Equilibrium and Pareto Efficiency in an exchange economy

Efficient economies

Definition: Pareto preferred allocation

The allocation $\{\hat{x}^h\}_{h\in\mathcal{H}}$ is Pareto preferred to $\{\overline{x}^h\}_{h\in\mathcal{H}}$ if all consumers weakly prefer $\{\hat{x}^h\}_{h\in\mathcal{H}}$ over $\{\overline{x}^h\}_{h\in\mathcal{H}}$ and at least one consumer strictly prefers $\{\hat{x}^h\}_{h\in\mathcal{H}}$.

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First welfare theorem for an exchange economy

If $U^h(x^h),\ h\in\mathcal{H}=\{1,...,H\}$ satisfies the non-satiation property and $\{\bar{x}^h\}_{h\in\mathcal{H}}$ is a Walrasian Equilibrium allocation, then $\{\bar{x}^h\}_{h\in\mathcal{H}}$ is Pareto Efficient.

2. Gains from exchange

Preliminary observation

Consider the standard utility maximization problem

with two commodities.

If the solution $\bar{x} >> 0$ then the marginal utility per

dollar must be the same for each commodity

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1}(\bar{x}) = \frac{1}{p_2}\frac{\partial U}{\partial x_2}(\bar{x}).$$

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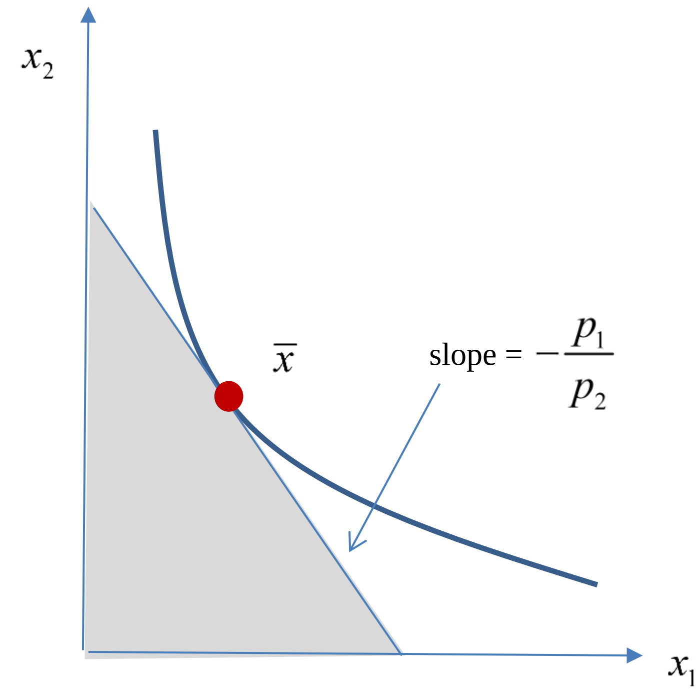

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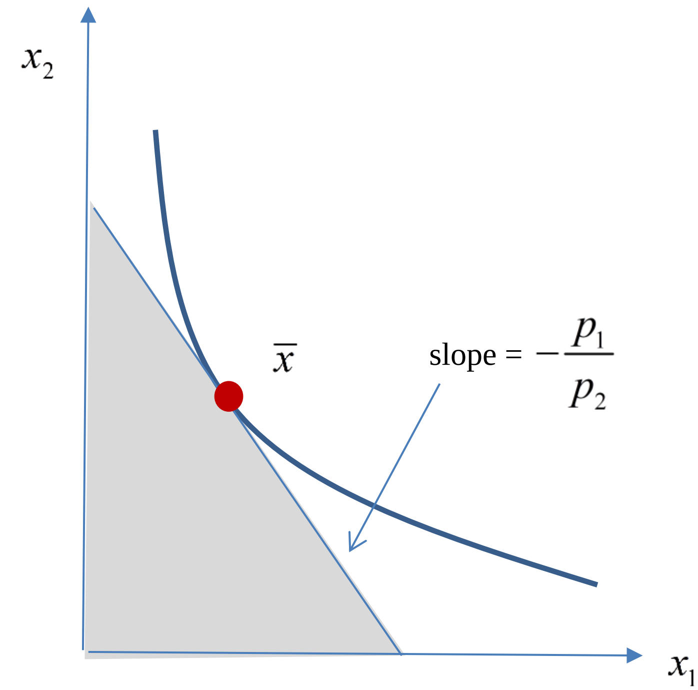
$$MRS(\bar{x}_1, \bar{x}_2) \equiv \frac{\dfrac{\partial U}{\partial x_1}}{\dfrac{\partial U}{\partial x_2}} = \frac{p_1}{p_2}$$



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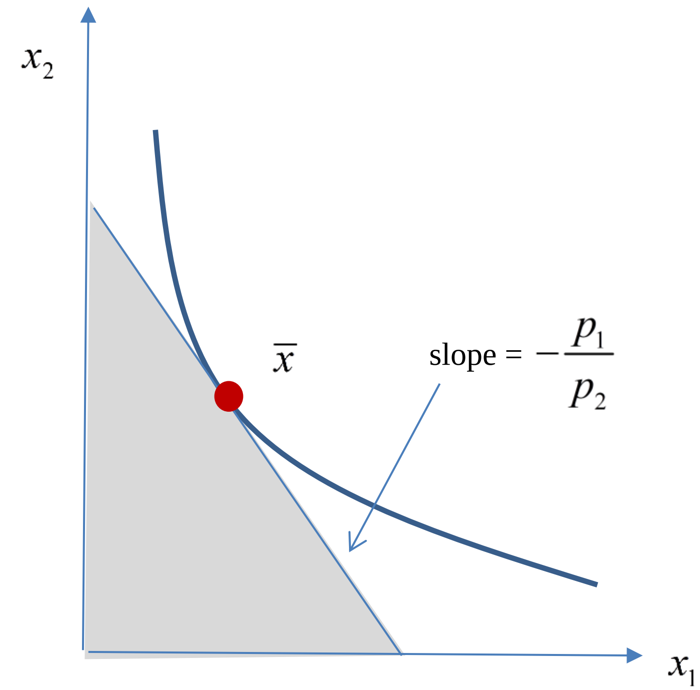

In the figure the slope of the budget line is $-\frac{p_1}{p_2}$.

At the maximum this slope is the same as the slope of the indifference curve.

Therefore $-MRS(\bar{x}_1,\bar{x}_2)$ is the slope of the indifference curve.

Pareto Efficient allocation in a 2 person 2 commodity economy

An allocation $\hat{x}^A$ and $\hat{x}^B$ is not a PE allocation if there

is an exchange of commodities $e=(e_1,e_2)$ such that

$U_A(\hat{x}^A+e)>U_A(\hat{x}^A)$ and $U_A(\hat{x}^B-e)>U_A(\hat{x}^B)$

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Proposition: If $\hat{x}^A >> 0$ and $\hat{x}^B >> 0$ then a necessary

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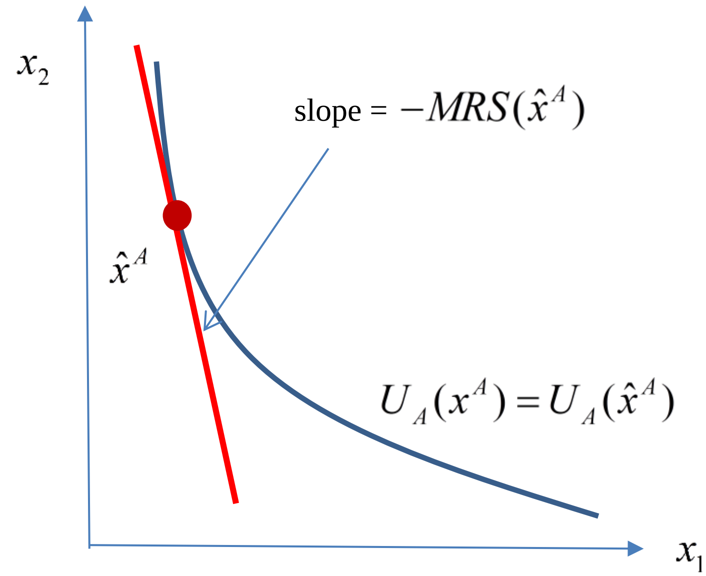
Pareto Efficient allocation in a 2 person 2 commodity economy



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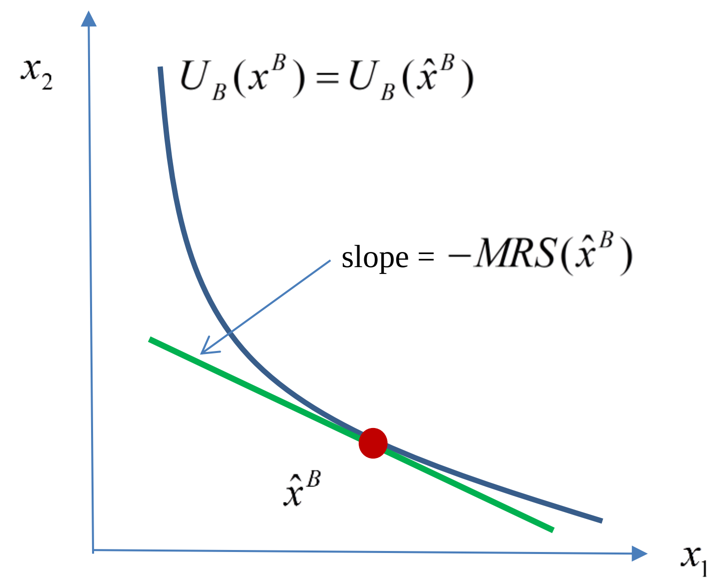
Proposition: If $\hat{x}^A >> 0$ and $\hat{x}^B >> 0$ then a necessary condition for an allocation to be a PE allocation is that marginal rates of substitution are equal.



Suppose instead that, as depicted,

$$MRS_A(\hat{x}^A) > MRS_B(\hat{x}^B)$$

Consider a proposal by Alex of $e = (e_1, e_2)$ where

$$e_1 > 0 > e_2$$

and the exchange rate lies between the two marginal rates of substitution

Such an exchange is depicted.

On the margin, Alex is willing to give up more

of commodity 2 In exchange for commodity 1.

Therefore Alex offers Bev some of commodity 2

In exchange for commodity 1.

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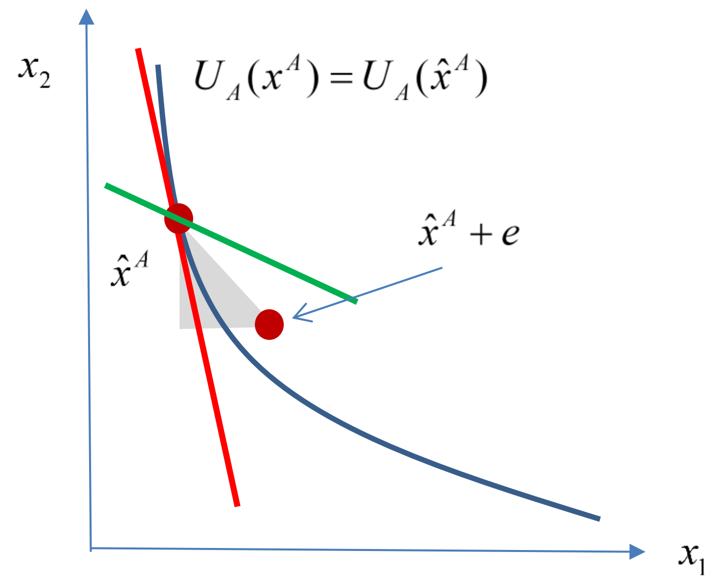

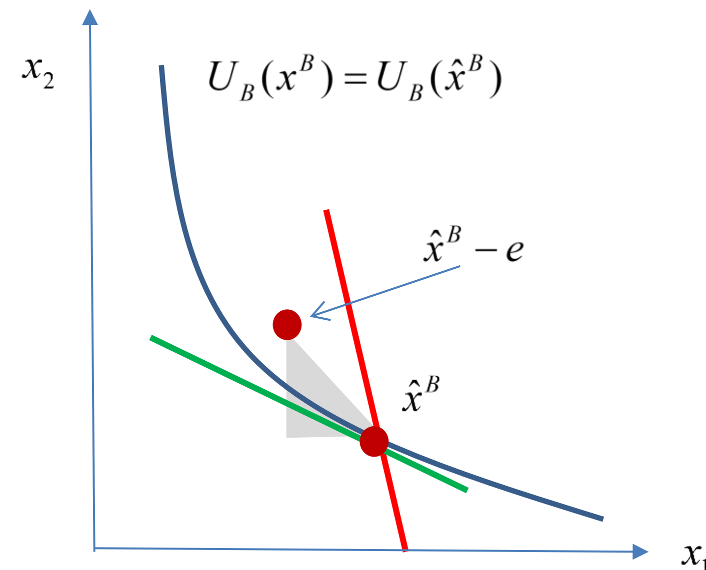

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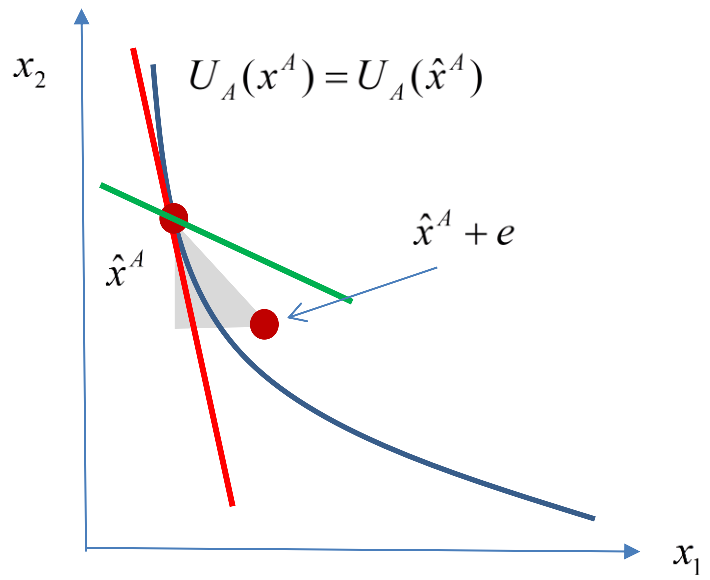
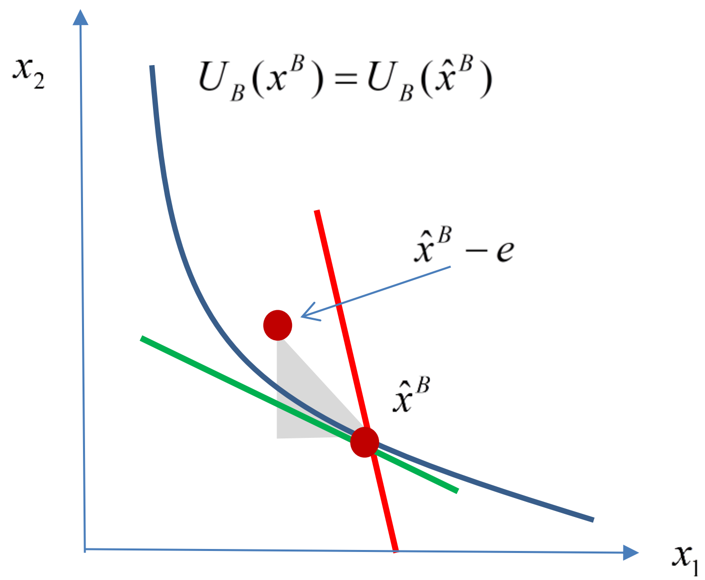


If the proposed trade is too large it may not be better

for both consumers due to the curvature of the level sets.



But for all sufficiently small θ , the proposed

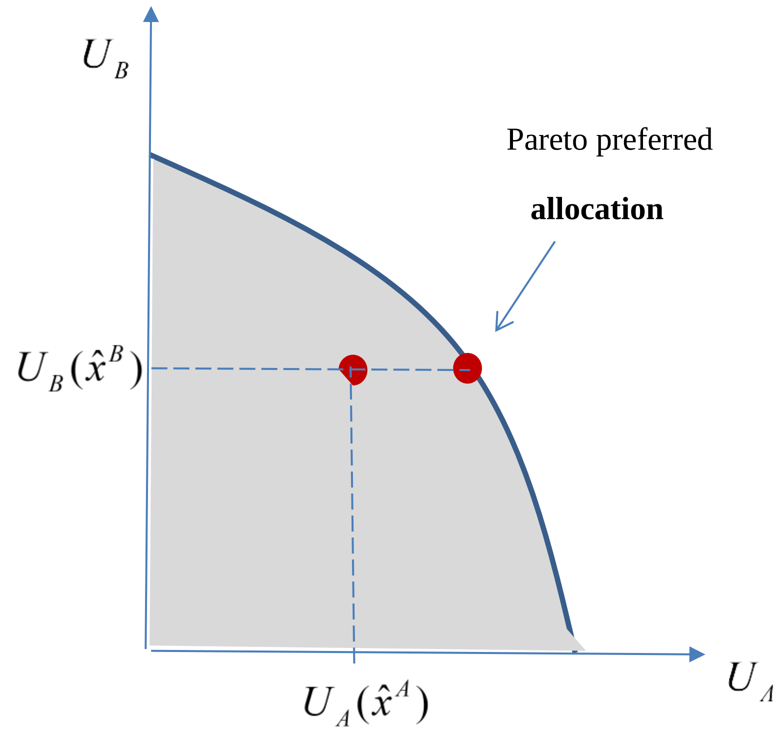
trade θe must raise the utility of both consumers.

So the initial allocation $\hat{x}^A, \hat{x}^B$ is not a Pareto efficient

allocation.

What if there are more than two commodities?

For all possible allocations we can, in principle compute the utilities and hence the set of feasible utilities.



For any point in the interior of this set there is another allocation such that Bev is no worse off and Alex is strictly better off.

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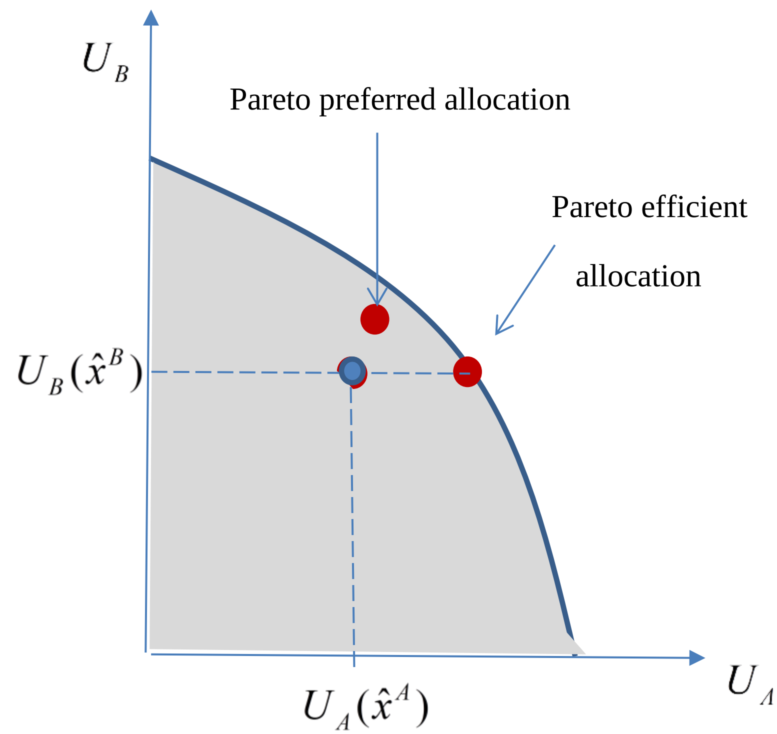


For any point in the interior of this set there is another allocation such that Bev is no worse off and Alex is strictly better off.

Consider the following maximization problem.

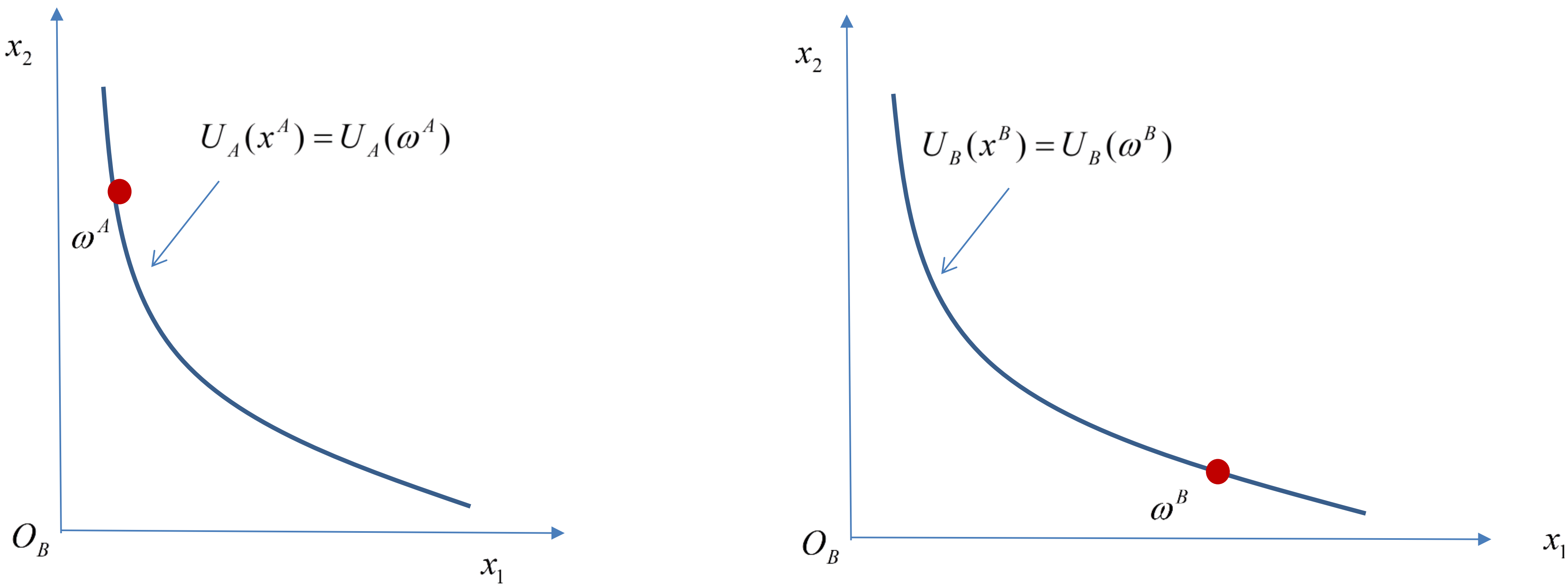
$$\underset{e}{Max}\{U_A(\hat{x}+e) \mid U_B(\hat{x}^B-e) \geq U_B(\hat{x}^B)\}$$

Class Exercise

What exchange e^* solves this problem if the allocation $\hat{x}^A, \hat{x}^B$ is Pareto efficient?

3. Efficiency in an Edgeworth-Box diagram

Consider Alex and Bev with endowments ω^A and ω^B.



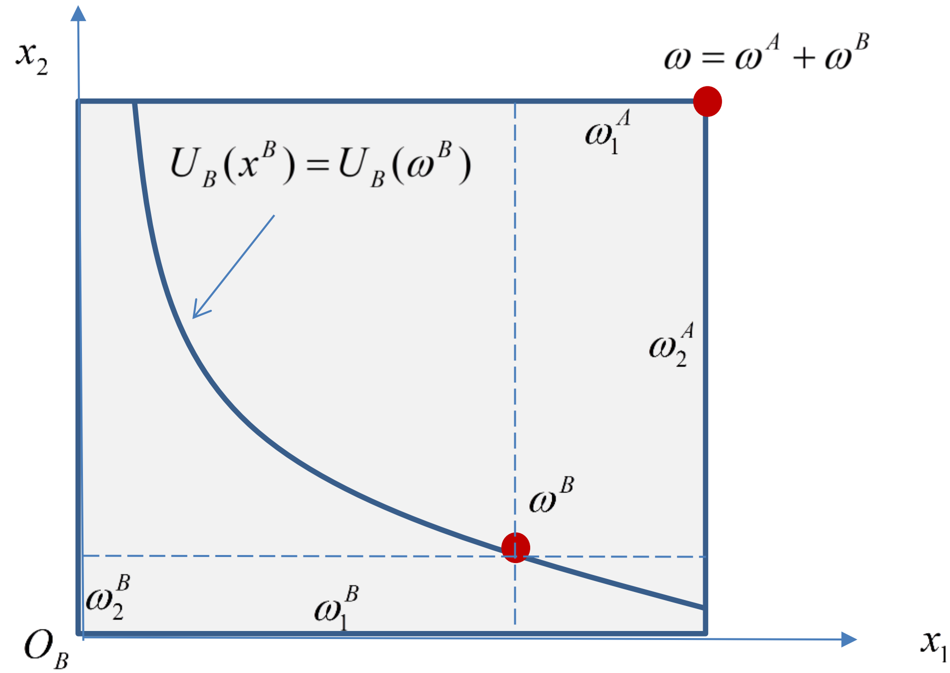
$MRS(\omega^A) > MRS(\omega^B)$ so there are gains from exchange.

Efficiency in an Edgeworth-Box diagram

If the endowments are ω^A and ω^B,

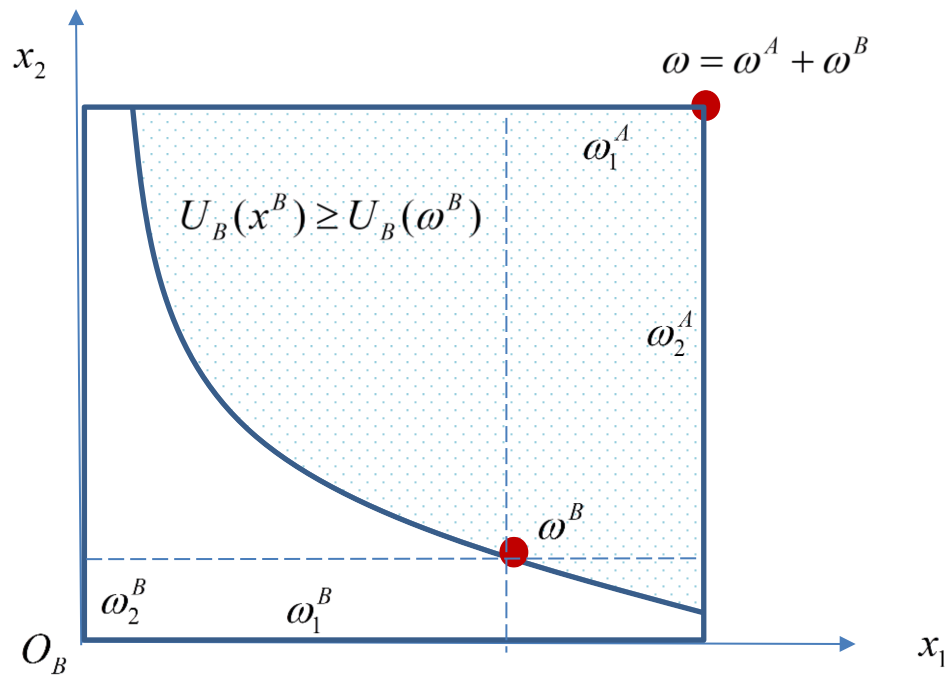
the set of feasible allocations for Bev is the

set of allocation in the rectangle or "box"



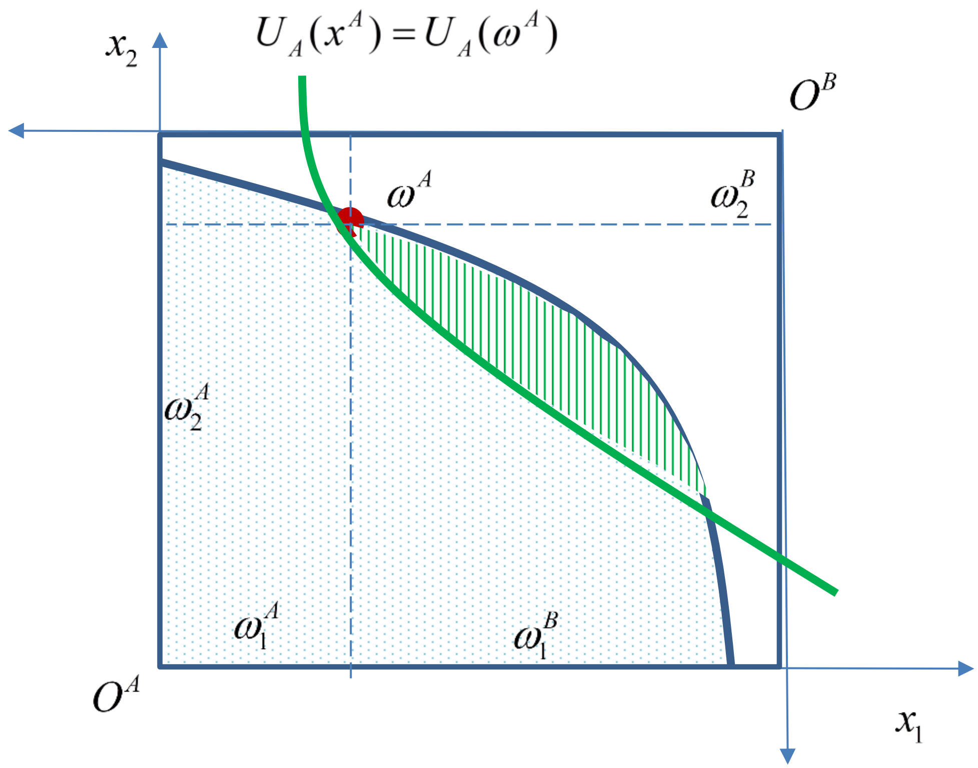
The set of allocations preferred by Bev

Is the dotted region in the lower box.



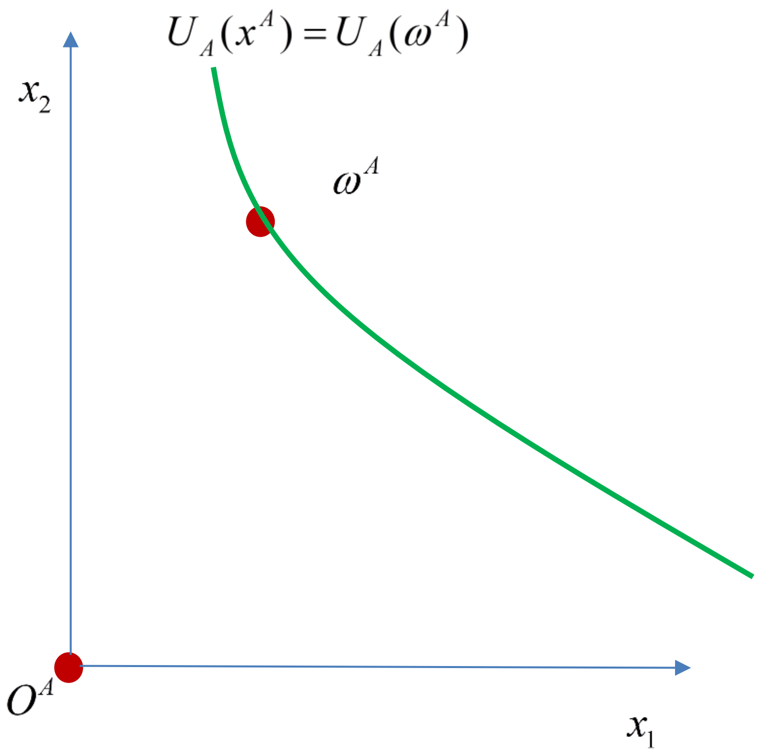
On the next slide we rotate the box 180°.

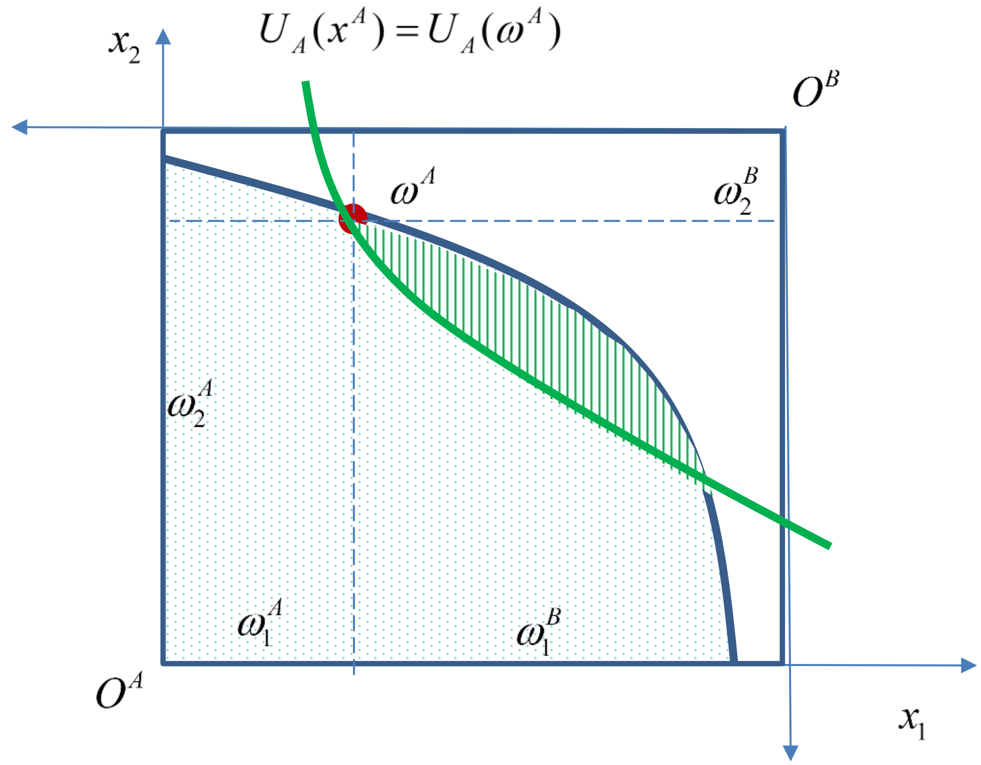
Box rotated 180°



We also add the level set for Alex through the endowment. Because $MRS_A(\omega^A) \neq MRS_B(\omega^B)$

there is a vertically lined region of Pareto preferred allocations

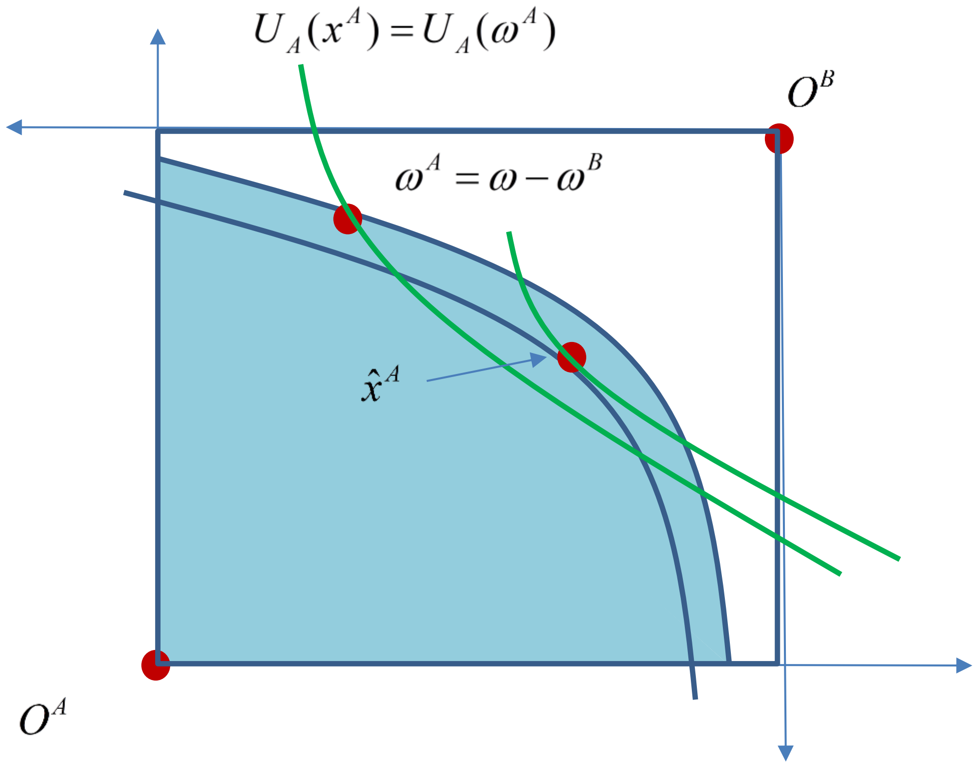




The allocation $\hat{x}^A$ and $\hat{x}^B = \omega - \hat{x}^A$

is Pareto- efficient since the

marginal rates of substitution are equal.



Group exercise

Suppose that $U_A(x^A) = 2(x_1^A)^{3/4} + 3(x_2^A)^{3/4}$ and $U_B(x^B) = 2(x_1^B)^{3/4} + 3(x_2^B)^{3/4}$

The aggregate endowment is $\omega = (100, 200)$.

(a) Show that for all allocation to be a PE allocation, both consumers are allocated twice as much of commodity 2.

(b) What is the MRS if an allocation is Pareto Efficient?

4. Properties of a consumer's choice

Non satiation property

For every x, there is at a commodity j such that for all sufficiently small $\delta > 0$,

$$U^h(x_1,....x_{j-1},x_j+\delta,x_{j+1},...,x_n) > U^h(x_1,....x_{j-1},x_j,x_{j+1},...,x_n)$$

Consider a consumer with a utility function whose utility satisfies this very weak property.

Let $\bar{x}^h$ be the choice of consumer h.

(i) If $U^h(\hat{x}^h) > U^h(\bar{x}^h)$ then $p\cdot\hat{x}^h > p\cdot\bar{x}^h$ and (ii) if $U^h(\hat{x}^h) = U^h(\bar{x}^h)$ then $p\cdot\hat{x}^h \geq p\cdot\bar{x}^h$.

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Proof of (i): If $U^h(\hat{x}^h) > U^h(\bar{x}^h)$ then $p\cdot\hat{x}^h > p\cdot\bar{x}^h$

Suppose instead that $U^h(\hat{x}^h) > U^h(\bar{x}^h)$ and $p\cdot\hat{x}^h \leq p\cdot\bar{x}^h$.

Then $\bar{x}^h$ is not the choice of the consumer since it does not maximize utility among commodity bundles in the budget set $p\cdot x \leq I$.

(ii) If $U^h(\hat{x}^h) \geq U^h(\bar{x}^h)$ then $p \cdot \hat{x}^h \geq p \cdot \bar{x}^h$

Proof: If $U^h(\hat{x}^h) > U^h(\bar{x}^h)$ this follows from (i)

Suppose instead that $U^h(\hat{x}^h) = U^h(\bar{x}^h)$ and $p \cdot \hat{x}^h < p \cdot \bar{x}^h$.

Define $\hat{\hat{x}}^h \equiv (\hat{x}_1, \ldots \hat{x}_{j-1}, \hat{x}_j + \delta, \hat{x}_{j+1}, \ldots, \hat{x}_n)$.

Then for some j and all small $\delta > 0$

$$U^h(\hat{\hat{x}}^h) = U^h(\hat{x}_1, \ldots \hat{x}_{j-1}, \hat{x}_j + \delta, \hat{x}_{j+1}, \ldots, \hat{x}_n) > U^h(\hat{x}_1, \ldots \hat{x}_{j-1}, \hat{x}_j, \hat{x}_{j+1}, \ldots, \hat{x}_n) .$$

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Since $p \cdot \hat{x}^h < p \cdot \overline{x}^h$ we can choose $\delta > 0$ such that $\hat{\hat{x}}^h$ is in the budget set and $U^h(\hat{\hat{x}}) > U^h(\hat{x}^h)$.

Then again $\overline{x}^h$ is not the choice of the consumer since it does not maximize utility among commodity bundles in the budget set $p \cdot x \leq I$.

3. Walrasian equilibria are Pareto Efficient

First welfare theorem for an exchange economy

If $U^h(x^h),\ h \in \mathcal{H} = \{1,...,H\}$ satisfies the non-satiation property and $\{\bar{x}^h\}_{h\in\mathcal{H}}$ is a Walrasian Equilibrium allocation, then $\{\bar{x}^h\}_{h\in\mathcal{H}}$ is Pareto Efficient.

Proof:

Remember that $\{\bar{x}^h\}_{h\in\mathcal{H}}$ is an equilibrium allocation.

Consider any Pareto preferred allocation $\{\hat{x}^h\}_{h\in\mathcal{H}}$

Step 1:

For some $h,\ U^h(\hat{x}^h) > U(\bar{x}^h)$.

By the non-satiation property (i)

$$p\cdot\hat{x}^h > p\cdot\omega^h \quad .$$

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$$p \cdot \hat{x}^h > p \cdot \omega^h \quad .$$

Step 2

For all h, $U^h(\hat{x}^h) \geq U(\bar{x}^h)$.

By the non-satiation properties (i) and (ii)

$$p \cdot \hat{x}^h \geq p \cdot \omega^h$$

Summarizing,

(a) $p\cdot\hat{x}^h > p\cdot\omega^h$ for some $h\in\mathcal{H}$

(b) $p\cdot\hat{x}^h \geq p\cdot\omega^h$ for all $h\in\mathcal{H}$

Summing over consumers,

$$p\cdot\hat{x}=p\cdot\sum_{h\in\mathcal{H}}\hat{x}^h > p\cdot\sum_{h\in\mathcal{H}}\omega^h = p\cdot\omega$$

Step 3:

For any feasible allocation $\{x^h\}_{h\in\mathcal{H}}$ the total consumption vector must satisfy

$$x=\sum_{h\in\mathcal{H}}x^h \leq \omega\ .$$

Since $p>0$ it follows that for any feasible allocation

$$p\cdot x\leq p\cdot\omega.$$

Since $p\cdot\hat{x}>p\cdot\omega$ it follows that $\hat{x}$ is not a feasible allocation.

Remark: An almost identical argument can be used to show that a Walrasian Equilibrium allocation for an economy with production is also Pareto Efficient