Module 1 Marginal analysis and single variable calculus

1.5 Maximization of a strictly concave function

Solving for the maximizing value of a differentiable function is especially easy if a function is concave. Consider the following problem.

$$\underset{x}{Max}\{f(x) \mid x \in X = [s,t]\} .$$

(a) Strictly concave differentiable function

Suppose that x^* is a critical point of such a function, i.e. $f'(x^*) = 0$. For a strictly concave differentiable function the slope is strictly decreasing on $X = [s,t]$. Therefore $f'(x) > 0$ for all $x < x^*$ and so $f(x)$ is strictly increasing on $[s, x^*)$. Also $f'(x) < 0$ for all $x > x^*$ and so $f(x)$ is strictly decreasing on $(x^*, b]$. Thus the critical point is unique and it is the maximizer. The three possible cases are depicted below

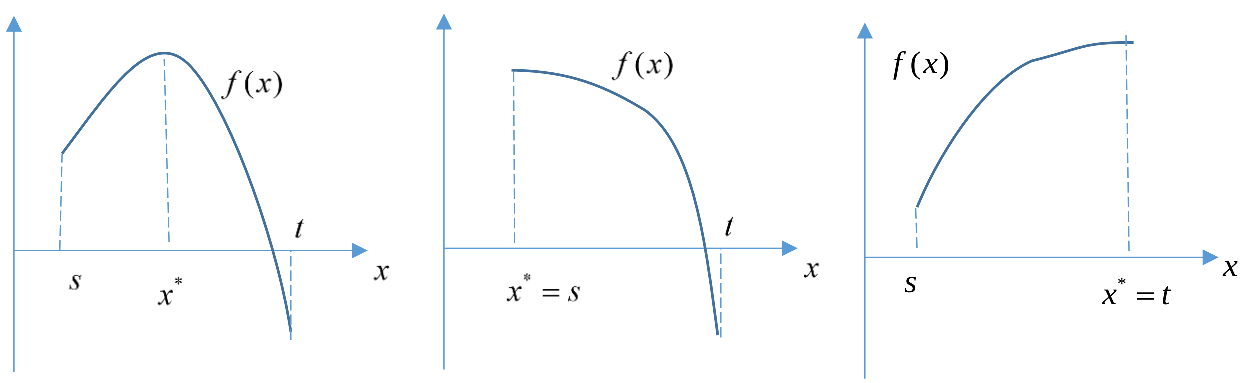
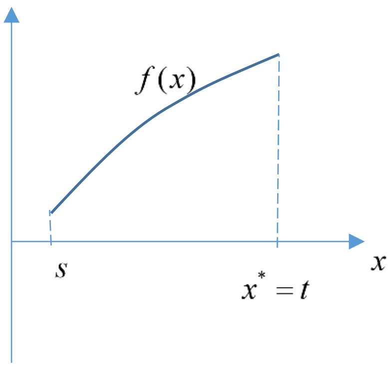


Fig. 1.5-1: Unique critical point is the maximizer.

Suppose next that there are no critical points. If $f'(t) > 0$ then $f'(x) > f'(t) > 0$ for all $x < t$. Then $f(x)$ takes on its maximum at $x^* = t$. Similarly, If $f'(s) < 0$ then $f'(x) < f'(s) < 0$ for all $x > s$. Then $f(x)$ takes on its maximum at $x^* = s$.

These two cases are depicted below.



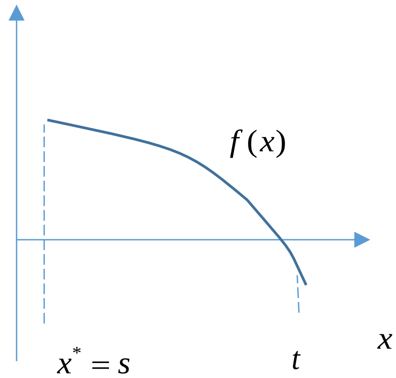


Fig. 1.5-2: Boundary solutions with no critical point

(a) Concave differentiable function

For a concave differentiable function the slope is decreasing but not necessarily strictly decreasing. For example $f(x) = a + 2x$, $g(x) = k$ and $h(x) = c - 3x$ are all concave functions on the interval $X = [s,t]$. Since $g'(x) = 0$, every point in X is a critical point and is a maximizer.

More generally there can be a subset of critical points. Suppose that x^* is the smallest and that x^{**} is the largest as depicted below. Then $f'(x^*) = f'(x^{**}) = 0$. Since the slope is decreasing it follows that the slope is zero everywhere on the interval $X^* = [x^*, x^{**}]$.

Thus the set of maximizers is an interval

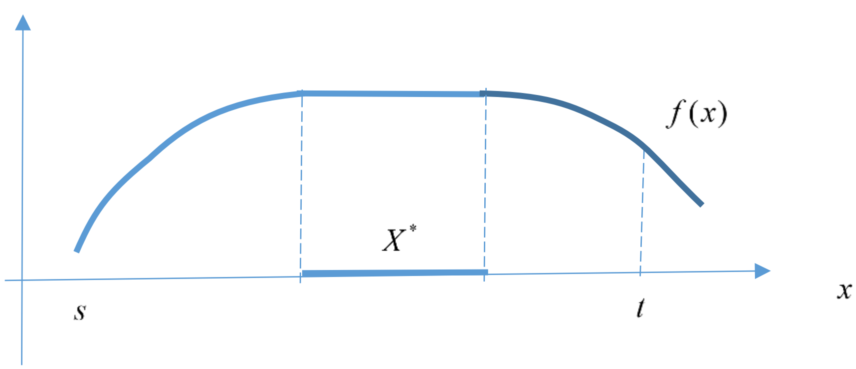


Fig. 1.5-3: The set of solutions is an interval

We can summarize these conclusions as follows:

Proposition: Maximizer of a strictly concave function

Suppose that $f(x)$ is strictly concave and continuously differentiable on $[s,t]$.

(i) If $f(x)$ has a critical point x^* in $X = [s,t]$, it is the unique maximizer for the following problem.

$$Max\{f(x) \mid x \in X = [s,t]\}.$$

(ii) If there are no critical points then either $f'(s) < 0$ and s is the unique maximizer or $f'(t) > 0$ and $x = t$ is the unique maximizer.

Proposition: Maximizer of a concave function

Suppose that $f(x)$ is concave and continuously differentiable on $[s,t]$.

(i) If $f(x)$ has a critical point x^* in $X = [s,t]$, then x^* is a maximizer for the following problem.

$$Max\{f(x) \mid x \in X = [s,t]\}.$$

(ii) If there are no critical points then either $f'(s) \le 0$ and s is the unique maximizer or $f'(t) \ge 0$ and $x = t$ is the unique maximizer.