{E_t} \exp (i_t^f) \right) \right] \right\} \) is a monotonic transformation of \( u_2 = E_t^\theta \left\{ - \exp \left[ -\gamma b_t \left( i_t - i_t^f - e_{t+1} + e_t \right) \right] \right\} \).

**Proof.** Suppose that under utility function \( u_1 \)

\[
(b_{t,1}, E_{t+1,1}, E_{t,1}, i_{t,1}^f) \succ (b_{t,2}, E_{t+1,2}, E_{t,2}, i_{t,2}^f)
\]

\[
\Leftrightarrow
u_1 = E_t^\theta \left\{ - \exp \left[ -\gamma b_t \left( \exp (i_t) - \frac{E_{t+1,1}}{E_{t,1}} \exp (i_{t,1}^f) \right) \right] \right\}
\]

\[
\geq u_2 = E_t^\theta \left\{ - \exp \left[ -\gamma b_t \left( \exp (i_t) - \frac{E_{t+1,2}}{E_{t,2}} \exp (i_{t,2}^f) \right) \right] \right\}
\]

for any measure \( \theta \).

\[
\Leftrightarrow \exp (i_t) - \frac{E_{t+1,1}}{E_{t,1}} \exp (i_{t,1}^f) \geq \exp (i_t) - \frac{E_{t+1,2}}{E_{t,2}} \exp (i_{t,2}^f)
\]

\[
\Leftrightarrow \text{dividing both sides by } \exp (i_t)
\]

\[
1 - \frac{E_{t+1,1}}{E_{t,1}} \exp (i_{t,1}^f - i_t) \geq 1 - \frac{E_{t+1,2}}{E_{t,2}} \exp (i_{t,2}^f - i_t)
\]

\[
\Leftrightarrow \exp \left( i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1} \right) \geq \exp \left( i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2} \right)
\]

\[
\Leftrightarrow i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1} \geq i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2}
\]

\[
\Leftrightarrow - \exp \left[ -\gamma b_t \left( i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1} \right) \right] \geq - \exp \left[ -\gamma b_t \left( i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2} \right) \right]
\]

\[
\tilde{u}_1 \equiv E_t^\theta \left\{ - \exp \left[ -\gamma b_t \left( i_t - i_{t,1}^f - e_{t+1,1} + e_{t,1} \right) \right] \right\}
\]

\[
\geq \tilde{u}_2 \equiv E_t^\theta \left\{ - \exp \left[ -\gamma b_t \left( i_t - i_{t,2}^f - e_{t+1,2} + e_{t,2} \right) \right] \right\}
\]

Therefore, \( (E_{t+1,1}, E_{t,1}, i_{t,1}^f) \succeq (E_{t+1,2}, E_{t,2}, i_{t,2}^f) \).