Choice under uncertainty

Part 1

Part 2

Part 3

4. Measure of risk aversion

Remark: Linear transformations of Von Neumann utility functions $v(x)$

Consider $u(x) = k_1 + k_2 v(x)$

$$\mathbb{E}[u(x)] = \pi_1 u(x_1) + \pi_2 u(x_2) = \pi_1(k_1 + k_2 v(x_1)) + \pi_2(k_1 + k_2 v(x_2))$$

$$= k_1 + k_2(\pi_1 v(x_1)) + \pi_2 v(x_2)) = k_1 + k_2 \mathbb{E}[v(x)]$$

Thus the ranking of lotteries is identical under linear transformations

Absolute aversion to risk

The bigger is $ARA(w) \equiv -\dfrac{v''(w)}{v'(w)}$ the bigger is $\alpha = (-\dfrac{v''(w)}{v'(w)})\dfrac{x}{4} = ARA(w)\dfrac{x}{4}$.

Thus an individual with a higher $ARA(w)$ requires the odds of a favorable outcome to be moved more. Thus $ARA(w)$ is a measure of an individual's aversion to risk.

$ARA(w) \equiv$ degree of absolute risk aversion

Examples: $v(x) = 3x^{1/2}$, $v(x) = \ln x$, $v(x) = 6 - 2x^{-1}$

$$ARA(x) = \frac{1}{2x} \qquad = \frac{1}{x} \qquad = \frac{2}{x}$$

Relative risk aversion

Betting on a small percentage of wealth

New risky alternative: $(w_1, w_2; \pi_1, \pi_2) = (\hat{w}(1+\beta), \hat{w}(1-\beta); \frac{1}{2}+\alpha, \frac{1}{2}-\alpha)$.

Choose α so that the consumer is indifferent between gambling and not gambling.

Note that we can rewrite the risky alternative as follows:

$(w_1, w_2; \pi_1, \pi_2) = (\hat{w}+x, \hat{w}-x; \frac{1}{2}+\alpha, \frac{1}{2}-\alpha)$ where $x = \beta\hat{w}$.

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Relative risk aversion

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From our earlier argument,

$$\alpha = \left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right)\frac{x}{4} = \left(-\frac{v''(\hat{w})}{v'(\hat{w})}\right)\frac{\beta\hat{w}}{4} = \left(-\frac{\hat{w}v''(\hat{w})}{v'(\hat{w})}\right)\frac{\beta}{4}.$$

*

Relative risk aversion

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Relative aversion to risk

The bigger is $RRA(w) \equiv -\dfrac{wv''(w)}{v'(w)}$ the bigger is $\alpha = \left(-\dfrac{wv''(w)}{v'(w)}\right)\dfrac{\beta}{4} = RRA(w)\dfrac{\beta}{4}$.

Thus an individual with a higher $RRA(w)$ requires the odds of a favorable outcome to be moved more. Thus $RRA(w)$ is a measure of an individual's aversion to risk.

$RRA(w) \equiv$ degree of relative risk aversion

Remark on estimates of relative risk aversion

$RRA(w) \equiv -\dfrac{wv''(w)}{v'(w)}$. Typical estimate between 1 and 2

Remark on estimates of absolute risk aversion

$$ARA(w) \equiv -\frac{v''(w)}{v'(w)} = \frac{1}{w} RRA(w)$$

Thus ARA is very small for anyone with significant life-time wealth

5. Insurance

A consumer with a wealth $\hat{w}$ has a financial loss of L with probability π_1. We shall call this outcome the "loss state" and label it state 1. With probability $\pi_2 = 1 - \pi_1$ the consumer is in the "no loss state" an label it state 2.

With no exchange the consumer's state contingent wealth is

$(x_1, x_2) = (\hat{w} - L, \hat{w})$.

This consumer wishes to exchange

wealth in state 2 for wealth in state 1.

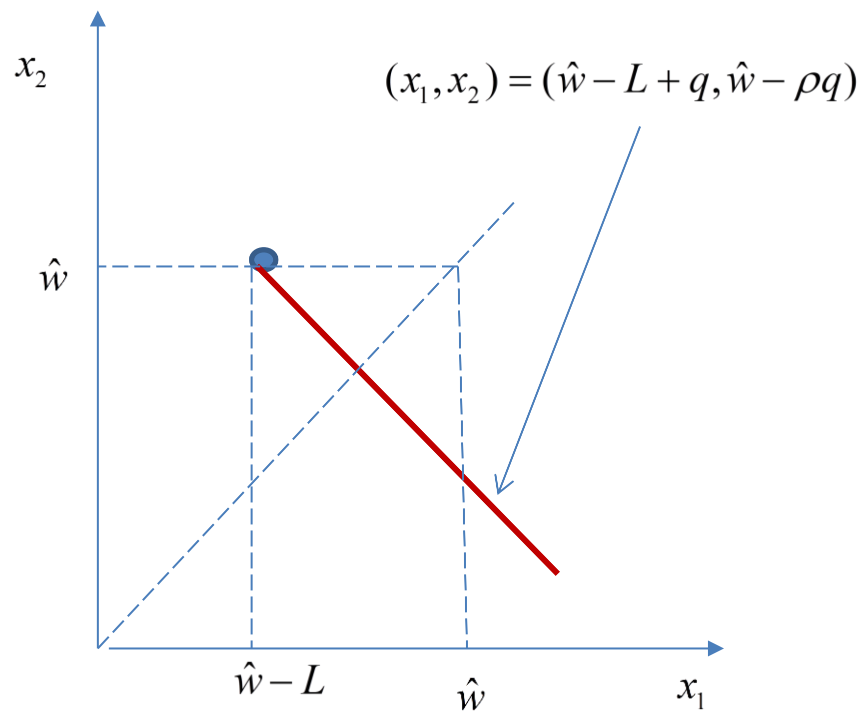
Suppose there is a market in which such an

exchange can take place. For each dollar of

coverage in the loss state, the consumer must pay

ρ dollars in the no loss state.

$$(x_1, x_2) = (\hat{w} - L + q, \hat{w} - \rho q)$$



The steepness of the line is rate at which the consumer

must exchange wealth in state 2 for wealth in state 1

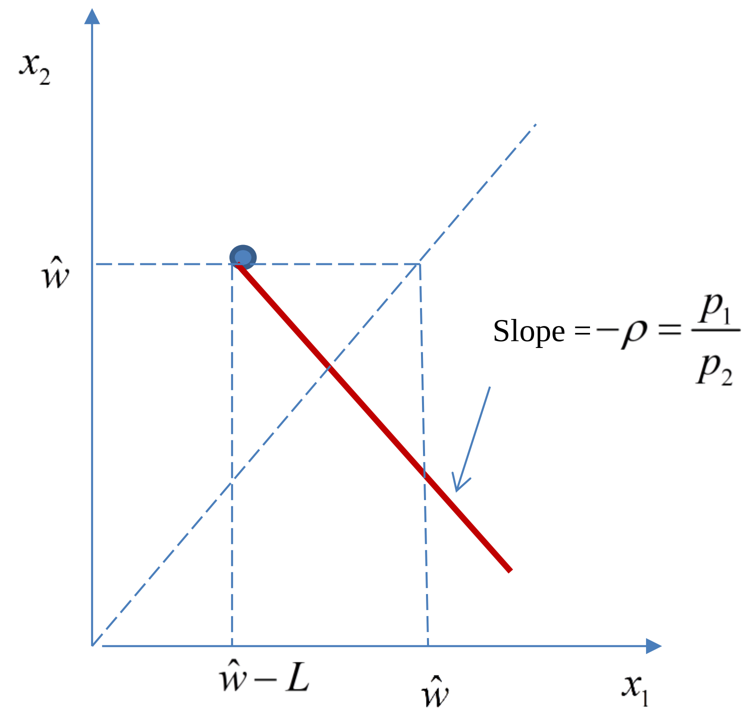
So ρ is a market exchange rate.

Then if there were prices for units of wealth in each state

$$\rho = \frac{p_1}{p_2}$$

Suppose that the consumer purchases q units of

insurance coverage.



$$x_1 = \hat{w} - L + q$$

$$x_2 = \hat{w} - \rho q = \hat{w} - \frac{p_1}{p_2} q$$

$$p_1 x_1 = p_1(\hat{w} - L) + p_1 q$$

$$p_2 x_2 = p_2 \hat{w} - p_1 q$$

Adding these equations,

$$p_1 x_1 + p_2 x_2 = p_1(\hat{w} - L) + p_2 \hat{w}$$

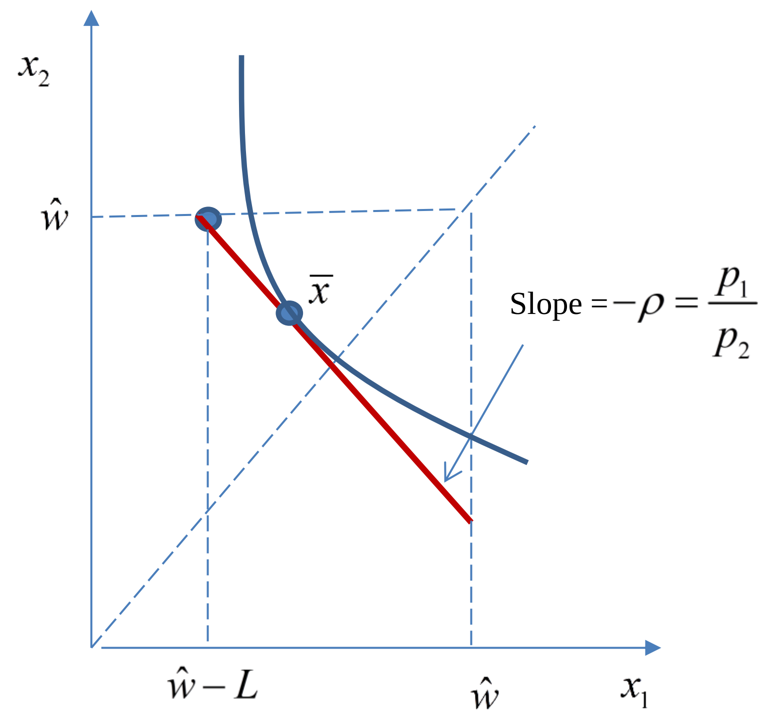
The consumer's expected utility is

$$U = \pi_1 v(x_1) + \pi_2 v(x_2)$$

We have argued that the consumer's choices are constrained to satisfy the following implicit budget constraint

$$p_1 x_1 + p_2 x_2 = p_1(\hat{w} - L) + p_2 \hat{w}.$$

This is the line depicted in the figure.



Implicit budget line though the "endowment"

Group Exercise: What must be the price ratio if the consumer purchases full coverage? (i.e. $\bar{x}_1 = \bar{x}_2$)

6. Sharing the risk on a South Pacific Island

Alex lives on the west end of the island and has 600 coconut palm trees. Bev lives on the East end and has 800 coconut palm trees. If the typhoon approaching the island makes landfall on the west end it will wipe out 400 of Alex's palm trees. If instead the typhoon makes landfall on the East end of the island it will wipe out 400 of Bev's coconut palms. The probability of each event is 0.5.

Let the West end typhoon landfall be state 1 and let the East end landfall be state 2. Then the risk facing Alex is $(200,600;\frac{1}{2},\frac{1}{2})$ while the risk facing Bev is $(800,400;\frac{1}{2},\frac{1}{2})$.

What should they do?

What would be the WE prices if they could trade state "contingent claims" provided by competitive insurance companies (in effect, market makers)?

What would be the WE outcome?

Let $v_h(\cdot)$ be h's VNM utility function so that

h's expected utility is

$$U^h(x^h) = \pi_1 v_h(x_1^h) + \pi_2 v_h(x_2^h).$$

where π_s is the probability of state s .

In state 1 Bev's "endowment " is $\omega_1^B = 800$

In state 2 the endowment is $\omega_2^B = 400$.

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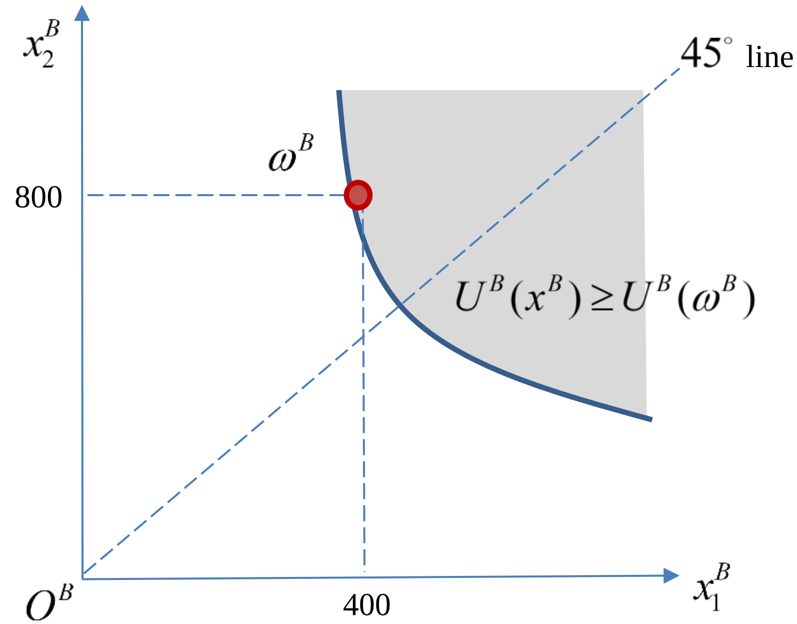
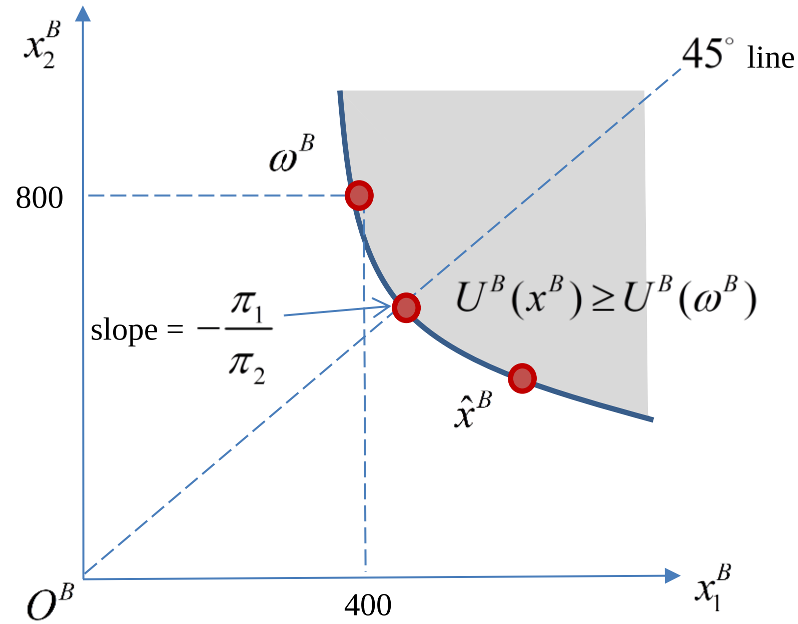




Let $v^h(\cdot)$ be Individual h's utility function so that

h's expected utility is

$$U^h(x^h) = \pi_1 v_h(x_1^h) + \pi_2 v_h(x_2^h).$$

where π_s is the probability of state s .

In state 1 Bev's "endowment " is $\hat{x}^B$

In state 2 the endowment is $\omega_2^B = 400$.

The level set for $U^B(x^B)$ through the

endowment point ω^B is depicted.

At a point $\hat{x}^B$ in the level set the steepness

of the level set is

$$MRS^B(\hat{x}^B) = \frac{MU_1}{MU_2} = \frac{\dfrac{\partial U^B}{\partial x_1^B}}{\dfrac{\partial U^B}{\partial x_2^B}} = \frac{\pi_1 v_B{}'(\hat{x}_1^B)}{\pi_2 v_B{}'(\hat{x}_2^B)} .$$

Note that along the 45° line the MRS is the probability ratio $\dfrac{\pi_1}{\pi_2}$ (equal probabilities so ratio is 1).

The level set for Alex is also depicted.

At each 45° line the steepness of the

Respective sets are both $\dfrac{\pi_1}{\pi_2}$.

Therefore

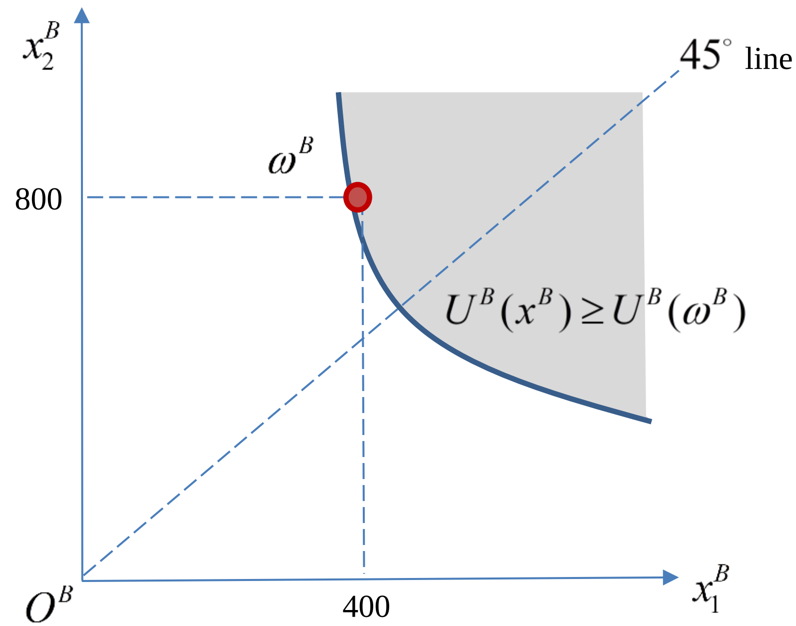
$$MRS^B(\omega^B) > \frac{\pi_1}{\pi_2} > MRS^A(\omega^A)$$

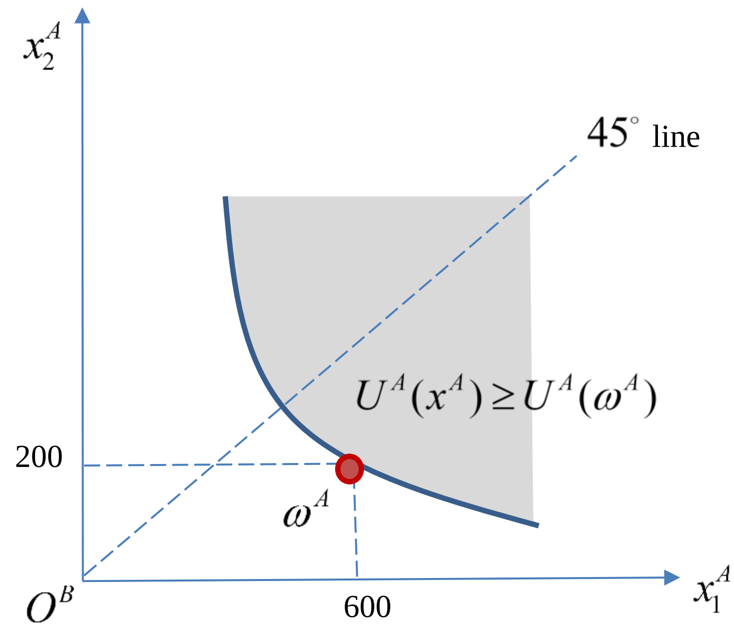
Therefore there are gains to be made from

trading state claims.

The consumers will reject any proposed exchange

that does not lie in their shaded superlevel sets.





Edgeworth Box diagram

Bev will reject any proposed

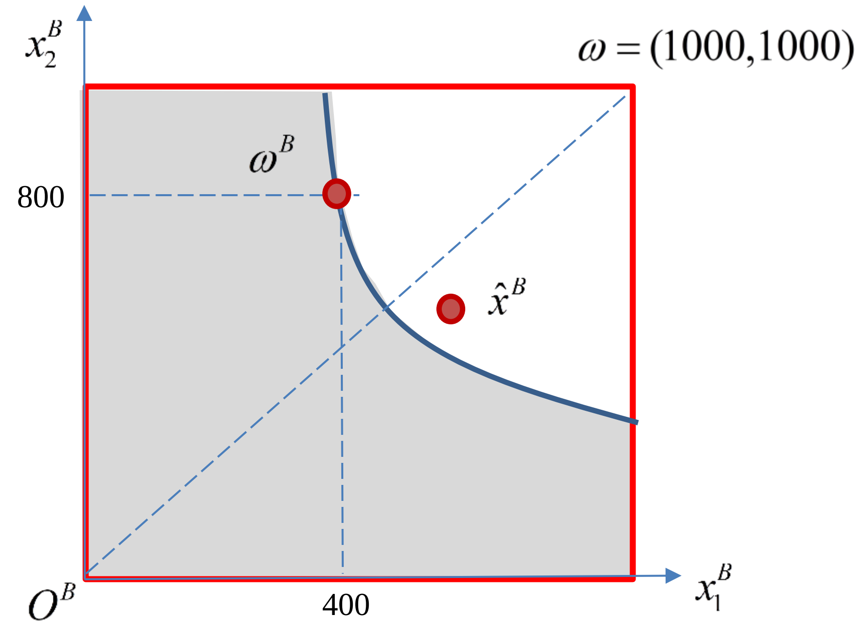
exchange that is in the shaded sublevel set.

Since the total supply of coconut palms is

1000 in each state, the set of potentially

acceptable trades must be the unshaded

region in the square "Edgeworth Box"



The rotated Edgeworth Box

Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$

Also added to the figure is the green level set

for Alex's utility function through ω^A.

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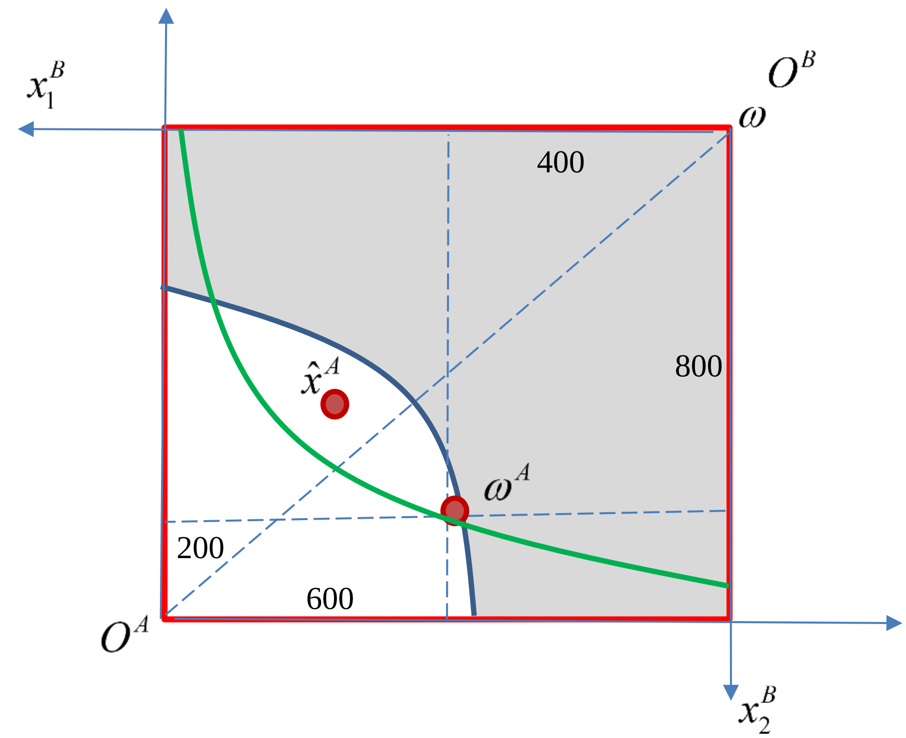

The rotated Edgeworth Box

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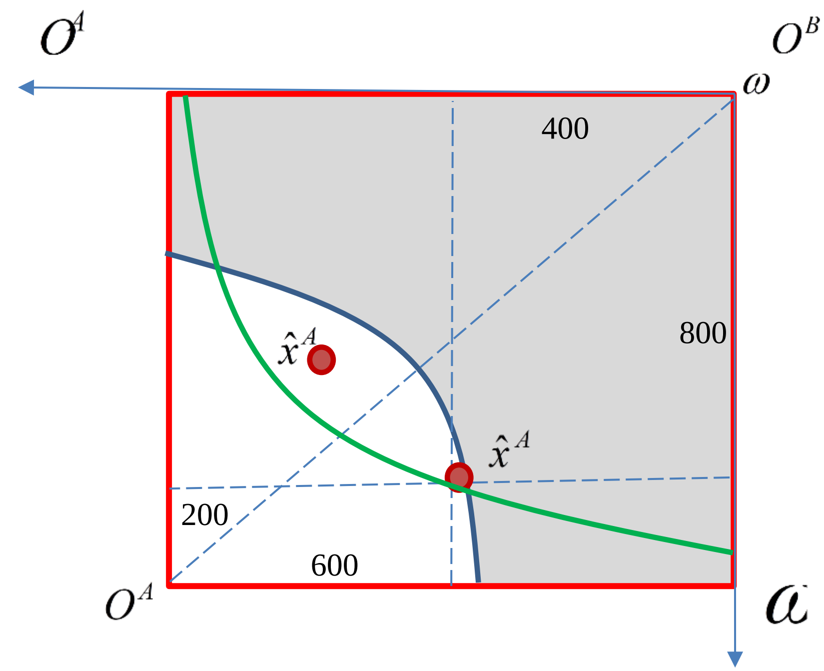

Also added to the figure is the green level set

for Alex's utility function through ω^A.

Any exchange must be preferred by both consumers

over the no trade allocation (the endowments).

Such an exchange must lie in the lens shaped

region to the right of Alex's level set and to the left

of Bev's level set.

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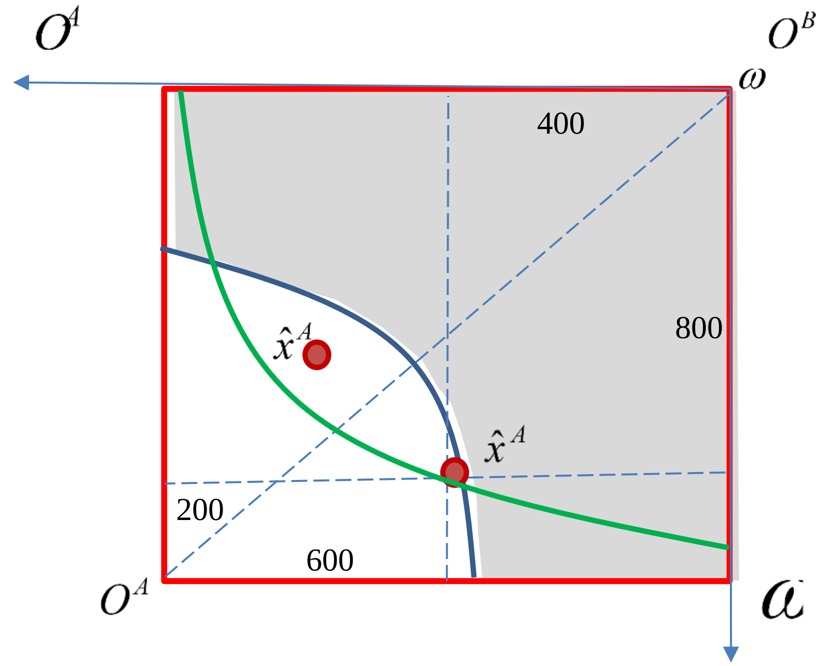
The rotated Edgeworth Box

Note that $\omega^A = \omega - \omega^B$ and $\hat{x}^A = \omega - \hat{x}^B$



Also added to the figure is the green level set

for Alex's utility function through ω^A.

Any exchange must be preferred by both consumers

over the no-exchange allocation (the endowment).

Such an exchange must lie in the lens shaped

region where both are better off.

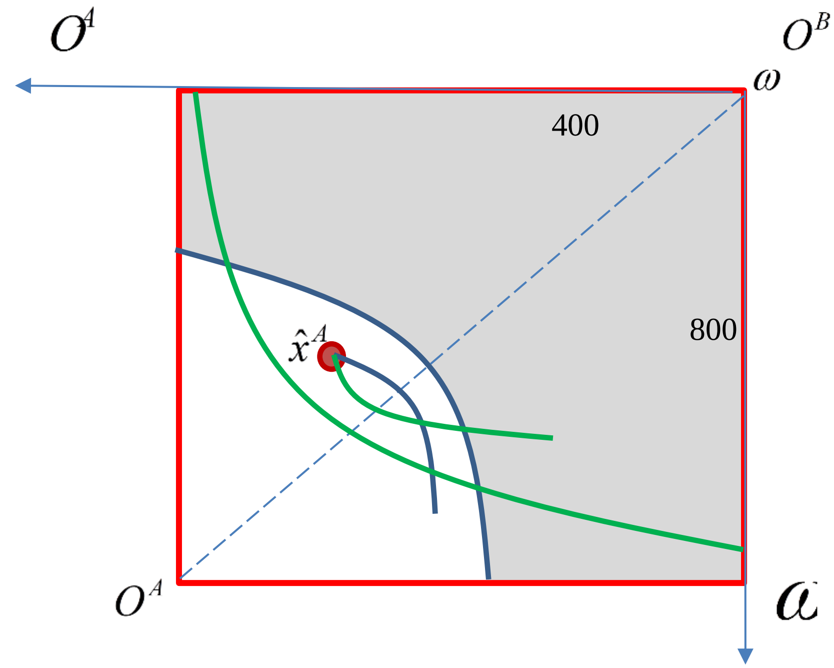
Pareto preferred allocations

If the proposed allocation is weakly preferred by both consumers and strictly preferred by at least one of the two consumers the new allocation is said to be Pareto preferred.

In the figure $\hat{x}^A$ (in the lens shaped region) is Pareto preferred to ω^A since Alex and Bev are both strictly better off.

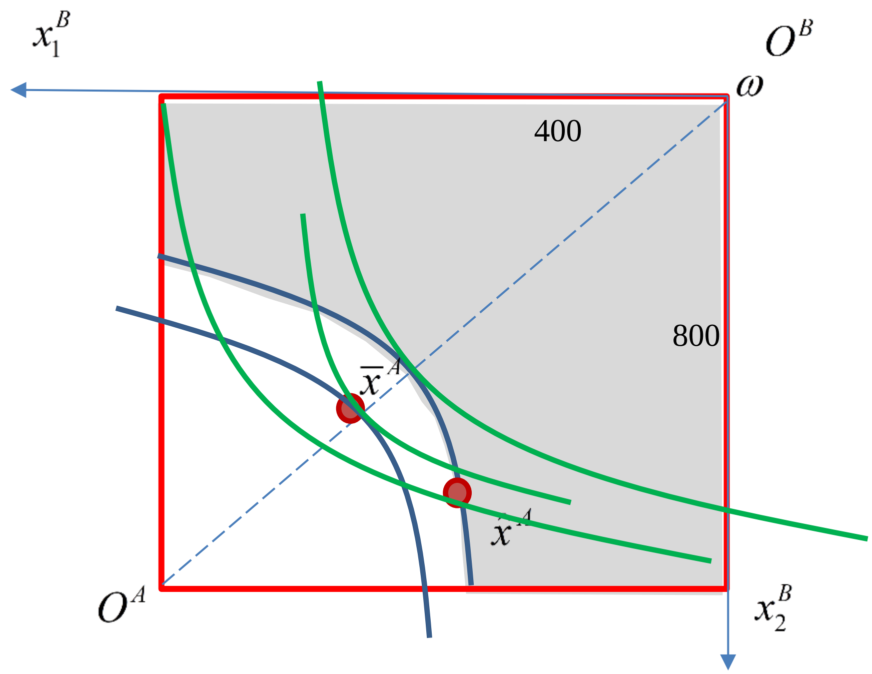
Consider any allocation such as $\hat{x}^A$

Where the marginal rates of substitution differ. From the figure there are exchanges that the two consumers can make and both have a higher utility.



Consider any allocation such as $\hat{\hat{x}}^A$

Where the marginal rates of substitution differ. From the figure there are exchanges that the two consumers can make and both have a higher utility.



Pareto Efficient Allocations

It follows that for an allocation

$$x^A \text{ and } x^B = \omega - x^A$$

to be Pareto efficient (i.e. no Pareto improving allocations)

$$MRS^A(x^A) = MRS^B(x^B)$$

Along the 45° line $MRS^A(\bar{x}^A) = \dfrac{\pi_1}{\pi_2} = MRS^B(\bar{x}^B)$.

Thus the Pareto Efficient allocations are all the allocations along 45° degree line.

Pareto Efficient exchange eliminates all individual risk.

Walrasian Equilibrium?

Suppose that insurance companies act as competitive intermediaries (effectively market makers) for people who want to trade the commodity in one state for more of the commodity in the other state. Let p_s be the price that a consumer must pay for delivery of a unit in state s, i.e. the price of "claim" in state s.

A consumer's endowment $\omega=(\omega_1,\omega_2)$, thus has a market value of $p\cdot\omega=p_1\omega_1+p_2\omega_2$. The consumer can then choose any outcome (x_1,x_2) satisfying

$$p\cdot x\leq p\cdot\omega$$

Given a utility function $u_h(x_s)$, the consumer chooses $\overline{x}^h$ to solve

$$\underset{x}{Max}\{U_h(x^h,\pi)\,|\,p\cdot x\leq p\cdot\omega^h\}$$

i.e.

$$\underset{x^h}{Max}\{\pi_1 v_h(x_1^h)+\pi_2 v_h(x_2^h)\,|\,p\cdot x^h\leq p\cdot\omega^h\}$$

FOC: $MRS_h(\overline{x}^h)=\dfrac{MU_1}{MU_2}=\dfrac{\pi_1 v_h{}'(\overline{x}_1^h)}{\pi_2 v_h{}'(\overline{x}_2^h)}=\dfrac{p_1}{p_2}$

(With equal probabilities, the price ratio is 1.)

Example 1: No aggregate (social) risk

$\omega^A = (200,600)$, $\omega^B = (800,400)$, $\omega = \omega^A + \omega^B = (1000,1000)$ $\pi = (\pi_1, \pi_2) = (\frac{1}{5}, \frac{4}{5})$

Group exercises

1. What is the WE state claims price ratio?
2. What is the WE allocation?
3. Normalizing so the sum of the state claims prices is 1, what is the value of each plantation?
4. What is the profit of the insurance companies?

Class exercise

1. What ownership of plantations would give the two consumers the WE outcome?
2. Could they trade shares in their plantations and achieve this outcome?

Class (or group) Exercise: What if the loss in state 1 is bigger than in state 2

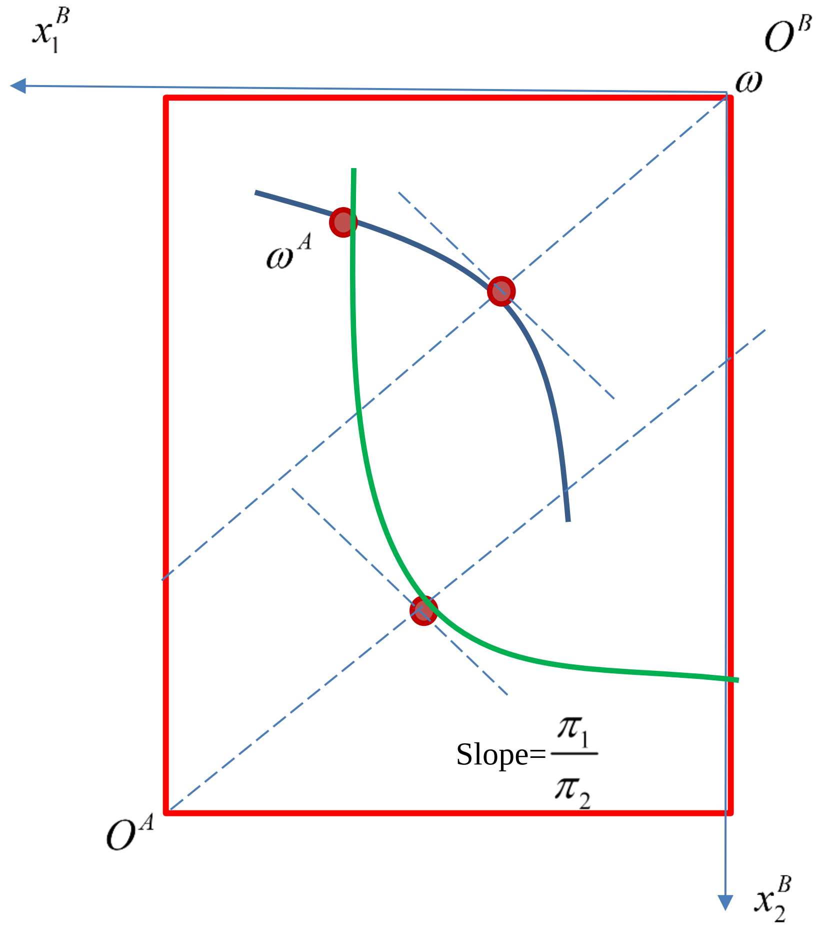
Suppose $v_A(x^h)$ and $v_B(x)$ are both strictly concave. The aggregate endowment is $\omega = (\omega_1, \omega_2)$ where $\omega_1 < \omega_2$. The probabilities are $\pi = (\pi_1, \pi_2)$

Consider the Edgeworth-Box.

Must the PE allocations lie between the two 45° lines?

What does the First Welfare Theorem tell us about the WE allocation?

Compare $\dfrac{p_1}{p_2}$ and $\dfrac{\pi_1}{\pi_2}$



Suppose that $U^h(x_1^h, x_2^h) = \pi_1 v(x_1^h) + \pi_2 v(x_2^h)$ is homothetic.

What can be said about the WE price ratio?

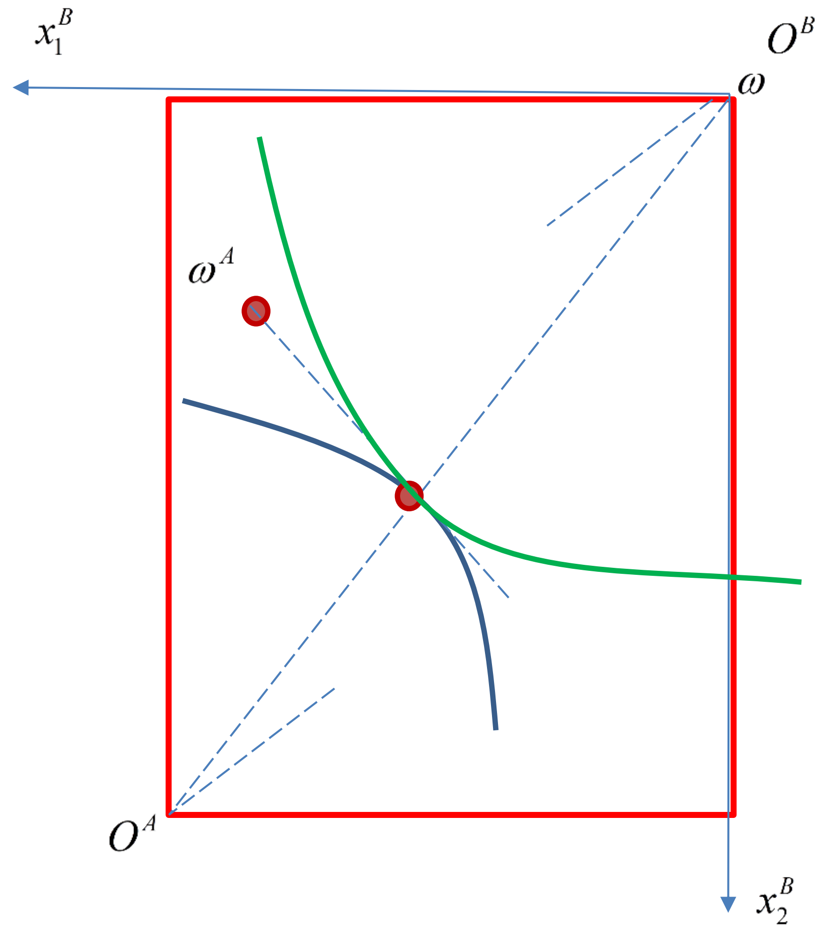
What can be said about the WE consumption ratio of each consumer?

Identical homothetic expected utility functions

Each consumer has a WE consumption

That is a fraction of the aggregate endowment.

$$f_A = \frac{p \cdot \omega^A}{p \cdot \omega}\ ,\ f_B = \frac{p \cdot \omega^B}{p \cdot \omega}.$$



What can be said about the set of PE allocations?

Example 2: Group exercise

Equal probabilities of each state

VNM utility function $v(x) = \ln x$

So

$$U(x) = \tfrac{1}{2}\ln x_1 + \tfrac{1}{2}\ln x_2$$

$$\omega^A = (200,800)\ ,\ \omega^B = (800,1200)$$

Solve for the equilibrium state claims prices and outcome.

7. Trading in financial markets.

Class discussion:

For example 2, consider trading shares in the assets (the plantations) rather than trading in state claims markets.

Example 3: $\pi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ $\omega = (100, 200, 200)$. There are three plantations

$\omega^A = (60, 60, 40)$, $\omega^B = (40, 40, 0)$, $\omega^C = (0, 100, 160)$

What is the market value of each plantation?

Can the plantation owners achieve the WE outcome by trading in asset markets?

7. A very general model: Consumers have different aversion to risk and different beliefs.

Suppose that there at least as many assets as states

State claims equilibrium with S states A assets and H consumers.

Consumer h has expected utility function $U^h(x) = \sum_{s=1}^{S} \pi_s^h v_h(x_s)$

Asset a, $a = 1,...,A$ has a total state contingent return $z_a = (z_{a1},...,z_{aS})$

(In the examples the returns are coconuts)

The aggregate return is $z = \sum_{z} z_a = (z_1,...,z_S)$

Simplifying assumption:

Consumer h has an almost homothetic expected utility function

$U(x^h) = \sum_{s=1}^{S} \pi_s v(\gamma^h + x_s^h)$ where γ^h is a parameter.

We will only consider the simplest case in which the parameter γ^h is zero so utility functions are identical and homothetic.

State claims prices (insurance)

p_s the price paid for a claim to wealth in state s (the eventuality that the outcome is s).

The choice of consumer h .

Let $p=(p_1,...,p_S)$ be the WE price vector for state claims

Then the value of asset a is $P_a = p \cdot z_a$.

Consumer h has shareholdings $\bar{q}^h=(\bar{q}_1^h,...,\bar{q}_A^h)$ and hence wealth $W^h = \sum_a \bar{q}_a^h P_a$

Consumer h chooses $\bar{x}^h$ to solve

$$\underset{x^h}{Max}\{U(x^h)=\sum_{s=1}^{S}\pi_s v(x_s^h) \mid p\cdot x^h \le W^h\}$$

This is a representative consumer economy

The representative consumer chooses $\bar{x}^R$ to solve

$$\underset{x}{Max}\{U(x)=\sum_{s=1}^{S}\pi_s v(x_s)\,|\,p\cdot x\le W=\sum_h W^h=p\cdot z\}$$

The WE prices are

$$p=(p_1,...,p_S)=(\frac{\partial U}{\partial x_1}(z),...,\frac{\partial U}{\partial x_S}(z))$$ (or any scalar multiple)

Then we can compute the value of each asset.

$$P_a=p\cdot z_a=\sum_{s=1}^{S}p_s z_{as}$$

The wealth of consumer h is $W^h=\sum_a \bar{q}_a^h P_a$

Consumer h has wealth share $f^h=\dfrac{W^h}{W}=\dfrac{\sum_a \bar{q}_a^h P_a}{\sum_a P_a}$.

Then $\bar{x}^h=f^h z$, a fraction of the market return.

All consumers happy to trade their asset holdings for a single S&P-A mutual fund.

Example 1: $\omega^A = (200,600)$, $\omega^B = (800,400)$, $\omega = (1000,1000)$, $\pi = (\pi_1, \pi_2) = (\frac{1}{5}, \frac{4}{5})$

Answers

What are the PE allocations?

$\omega = \omega^A + \omega^B = (1000,1000)$ so on the diagonal of the Edgeworth Box $MRS_A(x^A) = MRS_B(x^B) = \dfrac{\pi_1}{\pi_2}$

What is the WE price ratio? $\dfrac{p_1}{p_2} = MRS_A(\bar{x}^A) = MRS_B(\bar{x}^B) = \dfrac{\pi_1}{\pi_2}$. Thus $p = (\frac{1}{5}, \frac{4}{5})$ are WE prices

What is the value of each plantation?

$P_a = p \cdot \omega^A = 520$, $P_b = p \cdot \omega^B = 480$, $P_a + P_b = 1000$.

Alex has 52% of the total value so his consumption is $\bar{x}^A = \dfrac{52}{100}\omega = (520,520)$.

Bev has 48% of the total value so her consumption is $\bar{x}^B = \dfrac{48}{100}\omega = (480,480)$.

Alex can achieve this with a portfolio $q^A = (q_a^A, q_b^A) = (\frac{52}{100}, \frac{52}{100})$.

Bev can achieve this with a portfolio $q^B = (q_a^B, q_b^B) = (\frac{48}{100}, \frac{48}{100})$. Alex sells 48% of his plantation and purchases 52% of Bev's plantation.

Mutual Fund

S&P2

A firm sells a mutual fund that mimics the outcomes in the market portfolio. These are (1000,1000)

So the fund returns might be $z_{SP} = (10,10)$.

Then 100 units of this fund is the same as the market portfolio.

Instead of trading individual shares, Alex and Bev can sells their plantations and purchse shares in the mutual fund

Homework problem

$U(x^h) = \pi_1 v(x_1^h) + \pi_2 v(x_2^h) + \pi_3 v(x_3^h)$ where $\pi = (\pi_1, \pi_2, \pi_3) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $v(x) = x^{1/2}$

Assets

$z_1 = (50, 10, 100)$

$z_2 = (20, 10, 200)$

$z_3 = (30, 5, 100)$

Consumer h has an initial portfolio of $\bar{q}^h = (\bar{q}_1^h, \bar{q}_2^h, \bar{q}_3^h)$ (shares in the three assets.)

Solve for the WE asset prices.

Class discussion

8. Financial Engineering

The general model. Consumers have different aversion to risk and different beliefs.

Suppose that there at least as many assets as states

State claims equilibrium with S states A assets and H consumers.

Consumer h has expected utility function $U^h(x) = \sum_{s=1}^{S} \pi_s^h v_h(x_s)$

Asset a, $a = 1,...,A$ has a total return $z_a = (z_{a1},...,z_{aS})$

The aggregate return is $z = \sum_z z_a = (z_1,...,z_S)$

Let $p = (p_1,..., p_S)$ be the WE price vector for state claims

Then the value of asset a is

$$P_a = p \cdot z_a \,.$$

Consumer h has shareholdings $\overline{q}^h = (\overline{q}_1^h,...,\overline{q}_A^h)$ and hence wealth $W^h = \sum_a \overline{q}_a^h P_a$

Consumer h chooses $\overline{x}^h$ to solve

$$\underset{x^h}{Max}\{U(x^h) = \sum_{s=1}^{S} \pi_s^h v(x_s^h) \mid p \cdot x^h \le W^h\}$$

Let p be the WE (i.e. market clearing) state claims price vector.

Is it possible to create an asset that pays off only in state 1 using a portfolio q^* of financial assets?

Asset a returns

Two states Three states …… S states

$$z_a = \begin{bmatrix} z_{a1} \\ z_{a2} \end{bmatrix} \qquad z_a = \begin{bmatrix} z_{a1} \\ z_{a2} \\ z_{a3} \end{bmatrix}$$

a column vector of dimension S

Consider a two state example

Asset 1

$$z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \end{bmatrix}, \quad z_2 = \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 50 \\ 80 \end{bmatrix}$$

Portfolio (q_1^s, q_2^s) that pays off 1 unit in state s and 0 units in the other state.

A synthetically created state 1 claim.

$$q_1^1 \begin{bmatrix} 100 \\ 20 \end{bmatrix} + q_2^1 \begin{bmatrix} 50 \\ 80 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A synthetically created state 2 claim.

$$q_1^2 \begin{bmatrix} 100 \\ 20 \end{bmatrix} + q_2^2 \begin{bmatrix} 50 \\ 80 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Two equations and two unknowns

Group exercise

Solve for the two portfolios.

If this is possible we have created two synthetic assets.

Each pays off 1 unit in one state so is equivalent to a state claim.

NOTE: Some assets may have to be sold short

Let P_a be the price of asset a.

Then the value of a state s claim is the value of the portfolio $q^s = (q_1^s, q_2^s)$, $s = 1,2$

$$p_1 = q_1^1 P_1 + q_2^1 P_2 \;, \quad p_2 = q_1^2 P_1 + q_2^2 P_2 .$$

Therefore, instead of trading state claims, a consumer can trade in financial assets and achieve the same outcome.

General result with S states and $A \geq S$ assets:

Let $z_a = (z_{a1},...,z_{aS})$ be the vector of returns for asset a .

If, for all s there is a portfolio $q^s = (q_1^s,...,q_A^s)$ which pays off 1 in state s and zero in all other states, then there exists an asset market equilibrium which is equivalent to the contingent claims market (i.e. insurance market) equilibrium.

Example: Portfolio $q^1 = (q_1^1,...,q_A^1)$ equivalent to a state 1 claim.

$$q_1^1 \begin{bmatrix} z_{11} \\ \cdot \\ \cdot \\ z_{1S} \end{bmatrix} + q_2^1 \begin{bmatrix} z_{21} \\ \cdot \\ \cdot \\ z_{2S} \end{bmatrix} + \ldots + q_A^1 \begin{bmatrix} z_{A1} \\ \cdot \\ \cdot \\ z_{AS} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix}$$

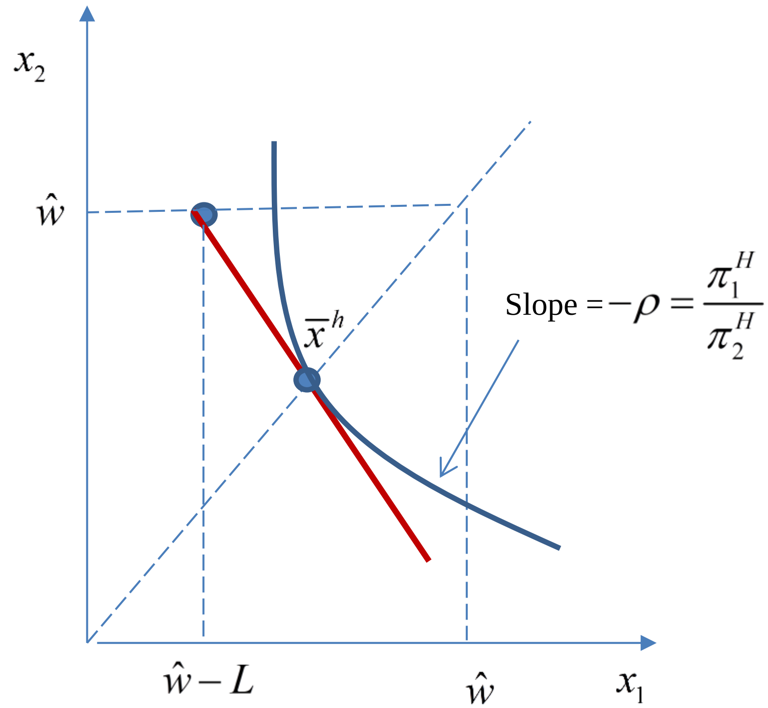
9. Adverse selection: Two risk classes

High risk class

Under perfect competition

The market odds $\dfrac{p_1}{p_2}$ is equal to the

probability ratio (the odds) $\dfrac{\pi_1^H}{\pi_2^H}$.



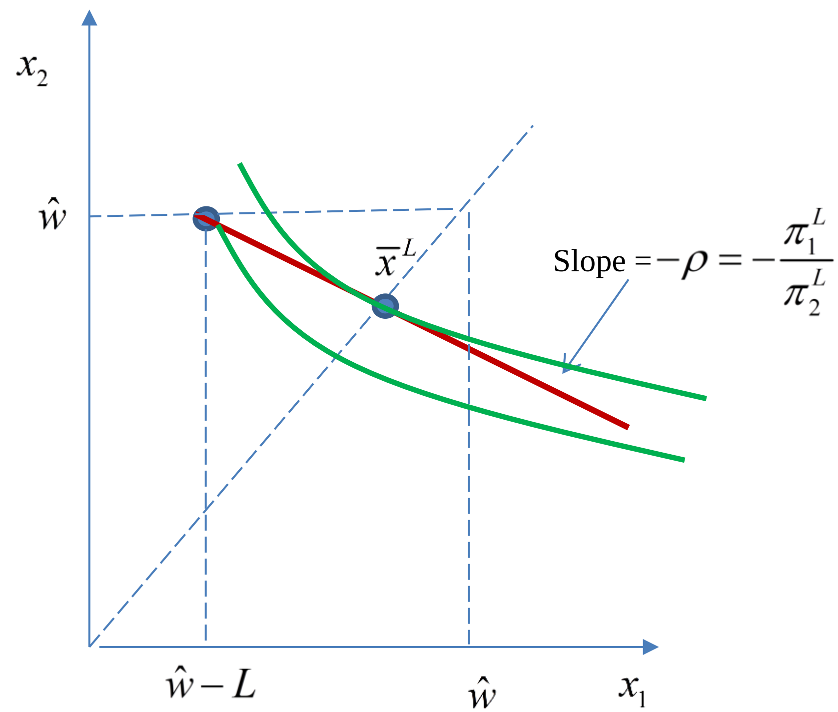
Implicit budget line though the "endowment"

Low risk class

Under perfect competition

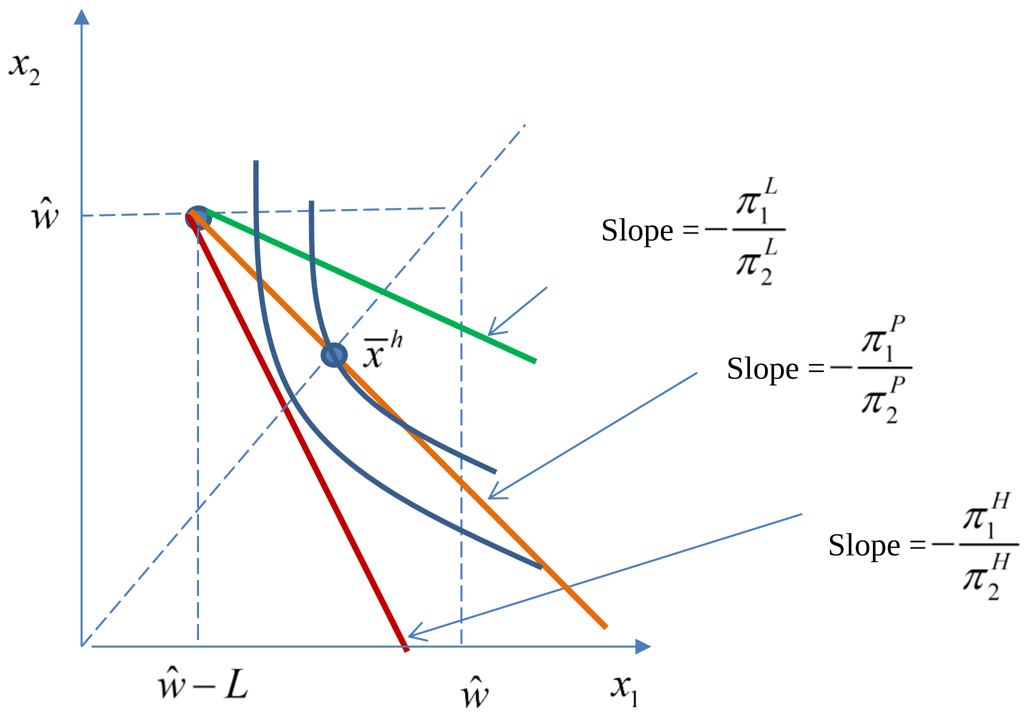
The market odds $\dfrac{p_1}{p_2}$ is equal to the

probability ratio (the odds) $\dfrac{\pi_1^L}{\pi_2^{L:}}$.

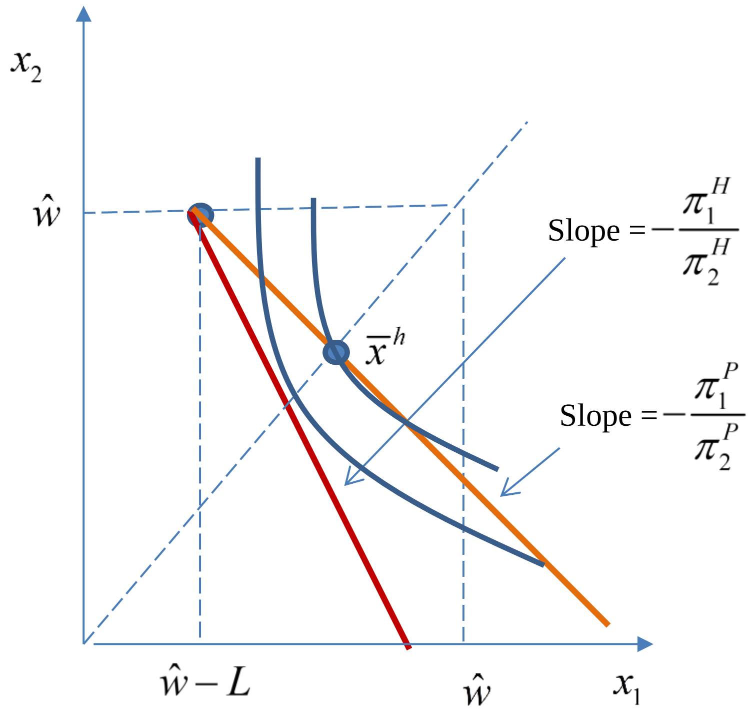


Implicit budget line though the “endowment”

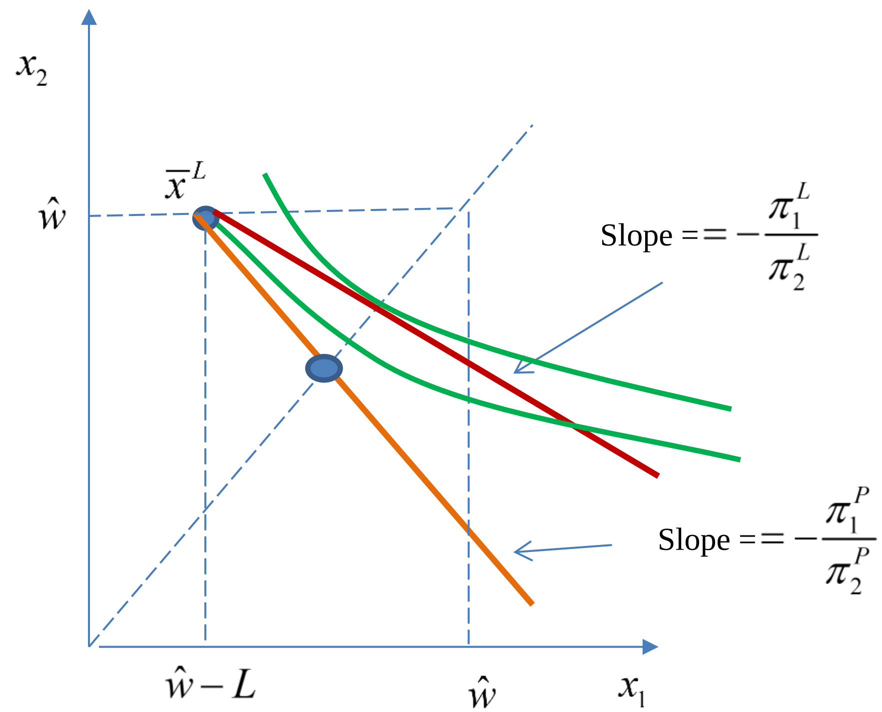
Suppose that the risk class in unobservable. If the classes are pooled the probability of loss (and hence no loss) is a population weighted average (π_1^P, π_2^P)



Pooled risk implicit budget line is between the high and low risk lines



High risks are subsidized at pooled odds

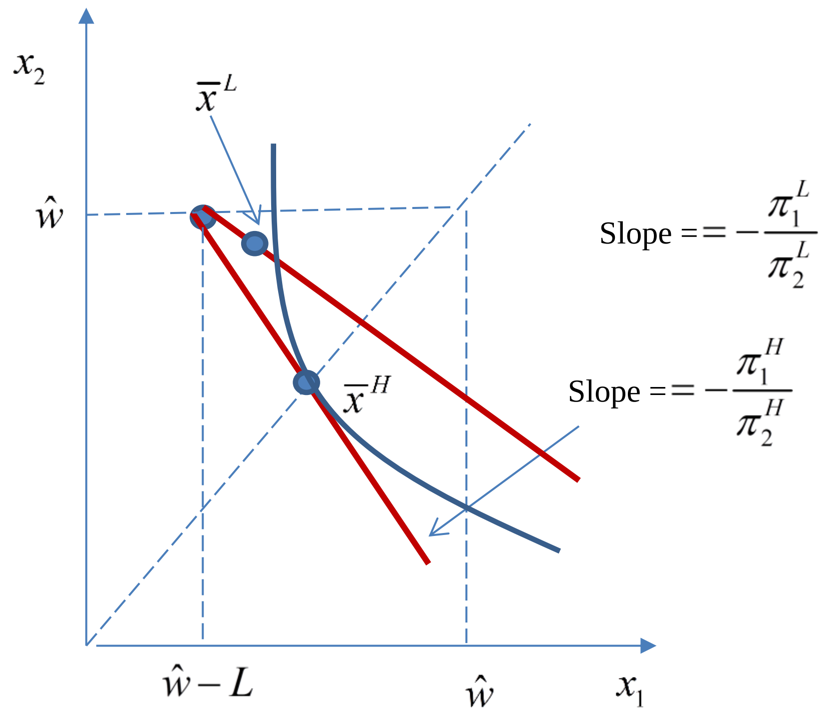


Low risks are better off purchasing no insurance

The quality of the risk pool is adversely selected. Only the high risks are insured and the equilibrium price reflects this.

Screening

Alternatively the insurers offer full coverage to high risk consumers and partial coverage to low risk consumers. If the coverage on the low risk insurance policy is sufficiently small, the high risk consumers choose the full coverage and the low risk the partial coverage.



Partial coverage for low risk consumers

10. Net demands and betting

Net demands

Consider a two commodity model.

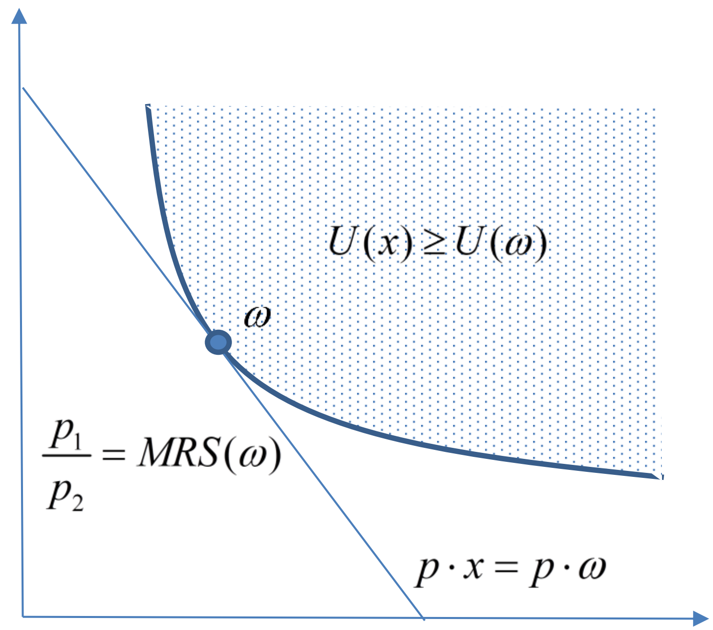
A consumer has an endowment $\omega = (\omega_1, \omega_2)$.

Let $x(p)$ be her market demand.

If $\dfrac{p_1}{p_2} = MRS(\omega)$ then $x(p) - \omega = 0$.

The consumer is better off not buying

more of either commodity.



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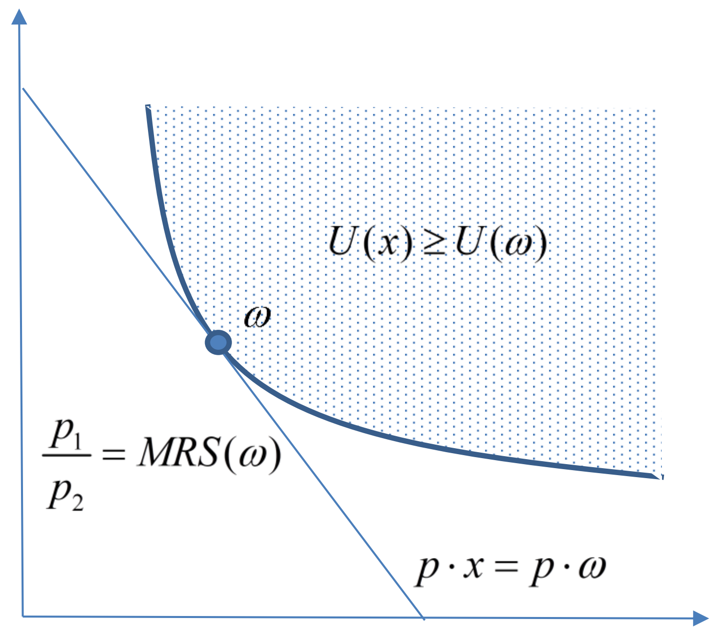


However, if the price ratio is not equal to $MRS(\omega)$, then the consumer will want to exchange some of one commodity for the other. These exchanges are called net trades

$$n_1(p) \equiv x_1(p) - \omega_1 \text{ and } n_2(p) \equiv x_2(p) - \omega_2$$

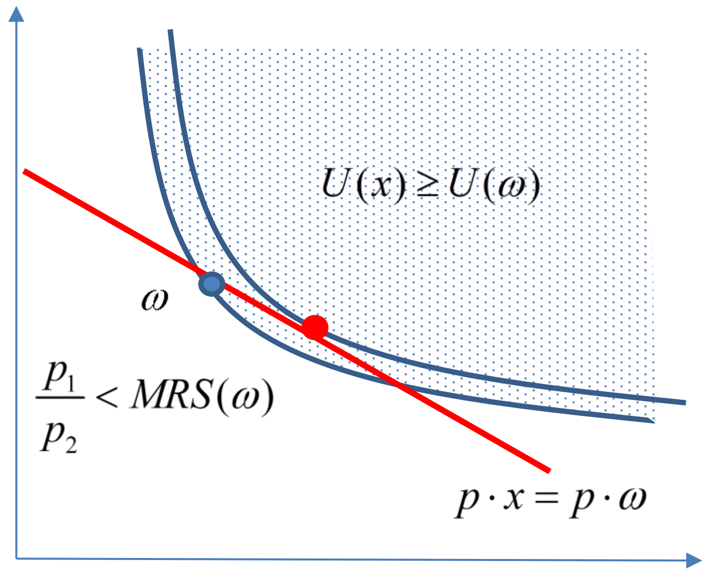
If $\frac{p_1}{p_2} < MRS(\omega)$

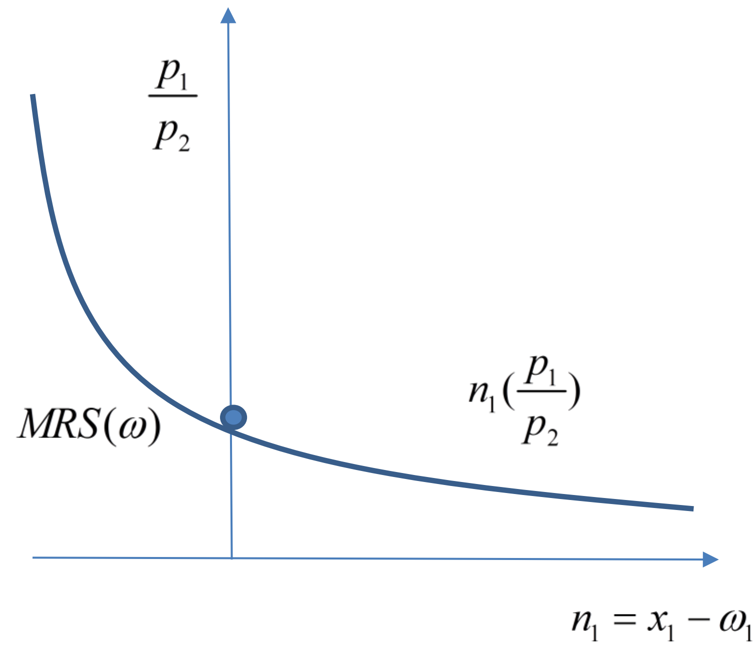
then the "net demands" are

$n_1(p) \equiv x_1(p) - \omega_1 > 0$ and $n_2(p) \equiv x_2(p) - \omega_2 < 0$

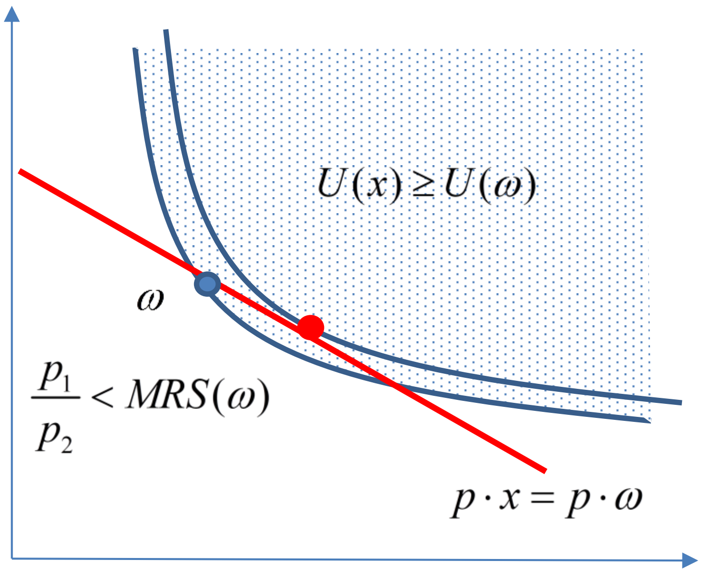
The net demand for commodity 1 is depicted in the lower diagram.

*





If $\dfrac{p_1}{p_2} < MRS(\omega)$



then the “net demands” are

$n_1(p) \equiv x_1(p) - \omega_1 > 0$ and $n_2(p) \equiv x_2(p) - \omega_2 < 0$

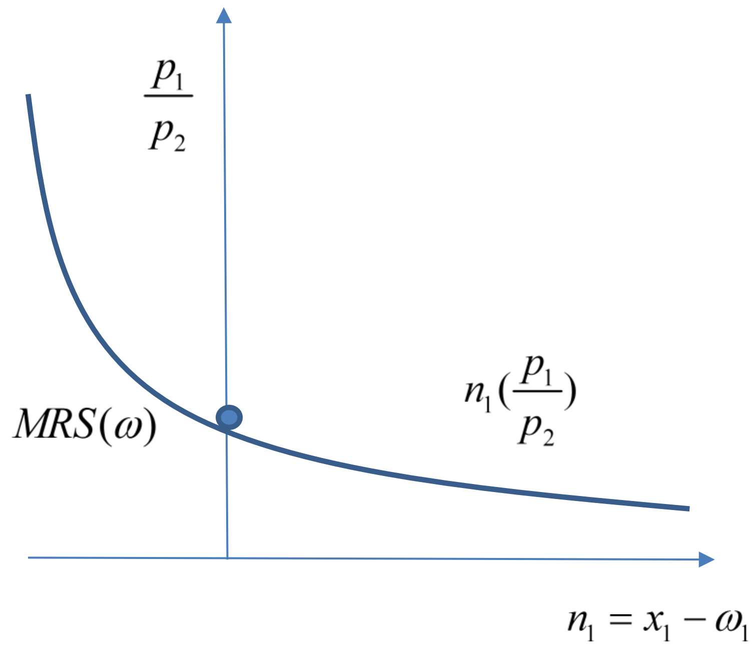
The net demand for commodity 1 is depicted in the lower diagram.

Note that

$p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) = 0$.

Therefore

$$n_2 = x_2 - \omega_2 = -\frac{p_1}{p_2} n_1$$



Example:

$$x(p) \text{ solves } \underset{x}{Max}\{U(x) = \alpha \ln x_1 + (1-\alpha)\ln x_2 \mid p \cdot \leq p \cdot \omega\}$$

As you may confirm,

$$p_1 x_1 = \alpha(p_1\omega_1 + p_2\omega_2)$$

*

Example:

$$x(p) \text{ solves } \underset{x}{Max}\{U(x) = \alpha \ln x_1 + (1-\alpha)\ln x_2 \mid p\cdot \leq p\cdot\omega\}$$

As you may confirm,

$$p_1 x_1 = \alpha(p_1\omega_1 + p_2\omega_2)$$

Therefore

$$x_1 = \alpha(\omega_1 + \frac{p_2}{p_1}\omega_2)$$

and so

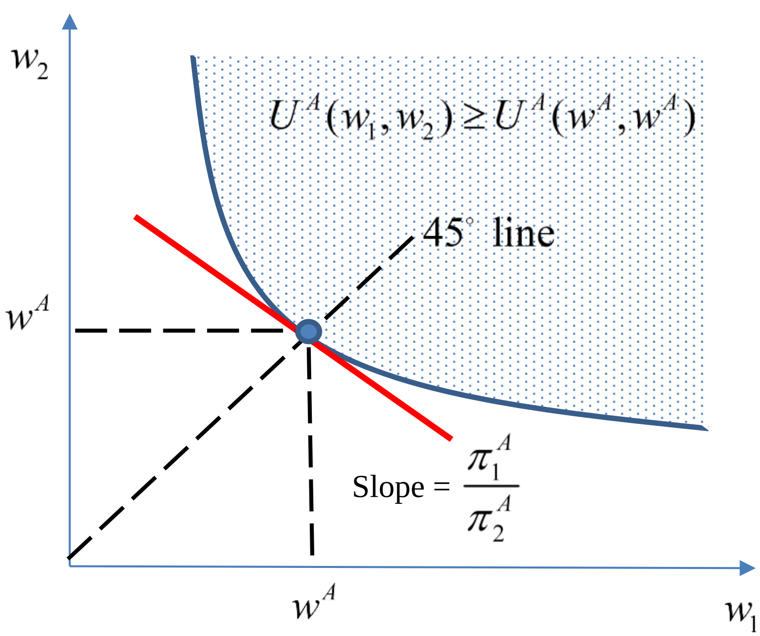
$$n_1 = x_1 - \omega_1 = -(1-\alpha)\omega_1 + \frac{p_2}{p_1}\omega_2$$

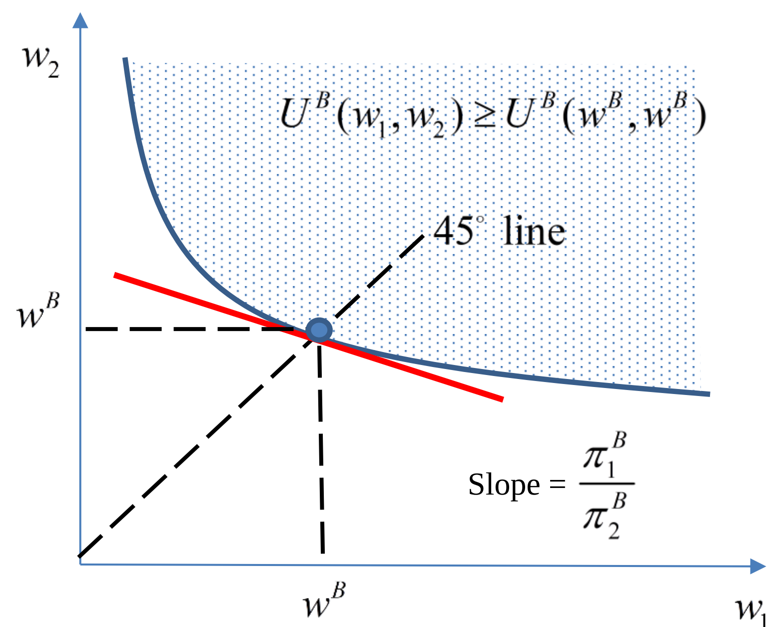
Betting on "The Game"

Alex is a rabid Bruins fan. He thinks that the probability of state 1 (Bruin victory) is high. Bev, who went to USC thinks that the probability of state 2 (Bruin defeat) is high. Alex's wealth is w^A and Bev's wealth is w^B.Their utility functions are as follows:

$$U_A(x;\pi^A) = \pi_1^A u_A(x_1^A) + \pi_2^A u_A(x_2^A) \text{ and } U_B(x^B;\pi^B) = \pi_1^B u_A(x_1^B) + \pi_2^B u_A(x_2^B)$$

$$MRS_A(x_1^A, x_2^A) = \frac{\pi_1^A}{\pi_2^A}\frac{u_A'(x_1^A)}{u_A'(x_2^A)} \qquad \text{and} \qquad MRS_B(x_1^B, x_2^B) = \frac{\pi_1^B}{\pi_2^B}\frac{u_B'(x_1^B)}{u_A'(x_2^B)}$$





Opportunities for mutual gain……

A monopoly "bookmaker"

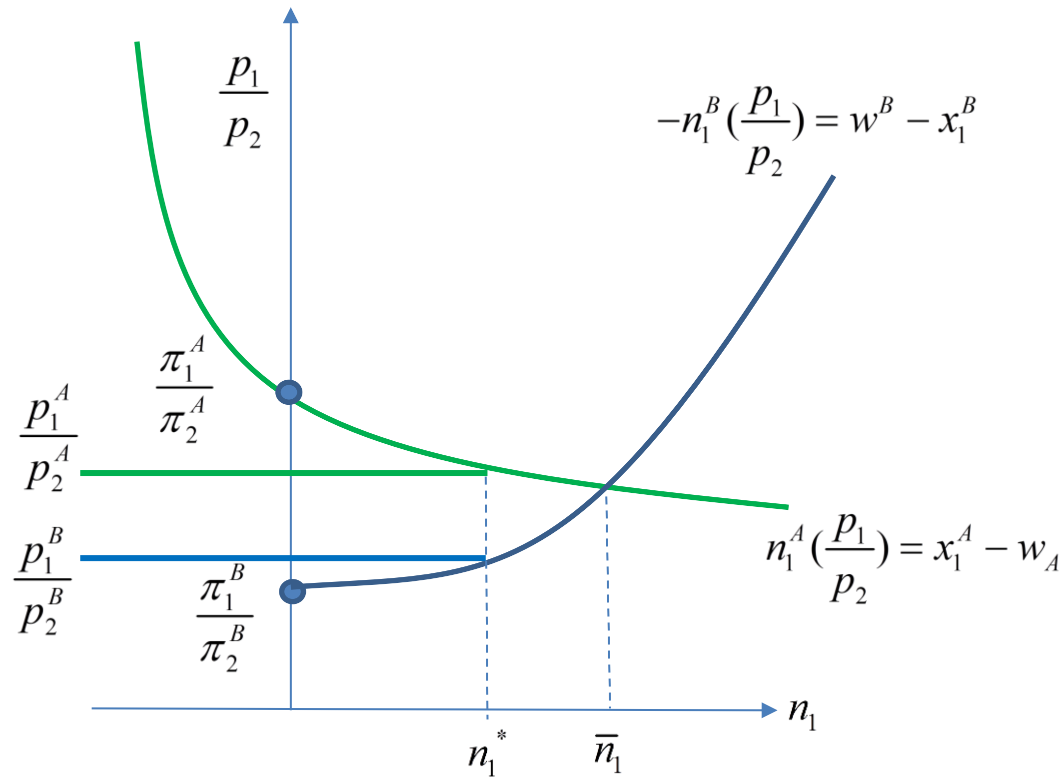
The bookmaker sets a price ratio

(the market odds for betting on a

Bruin victory) and another price ratio

(the market odds for betting on a

Trojan victory.



Competition among bookmakers lowers the difference in market odds. Ignoring bookmaker costs, the equilibrium volume of betting is $\bar{n}_1$

Group exercises

Assume a competitive betting market so the middlemen make a profit of zero

Left side: Alex (Bruin fan)

$U_A(x;\pi^A) = \frac{3}{4}\ln(x_1^A) + \frac{1}{4}\ln(x_2^A)$, wealth w^A .

Budget constraint $p_1(x_1^A - w^A) = p_2(w^A - x_2^A)$, where $\frac{p_1}{p_2}$ are the “market odds”

Solve for his demand for claims to wealth in state 1.

Right side: Bev (Trojan fan)

$U_B(x;\pi^B) = \frac{1}{2}\ln(x_1^B) + \frac{1}{2}\ln(x_2^B)$, Wealth w^B .

Budget constraint: $p_2(x_2^B - w^B) = p_1(w^B - x_1^B)$.

Solve for her demand for claims to wealth in state 1.

Exercises

1. Consumer choice

(a) If $u(x_s)=\ln x_s$ what is the consumer's degree of relative risk aversion?

(b) If there are two states, the consumer's endowment is ω and the state claims price vector is p, solve for the expected utility maximizing consumption.

(c) Confirm that if $\dfrac{p_1}{p_2}>\dfrac{\pi_1}{\pi_2}$ then the consumer will purchase more state 2 claims than state 1 claims.

2 . Consumer choice

(a), (b), (c) as in Exercise 1 except that $u(x_s)=x_s^{1/2}$.

(d) Compare the state claims consumption ratio in Exercise 1 with that in Exercise 2.

(e) Provide the intuition for your conclusion.

3. Equilibrium with social risk.

Suppose that both consumers have the same expected utility function

$$U_h(x,\pi)=\pi_1 \ln x_1^h+\pi_2 \ln x_2^h .$$

The aggregate endowment is $\omega=(\omega_1,\omega_2)$ where $\omega_1>\omega_2$.

(a) Solve for the WE price ratio $\dfrac{p_1}{p_2}$.

(b) Explain why $\dfrac{p_1}{p_2} < \dfrac{\pi_1}{\pi_2}$.

4. Equilibrium with social risk.

Suppose that both consumers have the same expected utility function

$$U_h(x,\pi) = \pi_1 (x_1^h)^{1/2} + \pi_2 (x_2^h)^{1/2} .$$

The aggregate endowment is $\omega = (\omega_1, \omega_2)$ where $\omega_1 > \omega_2$.

(a) Solve for the WE price ratio $\dfrac{p_1}{p_2}$.

(b) Compare the equilibrium price ratio and allocations in this and the previous exercise and provide some intuition.