1. Introduction

Four questions...

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Who are you going to learn most from at UCLA?
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What are the two great pillars of economic theory?

Who are you going to learn most from at UCLA?

What do economists do?

Discuss in 3 person groups
2. Profit-maximizing firm

An example

Cost function

\[ C(q) = 10q + q^2 \]

Demand price function

\[ p = 65 - \frac{1}{4}q \]

**Group exercise:** Solve for the profit maximizing output and price.
Two products

MODEL 1

Cost function

\[ C(q) = 10q_1 + 15q_2 + 2q_1^2 + 3q_1q_2 + 2q_2^2 \]

Demand price functions

\[ p_1 = 85 - \frac{1}{4}q_1 \text{ and } p_2 = 90 - \frac{1}{4}q_2 \]

Group 1 exercise: How might you solve for the profit maximizing outputs?

MODEL 2

Cost function

\[ C(q) = 10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2 \]

Demand price functions

\[ p_1 = 65 - \frac{1}{4}q_1 \text{ and } p_2 = 70 - \frac{1}{4}q_2 \]

Group 2 exercise: How might you solve for the profit maximizing outputs?
MODEL 1:

Revenue

\[ R_1 = p_1 q_1 = (85 - \frac{1}{4} q_1)q_1 = 85q_1 - \frac{1}{4} q_1^2, \quad R_2 = p_2 q_2 = (90 - \frac{1}{4} q_2)q_2 = 90q_2 - \frac{1}{4} q_2^2 \]

Profit

\[ \pi = R_1 + R_2 - C \]

\[ = 85q_1 - \frac{1}{4} q_1^2 + 90q_2 - \frac{1}{4} q_2^2 - (10q_1 + 15q_2 + 2q_1^2 + 3q_1q_2 + 2q_2^2) \]

\[ = 75q_1 + 75q_2 - \frac{9}{4} q_1^2 - \frac{9}{4} q_2^2 - 3q_1q_2 \]
Think on the margin

Marginal profit of increasing $q_1$

$$\frac{\partial \pi}{\partial q_1} = 75 - \frac{9}{2} q_1 - 3q_2 .$$

Therefore the profit-maximizing choice is

$$q_1 = m_1(q_2) = \frac{2}{3} (75 - 3q_2) = \frac{2}{3} (25 - q_2) .$$

$$\pi = 75q_1 + 75q_2 - \frac{9}{4} q_1^2 - \frac{9}{4} q_2^2 - 3q_1q_2$$

Marginal profit of increasing $q_2$

$$\frac{\partial \pi}{\partial q_2} = 75 - 3q_1 - \frac{9}{2} q_2 .$$

Therefore the profit-maximizing choice is

$$q_2 = m_2(q_1) = \frac{2}{3} (75 - 3q_1) = \frac{2}{3} (25 - q_1) .$$

The two profit-maximizing lines are depicted.
\[ q_1 = m_1(q_2) = \frac{2}{3} (25 - q_2), \quad q_2 = m_2(q_1) = \frac{2}{3} (25 - q_1) \]

If you solve for \( \bar{q} \) satisfying both equations you will find that the unique solution is \( \bar{q} = (\bar{q}_1, \bar{q}_2) = (10,10) \).
MODEL 2

Cost function

\[ C(q) = 10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2 \]

Demand price functions

\[ p_1 = 65 - \frac{1}{4}q_1 \text{ and } p_2 = 70 - \frac{1}{4}q_2 \]

Revenue

\[ R_1 = p_1q_1 = (65 - \frac{1}{4}q_1)q_1 = 65q_1 - \frac{1}{4}q_1^2, \quad R_2 = p_2q_2 = (70 - \frac{1}{4}q_2)q_2 = 70q_2 - \frac{1}{4}q_2^2 \]

Profit

\[ \pi = R_1 + R_2 - C \]
\[ = 65q_1 - \frac{1}{4}q_1^2 + 70q_2 - \frac{1}{4}q_2^2 - (10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2) \]
\[ = 55q_1 + 55q_2 - \frac{5}{4}q_1^2 - \frac{5}{4}q_2^2 - 3q_1q_2 \]
Think on the margin

Marginal profit of increasing $q_1$

$$\frac{\partial \pi}{\partial q_1} = 55 - \frac{5}{2} q_1 - 3q_2 \cdot$$

Therefore, for any $q_2$ the profit-maximizing $q_1$ is

$$q_1 = m_1(q_2) = \frac{2}{5} (55 - 3q_2) \cdot$$

Marginal profit of increasing $q_2$

$$\frac{\partial \pi}{\partial q_2} = 55 - 3q_1 - \frac{5}{2} q_2 \cdot$$

Therefore, for any $q_1$ the profit-maximizing $q_2$ is

$$q_2 = m_2(q_1) = \frac{2}{5} (55 - 3q_1)$$

The two profit-maximizing lines are depicted.

If you solve for $\bar{q}$ satisfying both equations you will find that the unique solution is

$$\bar{q} = (\bar{q}_1, \bar{q}_2) = (10, 10) \cdot$$
These look very similar to the profit-maximizing lines in Model 1. However now the profit-maximizing line for $q_2$ is steeper (i.e. has a more negative slope).

As we shall see, this makes a critical difference.
Is $\bar{q}$ the profit-maximizing output bundle (output vector)?

**Group Exercise:** For model 2 solve for maximized profit if only one commodity is produced.

Compare this with the profit if $\bar{q} = (10,10)$ is produced.
Model 1

Suppose we alternate, first maximizing with respect to $q_1$, then $q_2$ and so on.

There are four zones.

$Z(+,+)$:

The zone in which $q_1$ is increasing and $q_2$ is increasing

$Z(+,-)$:

The zone in which $q_1$ is increasing and $q_2$ is decreasing

and so on...

If you pick any starting point you will find this process leads to the intersection point $\bar{q} = (10,10)$. 
The profit is depicted below (using a spread-sheet)
MODEL 2

Suppose we alternate,

first maximizing with respect to $q_1$, then $q_2$ and so on.

There are four zones.

$Z(+, +)$:

The zone in which $q_1$ is increasing and $q_2$ is increasing

$Z(+, -)$:

The zone in which $q_1$ is increasing and $q_2$ is decreasing

and so on...

If you pick any starting point you will find this process

never leads to the intersection point $\bar{q} = (10, 10)$. 
The profit function has the shape of a saddle. The output vector $\vec{q}$ where the slope in the direction of each axis is zero is called a saddle-point.
3. General results

Consider the two variable problem

$$\max_{q} f(q_1, q_2)$$

Necessary conditions

Consider any $\bar{q} \gg 0$. If the slope in the cross section parallel to the $q_1$-axis is not zero, then by standard one variable analysis, the function is not maximized. The same holds for the cross section parallel to the $q_2$-axis. Thus for $\bar{q}$ to be a maximizer, the slope of both cross sections must be zero.

First order necessary conditions for a maximum

For $\bar{q} \gg 0$ to be a maximizer the following two conditions must hold

$$\frac{\partial f}{\partial q_1}(\bar{q}) = 0 \quad \text{and} \quad \frac{\partial f}{\partial q_2}(\bar{q}) = 0 \quad (3-1)$$
Suppose that the first order necessary conditions hold at $\bar{q}$. Also, if the slope of the cross section parallel to the $q_i$-axis is strictly increasing in $q_i$ at $\bar{q}$, then $\bar{q}_i$ is not a maximizer. Thus a necessary condition for a maximum is that the slope must be decreasing. Exactly the same argument holds for $\bar{q}_2$.

We therefore have a second set of necessary conditions for a maximum. Since they are restrictions on second derivatives they are called the second order conditions.

**Second order necessary conditions for a maximum**

If $\bar{q} >> 0$ is a maximizer of $f(q)$, then

$$\frac{\partial}{\partial q_i} \frac{\partial f}{\partial q_1}(\bar{q}) \leq 0 \text{ and } \frac{\partial}{\partial q_2} \frac{\partial f}{\partial q_2}(\bar{q}) \leq 0$$  \hspace{1cm} (3-2)
As we have seen, these conditions are necessary for a maximum but they do not, by themselves guarantee that $\bar{q}$ satisfying these conditions is the maximum.

However, if the step by step approach does lead to $\bar{q}$ then this point is a least a local maximizer.

**Proposition: Sufficient conditions for a local maximum**

If in the neighborhood of $\bar{q} \gg 0$ satisfying the first order conditions (FOC) the step by step approach leads to $\bar{q}$, then the function $f(q)$ has a local maximum at $\bar{q}$

**Proposition: Sufficient conditions for a global maximum**

If the conditions above hold at $\bar{q}$ and the FOC hold only at $\bar{q}$, then this is the global maximizer.

If there is a unique $\bar{q}$ it is the global maximizer.
4. Model of a private ownership economy

Commodities:

The set of commodities is $\mathcal{N} = \{1,\ldots,n\}$

Endowments:

$\omega_j \geq 0$ the initial endowment of commodity $j$ (land, labor, coconuts...)

Consumers:

The set of consumers is $\mathcal{H} = \{1,\ldots,H\}$.

Each consumer’s preferences can be represented by a continuously differentiable strictly increasing function $U(x^h)$, $h = 1,\ldots,H$ where $x^h = (x_1^h,\ldots,x_n^h)$

Notation: If $\bar{x}_i \geq x_i$ for $i = 1,\ldots,n$ and the inequality is strict for some $i$ we write $\bar{x} > x$.

Then utility is increasing if $\bar{x} \geq x$ implies that $U^h(\bar{x}) \geq U^h(x)$. 
Firms

The set of firms is \( F = \{1, \ldots, F\} \).

Transformers of inputs into outputs:

Let \( z^f = (z_1^f, \ldots, z_n^f) \) be a vector of firm \( f \)'s inputs.

Each component is a quantity of one of the commodities.

Let \( q^f \) be a vector of the outputs.

Production vector

\( y^f \equiv (z^f, q^f) \) is an ordered list of all the inputs and outputs.

Each firm must choose among the production vectors that are feasible.

We write this set of feasible plans as \( Y^f \).

Private ownership:

Shareholdings: \( k_{hf} \) is the share-holding of consumer \( h \) in firm \( f \).
Feasible consumption vectors

\[
\sum_{h=1}^{h} x_j^h \leq \sum_{h=1}^{h} \omega_j^h - \sum_{f=1}^{F} z_j^f + \sum_{f=1}^{F} q_j^f , \quad \text{where } y^f \equiv (z^f, q^f) \in Y^f \quad \text{(production plans are feasible)}
\]

total consumption \leq total endowment - total inputs + total outputs

Price taking equilibrium allocation

An allocation to consumers, \( \{ \bar{x}^h \}_{h=1}^{H} \), feasible production plans \( \{ \bar{y}^f \}_{f=1}^{F} \) and a price vector \( p \geq 0 \)

Such that

(i) No consumer has a strictly preferred point in his/her budget set.

(ii) There is no strictly more profitable feasible plan for any firm.

(iii) All markets clear.

\[
\sum_{h=1}^{h} \bar{x}_j^h \leq \sum_{h=1}^{h} \omega_j^h - \sum_{f=1}^{F} \bar{z}_j^f + \sum_{f=1}^{F} \bar{q}_j^f \quad \text{and } p_j = 0 \text{ if the inequality is strict.}
\]
We will look at simple economies to gain insights into how commodities are allocated via markets.

For now consider one example.

Alex likes only bananas. Bev likes only coconuts.

Alex has an endowment of 4 bananas and 6 coconuts \( \omega^A = (4,6) \).

Bev has an endowment of 10 bananas and 7 coconuts \( \omega^B = (10,7) \).

**Group exercise:**

What are market clearing prices in this economy?
5. Consumers

We begin by considering a consumer maximization problem.

\[
\max_{x \geq 0} \{ U^h(x) \mid p_1 x_1 + ... + p_n x_n \leq I \}
\]

Using the sumproduct notation \( a \cdot b = a_1 b_1 + ... + a_n b_n \), we can write this as follows:

\[
\max_{x \geq 0} \{ U^h(x) \mid p \cdot x \leq I \}
\]

We will assume that \( U(x^h) \) is a strictly increasing function.

This is an example of a maximization with a single linear resource constraint.

As you will see by looking at sections A-C in the following slides, an almost identical argument holds for any maximization problem with resource constraints.

\[ \max_{x \geq 0} \{U^h(x) \mid p \cdot x \leq I \} . \]

Let \( \bar{x} \) be a solution. Note that, since \( U(x) \) is strictly increasing, \( p \cdot \bar{x} = I \)

Example with 2 commodities: \( \max_{x \geq 0} \{U(x) = x_1^{\alpha_1} x_2^{\alpha_2} \mid p_1 x_1 + p_2 x_2 \leq I \} \)

Note that utility is zero if consumption of either commodity is zero. Therefore every component of the solution \( \bar{x} \) is strictly positive. (We write \( \bar{x} >> 0 \) ).
Geometry

\[ \max_{x \geq 0} \{ U(x) = x_1^{\alpha_1} x_2^{\alpha_2} \mid p_1 x_1 + p_2 x_2 \leq I \} \]

In the 2 commodity case we can represent preferences by depicting points for which utility has the same value. Such a set of points is called a level set. In the figure the level sets are the boundaries of the blue, red and green shaded regions.

Note that utility is zero if consumption of either commodity is zero. Therefore every component of the solution \( \bar{x} \) is strictly positive. (We write \( \bar{x} >> 0 \)).
Example with 2 commodities: \( \max_{x \geq 0} \{ U(x) = x_1^{\alpha_1} x_2^{\alpha_2} \mid p_1 x_1 + p_2 x_2 \leq I \} \)

**Level sets**

In mathematical notation the 4 level sets are

\[
\{ x \mid U(x) = 0 \}, \{ x \mid U(x) = 1 \}, \{ x \mid U(x) = 2 \}, \{ x \mid U(x) = 3 \}. 
\]

Three of them are what economists often call indifference curves.

**Superlevel sets**

The set of points on or above a level set

\[
\{ x \mid U(x) \geq k \}
\]

is called a superlevel set.
The set of points satisfying

$$p_1 x_1 + p_2 x_2 = I$$

Can be rewritten as

$$x_2 = \frac{I}{p_2} - \frac{p_1}{p_2} x_1$$

This is a line of slope

$$-\frac{p_1}{p_2}$$

In the figure both components of the solution

$$\bar{x} = (\bar{x}_1, \bar{x}_2)$$

are strictly positive.

(Mathematical shorthand $\bar{x} \gg 0$.)

So the slope of the budget line is

equal to the slope of the level set.
Necessary conditions for a maximum

To a first approximation, if a consumer currently, choosing $\bar{x}$ can increase consumption of commodity $j$ by $\Delta x_j$, the change in utility is

$$\Delta U = \frac{\partial U}{\partial x_j}(\bar{x}) \Delta x_j .$$

This is depicted in the figure.

The slope of the tangent line at $\bar{x}$ is

$$\frac{\partial U}{\partial x_j}(\bar{x}) .$$

Fig. 4-1: Utility as a function of $x_j$
Necessary conditions for a maximum

To a first approximation, if a consumer currently, choosing $\bar{x}$ can increase consumption of commodity $j$ by $\Delta x_j$, the change in utility is

$$\Delta U = \frac{\partial U}{\partial x_j}(\bar{x})\Delta x_j .$$

This is depicted in the figure.

The slope of the tangent line at $\bar{x}$ is

$$\frac{\partial U}{\partial x_j}(\bar{x}) .$$

If the consumer has an additional $\Delta E$ dollars then $\Delta E = p_j \Delta x_j$ and so $\Delta x_j = \frac{\Delta E}{p_j}$. 
**Necessary conditions for a maximum**

To a first approximation, if a consumer currently, choosing $\bar{x}$ can increase consumption of commodity $j$ by $\Delta x_j$, the change in utility is

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This is depicted in the figure.

The slope of the tangent line at $\bar{x}$ is

$$\frac{\partial U}{\partial x_j}(\bar{x}).$$

If the consumer has an additional $\Delta E$ dollars then $\Delta E = p_j \Delta x_j$ and so $\Delta x_j = \frac{\Delta E}{p_j}$. 

The increase in utility is therefore

$$\Delta U = \frac{\partial U}{\partial x_j}(\bar{x})\Delta x_j = \frac{\partial U}{\partial x_j}(\bar{x}) \frac{\Delta E}{p_j} = \frac{1}{p_j} \frac{\partial U}{\partial x_j}(\bar{x})\Delta E.$$
We have seen that

$$\Delta U = \frac{1}{p_j} \frac{\partial U}{\partial x_j}(\bar{x}) \Delta E$$

Therefore

$$\frac{\Delta U}{\Delta E} = \frac{1}{p_j} \frac{\partial U}{\partial x_j}(\bar{x})$$

In the limit as $\Delta E$ approaches zero, this becomes the rate at which utility rises as expenditure on commodity $j$ rises.

$$\frac{1}{p_j} \frac{\partial U}{\partial x_j}(\bar{x})$$ is the marginal utility per dollar as expenditure on commodity $j$ rises.
Suppose that the consumer spends 1 dollar less on commodity \( j \). His change in utility is

\[-\frac{1}{p_j} \frac{\partial U}{\partial x_j}(\vec{x})\].

He then spends the dollar on commodity \( i \).

The change in utility is \( \frac{1}{p_i} \frac{\partial U}{\partial x_i}(\vec{x}) \). The net change in utility is therefore

\[\frac{1}{p_i} \frac{\partial U}{\partial x_i}(\vec{x}) - \frac{1}{p_j} \frac{\partial U}{\partial x_j}(\vec{x})\]

*
Suppose that the consumer spends 1 dollar less on commodity \( j \). His change in utility is

\[- \frac{1}{p_j} \frac{\partial U}{\partial x_j} (\bar{x}) \].

He then spends the dollar on commodity \( i \).

The change in utility is \( \frac{1}{p_i} \frac{\partial U}{\partial x_i} (\bar{x}) \). The net change in utility is therefore

\[ \frac{1}{p_i} \frac{\partial U}{\partial x_i} (\bar{x}) - \frac{1}{p_j} \frac{\partial U}{\partial x_j} (\bar{x}) \]

Case (i) \( \bar{x}_i, \bar{x}_j > 0 \)

If the change in utility is strictly positive the current utility can be increased by consuming more of commodity \( i \) and less of commodity \( j \). If it is negative, utility can be increased by spending less commodity \( j \) and more on commodity \( i \). Thus a necessary condition for \( \bar{x} \) to be utility maximizing is that

\[ \frac{1}{p_i} \frac{\partial U}{\partial x_i} (\bar{x}) = \frac{1}{p_j} \frac{\partial U}{\partial x_j} (\bar{x}) \]
Case (ii) $x_j > x_i = 0$

If the difference in marginal utilities is positive current $U(x)$ can be increased by spending a positive amount on commodity $j$. Thus a necessary condition for $x$ to be utility maximizing is that

$$\frac{1}{p_i} \frac{\partial U}{\partial x_i}(x) \leq \frac{1}{p_j} \frac{\partial U}{\partial x_j}(x)$$
Case (ii) \( \bar{x}_j > \bar{x}_i = 0 \)

If the difference in marginal utilities is positive current \( U(\bar{x}) \) can be increased by spending a positive amount on commodity \( j \). Thus a necessary condition for \( \bar{x} \) to be utility maximizing is that

\[
\frac{1}{p_i} \frac{\partial U}{\partial x_i}(\bar{x}) \leq \frac{1}{p_j} \frac{\partial U}{\partial x_j}(\bar{x})
\]

Let \( \lambda \) be the common marginal utility per dollar for all those commodities that are consumed in strictly positive amounts. We can therefore summarize the necessary conditions as follows:

**Necessary conditions for a maximum**

If \( \bar{x}_j > 0 \) then \( \frac{1}{p_j} \frac{\partial U}{\partial x_j}(\bar{x}) = \lambda \)

If \( \bar{x}_j = 0 \) then \( \frac{1}{p_j} \frac{\partial U}{\partial x_j}(\bar{x}) \leq \lambda \)

Note: Since \( \lambda \) is the rate at which utility rises with income it is called the marginal utility of income
For the general problem,

\[ \max_{x} \{ f(x) \mid h(x) = b - g(x) \geq 0 \} \]

\[ \Delta b = \frac{\partial g}{\partial x_j} \Delta x_j \] is the extra resources needed to increase commodity \( j \) by \( \Delta x_j \).

Therefore \( \Delta x_j = \frac{1}{\frac{\partial g}{\partial x_j}} \Delta b \) is the extra output you can get using an extra \( \Delta b \) of the resource.

An almost identical argument then yields the following necessary conditions.

If \( \bar{x}_j > 0 \) then \( \frac{\partial f}{\partial g} (\bar{x}) = \lambda \). If \( \bar{x}_j = 0 \) then \( \frac{\partial f}{\partial g} (\bar{x}) \leq \lambda \).
Use the Lagrangian to write down the necessary conditions

Problem

\[ \max_{x} \{ f(x) | h(x) = b - g(x) \geq 0 \} \]

The Lagrangian for the maximization problem is defined as follows:

\[ \mathcal{L}(x, \lambda) = f(x) + \lambda h(x) = f(x) + \lambda (b - g(x)) , \text{ where } \lambda \geq 0 \]

**
Use the Lagrangian to write down the necessary conditions

Problem

\[ \max_{x} \{ f(x) \mid h(x) = b - g(x) \geq 0 \} \]

The Lagrangian for the maximization problem is defined as follows:

\[ \mathcal{L}(x, \lambda) = f(x) + \lambda h(x) = f(x) + \lambda (b - g(x)) \], where \( \lambda \geq 0 \)

Necessary conditions for \( \bar{x} \) to be a solution to this maximization problem.

\[ \frac{\partial \mathcal{L}}{\partial x_j} (x, \lambda) = \frac{\partial f}{\partial x_j} + \lambda \frac{\partial h}{\partial x_j} (x) = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} (x) \leq 0, \text{ with equality if } \bar{x}_j > 0. \ j = 1, \ldots, n \quad (1) \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda} (x, \lambda) = h(\bar{x}) = b - g(\bar{x}) \geq 0, \text{ with equality if } \lambda > 0. \quad (2) \]
Use the Lagrangian to write down the necessary conditions

**Problem**

\[
\max_x \{ f(x) \mid h(x) = b - g(x) \geq 0 \}
\]

The Lagrangian for the maximization problem is defined as follows:

\[
\mathcal{L}(x, \lambda) = f(x) + \lambda h(x) = f(x) + \lambda (b - g(x))\, , \text{ where } \lambda \geq 0
\]

Necessary conditions for \( \bar{x} \) to be a solution to this maximization problem.

\[
\frac{\partial \mathcal{L}}{\partial x_j}(x, \lambda) = \frac{\partial f}{\partial x_j} + \lambda \frac{\partial h}{\partial x_j}(x) = \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j}(x) \leq 0\, , \text{ with equality if } \bar{x}_j > 0\, . \, j = 1, \ldots, n \tag{1}
\]

\[
\frac{\partial \mathcal{L}}{\partial \lambda}(x, \lambda) = h(\bar{x}) = b - g(\bar{x}) \geq 0\, , \text{ with equality if } \lambda > 0\, . \tag{2}
\]

From (1), note that

if \( \bar{x}_i \) and \( \bar{x}_j > 0 \), then

\[
\lambda = \frac{\frac{\partial f}{\partial x_j}}{\frac{\partial g}{\partial x_j}} = \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial g}{\partial x_i}} . \quad \text{And if } \bar{x}_j > 0 = \bar{x}_i, \text{ then } \lambda = \frac{\frac{\partial f}{\partial x_j}}{\frac{\partial g}{\partial x_j}} \geq \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial g}{\partial x_i}} .
\]
**An alternative approach**

From the argument above, if both commodity $i$ and commodity $j$ are consumed, then the ratio of their marginal utilities must be equal to the price ratio.

To understand this consider a change in $x_i$ and $x_j$ that leaves the consumer on the same level set. i.e.

$$U(\bar{x}_1 + \Delta x_1, \bar{x}_2 + \Delta x_2) = U(\bar{x}_1, \bar{x}_2)$$
An alternative approach

From the argument above, if both commodity $i$ and commodity $j$ are consumed, then the ratio of their marginal utilities must be equal to the price ratio.

To understand this consider a change in $x_i$ and $x_j$ that leaves the consumer on the same level set. i.e.

$$U(\bar{x}_1 + \Delta x_1, \bar{x}_2 + \Delta x_2) = U(\bar{x}_1, \bar{x}_2)$$

Above we showed that, to a first approximation,

$$\Delta U = \frac{\partial U}{\partial x_j}(\bar{x})\Delta x_j$$

If we increase the quantity of commodity $j$ and reduce the quantity of commodity $i$, then the net change in utility is

$$\Delta U = \frac{\partial U}{\partial x_j}(\bar{x})\Delta x_j - \frac{\partial U}{\partial x_i}(\bar{x})\Delta x_i$$
We have argued that \( \Delta U = \frac{\partial U}{\partial x_j} (\bar{x}) \Delta x_j - \frac{\partial U}{\partial x_i} (\bar{x}) \Delta x_i \)

For this net change to be zero,

\[ \frac{\Delta x_j}{\Delta x_i} = \frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}} \]
We have argued that \[ \Delta U = \frac{\partial U}{\partial x_j} (\bar{x}) \Delta x_j - \frac{\partial U}{\partial x_i} (\bar{x}) \Delta x_i \]

For this net change to be zero,

\[ \Delta x_j \over \Delta x_i = \frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}} \]

In the figure, \( \frac{\Delta x_2}{\Delta x_1} \) is the slope of the level set at \( \bar{x} \).

The ratio is the rate at which \( x_i \) must be substituted into the consumption bundle to compensate for a reduction in \( x_2 \).

Hence we call it the marginal rate of substitution of \( x_1 \) for \( x_2 \).

**Definition: Marginal rate of substitution**

\[ MRS(x_i, x_j) = \frac{\frac{\partial U}{\partial x_i}}{\frac{\partial U}{\partial x_j}} \]
For \( \bar{x} \) to be the maximizer the rate at which \( x_1 \) can be substituted into the budget as \( x_2 \) is reduced must leave total expenditure on the two commodities constant, i.e.,

\[
p_i \Delta x_i + p_j \Delta x_j = 0
\]

Then along the budget line,

\[
\frac{\Delta x_j}{\Delta x_i} = -\frac{p_i}{p_j}
\]

Graphically, the slope of the budget line must be equal to the slope of the indifference curve at \( \bar{x} \) i.e.,

\[
MRS(\bar{x}_i, \bar{x}_j) = \frac{\partial U}{\partial x_i} \frac{\partial x_i}{\partial U} = \frac{p_i}{p_j}
\]
The example: Cobb-Douglas utility function: 

\[ \text{Max}\{U(x) = x_1^{\alpha_1} x_2^{\alpha_2} \mid p_1 x_1 + p_2 x_2 \leq I \} \]

Necessary conditions for a maximum

**Method 1: Equalize the marginal utility per dollar**

To make differentiation simple, try to find an increasing function of the utility function that is simple.
The example: Cobb-Douglas utility function: 

$$\max_{x \geq 0} \{ U(x) = x_1^{\alpha_1} x_2^{\alpha_2} \mid p_1 x_1 + p_2 x_2 \leq I \}$$

Necessary conditions for a maximum

**Method 1: Equalize the marginal utility per dollar**

To make differentiation simple, try to find an increasing function of the utility function that is simple.

Define the new utility function $u(x) = \ln U(x)$

The new maximization problem is

$$\max_{x \geq 0} \{ u(x) = \ln U(x) \mid p_1 x_1 + p_2 x_2 \leq I \}$$

That is

$$\max_{x \geq 0} \{ \alpha_1 \ln x_1 + \alpha_2 \ln x_2 \mid p_1 x_1 + p_2 x_2 \leq I \}$$

Note that

$$\frac{\partial u}{\partial x_j} = \frac{\alpha_j}{x_j}.$$
Necessary conditions

\[ \frac{1}{p_1} \frac{\partial u}{\partial x_1} = \frac{1}{p_2} \frac{\partial u}{\partial x_2} = \lambda. \]

For the example it follows that \( \frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \lambda. \)

Also \( p_1 x_1 + p_2 x_2 = I. \)

**Technical tip**

**Ratio Rule:**

If \( \frac{a_1}{b_1} = \frac{a_2}{b_2} \) and \( b_1 + b_2 \neq 0 \) then \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}. \)
Necessary conditions

\[ \frac{1}{p_1} \frac{\partial u}{\partial x_1} = \frac{1}{p_2} \frac{\partial u}{\partial x_2} = \lambda. \]
For the example it follows that \[ \frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \lambda. \]

Also \( p_1 x_1 + p_2 x_2 = I \).

**Technical tip**

**Ratio Rule:**

If \( \frac{a_1}{b_1} = \frac{a_2}{b_2} \) and \( b_1 + b_2 \neq 0 \) then \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2} \).

Therefore

\[ \frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \frac{\alpha_1 + \alpha_2}{p_1 x_1 + p_2 x_2} = \frac{\alpha_1 + \alpha_2}{I} \]

*
Necessary conditions

\[ \frac{1}{p_1} \frac{\partial u}{\partial x_1} = \frac{1}{p_2} \frac{\partial u}{\partial x_2} = \lambda. \]  

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**Technical tip**

**Ratio Rule:**

If \( \frac{a_1}{b_1} = \frac{a_2}{b_2} \) and \( b_1 + b_2 \neq 0 \) then \( \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}. \)

Therefore

\[ \frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \frac{\alpha_1 + \alpha_2}{p_1 x_1 + p_2 x_2} = \frac{\alpha_1 + \alpha_2}{I}. \]

We can then solve for \( x_1 \) and \( x_2 \)

**Cobb-Douglas demands**

\[ x_j = \frac{\alpha_j}{\alpha_1 + \alpha_2} \frac{I}{p_j}, \]
Method 2: Equate the MRS and price ratio

\[ U(x) = x_1^{\alpha_1} x_2^{\alpha_2}. \] Then \( \frac{\partial U}{\partial x_1} = \alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} \) and \( \frac{\partial U}{\partial x_2} = \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1} \)

\[ \text{MRS}(x_1, x_2) = \frac{\partial U}{\partial x_1} \frac{\partial x_1}{\partial x_2} = \frac{\alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2}}{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1}} = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1}. \]

Then to be the maximizer,

\[ \text{MRS}(x_1, x_2) = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1} = \frac{p_1}{p_2} \]

As we have seen, it is helpful to rewrite this as follows:

\[ \frac{p_1 x_1}{\alpha_1} = \frac{p_2 x_2}{\alpha_2}. \]

Then proceed as before.
Data Analytics (Taking the model to the data)

\[ x_1(p, I) = \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{I}{p_j}. \]

Take the logarithm

\[ \ln x_j = \ln\left(\frac{\alpha_1}{\alpha_1 + \alpha_2}\right) + \ln I - \ln p_j. \]

The model is now linear. We can then use least squares estimation

\[ \ln x_j = a_0 + a_1 (\ln I - \ln p_j) \]

or

\[ \ln x_j = a_0 + a_1 \ln I + a_2 \ln p_j. \]

Exercise: If \( U(x) = (a_1 + x_1)^{\alpha_1} (a_2 + x_2)^{\alpha_2} \), solve for the demand function \( x_1(p, I) \)
6. Market demand in a Cobb-Douglas economy with no production

Consumer \( h \in \mathcal{H} \) has some initial endowment \( \omega = (\omega_1, \ldots, \omega_n) \). With a price vector \( p \), the market value of this endowment is \( I^h = p \cdot \omega^h \). If the consumer sells his entire endowment he can then purchase any consumption bundle \( x^h \) satisfying

\[
p \cdot x^h \leq I^h = p \cdot \omega^h.
\]

**
6. Market demand in a Cobb-Douglas economy with no production

Consumer \( h \in H \) has some initial endowment \( \omega = (\omega_1, \ldots, \omega_n) \). With a price vector \( p \), the market value of this endowment is \( I^h = p \cdot \omega^h \). If the consumer sells his entire endowment he can then purchase any consumption bundle \( x^h \) satisfying

\[
p \cdot x^h \leq I^h = p \cdot \omega^h.
\]

Consider a 2-commodity economy in which each consumer has the same Cobb-Douglas preferences.

\[
U(x^h) = (x_1^h)^{\alpha_1} (x_2^h)^{\alpha_2}.
\]

*
5. Market demand in a Cobb-Douglas economy with no production

Consumer $h \in \mathcal{H}$ has some initial endowment $\omega = (\omega_1, \ldots, \omega_n)$. With a price vector $p$, the market value of this endowment is $I^h = p \cdot \omega^h$. If the consumer sells his entire endowment he can then purchase any consumption bundle $x^h$ satisfying

$$p \cdot x^h \leq I^h = p \cdot \omega^h.$$ 

Consider a 2-commodity economy in which each consumer has the same Cobb-Douglas preferences.

$$U(x^h) = (x_1^h)^{\alpha_1} (x_2^h)^{\alpha_2}.$$ 

From section 4, demand for commodity 1 is

$$x_1^h(p, I^h) = \theta_1 \frac{I^h}{p_1} \text{ where } \theta_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$
Suppose that all consumers have the same preferences

Then a consumer with a budget of 1 has a consumption of \( x_i(p,1) = \frac{\alpha_1}{\alpha_1 + \alpha_2} \). Therefore consumer \( h \)'s demand is simply scaled up by her income.

\[
x_i^{h}(p,I^{h}) = I^{h}x(p,1)
\]

\[
x_i(p,1) = \frac{\theta_1}{p_1} \text{ where } \theta_1 \equiv \frac{\alpha_1}{\alpha_1 + \alpha_2}
\]

\[
x_i^{h}(p,I^{h}) = I^{h}x(p,1)
\]

Summing over consumers, Cobb-Douglas market demand for two consumers is

\[
x_i(p) = x_i^{1}(p,I^{1}) + x_i^{1}(p,I^{h}) = x_i(p,1)I^{1} + x_i(p,1)I^{2} = x_i(p,1)(I^{1} + I^{2})
\]

For \( H \) consumers, market demand is

\[
x_i(p) = \sum_{h=1}^{H} x_i^{h}(p,I^{h}) = x_i(p,1)I^{1} + ... | x_i(p,1)I^{H} = x_i(p,1)(I^{1} + ... + I^{H})
\]
Consider a very simple exchange economy

No production. On one end of the island there are banana plantations. At the other end there are coconut plantations.

Given a price vector $p$, if consumer $h$ has an endowment of $\omega^h = (\omega_1^h, \omega_2^h)$, this has market value

$$I^h = p \cdot \omega^h = p_1 \omega_1^h + p_2 \omega_2^h.$$ 

She can trade this for any other bundle of bananas and coconuts costing less. So her maximization problem is

$$\max \{ U^h(x^h) \mid p \cdot x^h \leq p \cdot \omega^h = I^h \}.$$ 

The value of all the endowment is $I^1 + \ldots + I^h = p \cdot \omega$ where $\omega = \sum_{h=1}^{H} \omega^h$.

Therefore if all consumers have the same Cobb-Douglas preferences

$$x_1(p) = \frac{\theta_1}{p_1} p \cdot \omega.$$
Equilibrium in an exchange economy with Cobb-Douglas utility \( U^h = (x_1^h)^{\alpha_1}(x_2^h)^{\alpha_2} \)

The total supply of commodity 1 is \( \omega_1 \). The market commodity 1 clears if

\[
x_1(p) - \omega_1 = \frac{\theta_1}{p_1} p \cdot \omega - \omega_1 = \frac{1}{p_1} (\theta_1 p \cdot \omega - p_1 \omega_1) = 0
\]

Solving

\[
p_1 \omega_1 = \theta_1 (p_1 \omega_1 + p_2 \omega_2)
\]

\[
\frac{p_1}{p_2} = \frac{1 - \theta_1 \omega_2}{\theta_1 \omega_1} = \frac{\alpha_1 \omega_2}{\alpha_2 \omega_1}
\]
Additional example: (A bit technical for a closed book exam)

\[ U(x) = -\frac{1}{x_1} - \frac{9}{x_2}, \quad \frac{\partial U}{\partial x_1}(x) = \frac{1}{x_1^2}, \quad \frac{\partial U}{\partial x_2}(x) = \frac{9}{x_2^2}. \]

Solve the budget problem: \( \text{Max} \{ U(x) \mid x_1 + 4x_2 \leq 21 \)
Additional example: (A bit technical for a closed book exam)

\[ U(x) = -\frac{1}{x_1} - \frac{9}{x_2}, \quad \frac{\partial U}{\partial x_1} (x) = \frac{1}{x_1^2}, \quad \frac{\partial U}{\partial x_1} (x) = \frac{9}{x_2^2}. \]

Solve the budget problem: \( \text{Max}\{U(x) \mid x_1 + 4x_2 \leq 21 \} \)

Step 1: \( \bar{x} >> 0 \) (why?)

\[
\frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x_2} = \frac{1}{x_1^2} = \frac{9}{4x_2^2}.
\]

***
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Solve the budget problem: \( \text{Max}\{U(x)\} \mid x_1 + 4x_2 \leq 21 \)

Step 1: If \( \bar{x} \gg 0 \)

\[
\frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x_2} = \frac{1}{x_1^2} = \frac{9}{4x_2^2}.
\]

Step 2: Make the denominators linear in each commodity by taking the square root.

\[
\frac{1}{x_1} = \frac{3}{2x_2}.
\]

**
Additional example: (A bit technical for a closed book exam)

\[ U(x) = -\frac{1}{x_1} - \frac{9}{x_2}, \quad \frac{\partial U}{\partial x_1}(x) = \frac{1}{x_1^2}, \quad \frac{\partial U}{\partial x_2}(x) = \frac{9}{x_2^2}. \]

Solve the budget problem: \( \text{Max}\{U(x) | x_1 + 4x_2 \leq 21 \} \)

Step 1: If \( x \gg 0 \)

\[ \frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x_2} = \frac{1}{x_1^2} = \frac{9}{4x_2^2}. \]

Step 2: Make the denominators linear in \( x_j \) by taking the square root.

\[ \frac{1}{x_1} = \frac{3}{2x_2}. \]

Step 3: Convert the denominators into expenditure and appeal to the Ratio Rule

\[ \frac{1}{x_1} = \frac{6}{4x_2} = \frac{7}{x_1 + 4x_2} = \frac{7}{21}. \]
Additional example: (A bit technical for a closed book exam)

\[ U(x) = -\frac{1}{x_1} - \frac{9}{x_2}, \quad \frac{\partial U}{\partial x_1}(x) = \frac{1}{x_1^2}, \quad \frac{\partial U}{\partial x_2}(x) = \frac{9}{x_2^2}. \]

Solve the budget problem: \( \text{Max}\{U(x)\} \mid x_1 + 4x_2 \leq 21 \)

Step 1: If \( \bar{x} >> 0 \)

\[ \frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x_2} = \frac{1}{x_1^2} = \frac{9}{4x_2^2}. \]

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\[ \frac{1}{x_1} = \frac{6}{4x_2} = \frac{7}{x_1 + 4x_2} = \frac{7}{21}. \]

Step 4: Solve for the demands

\[ \bar{x}_1 = 3 \quad \bar{x}_2 = 4.5 \]