Econ 401a Final Examination

Sketch of answers

1. Choice over time

(a) $x_1 = y_1 - s$,

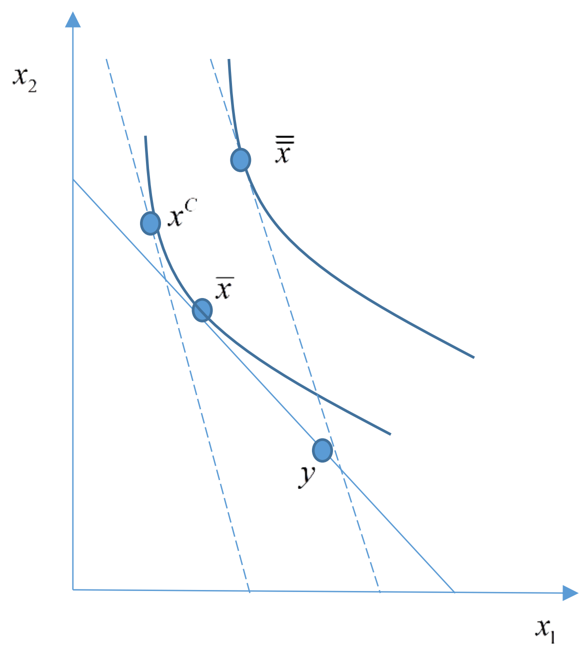
$x_2 = y_2 + (1+r)s$

Then

$$\frac{x_2}{1+r} = \frac{y_2}{1+r} + s$$

Adding,

$$x_1 + \frac{x_2}{1+r} = y_1 + \frac{y_2}{1+r} \ .$$



(b) The slope of the life-time budget line is $1+r$. The initial optimum is $\bar{x}$.

When r rises the compensated consumption vector is x^C . The saver is better off with a higher interest rate so the consumer has to be taxed to consume at x^C . Give the tax back and the consumer moves from x^C to $\bar{\bar{x}}$.

Note that the substitution effect and (normal) income effects are reinforcing for commodity 2 but offsetting for commodity 1.

(c) For a borrower the consumer has to be subsidized at x^C . Taking the subsidy back reduces demand for both commodities in the normal case. Now the two effects are reinforcing for commodity 1 and offsetting for commodity 2.

$$Max\{U(x) = x_1^{1/2} + x_2^{1/2} \mid x_1 + \frac{x_2}{1+r} = y_1 + \frac{y_2}{1+r}\} \ .$$

FOC

$$\frac{MU_1}{p_1}=\frac{MU_2}{p_2}:\ \frac{MU_1}{1}=\frac{(1+r)MU_2}{1};\quad \frac{\frac{1}{2}}{x_1^{1/2}}=\frac{\frac{1}{2}(1+r)}{x_2^{1/2}}\ .$$

Then

$$\frac{1}{x_1}=\frac{(1+r)^2}{x_2}=\frac{1+r}{\frac{x_2}{1+r}}=\frac{2+r}{x_1+\frac{x_2}{1+r}}=\frac{2+r}{W}$$

Then $\bar{x}=(\frac{W}{2+r},\frac{(1+r)^2W}{2+r})$.

With three periods

FOC

$$\frac{\frac{1}{2}}{x_1^{1/2}}=\frac{\frac{1}{2}(1+r)}{x_2^{1/2}}=\frac{\frac{1}{2}(1+r)^2}{x_3^{1/2}}\ .$$

$$\frac{1}{x_1}=\frac{(1+r)^2}{x_2}=\frac{(1+r)^4}{x_3}$$

$$\frac{1}{x_1}=\frac{1+r}{\frac{x_2}{1+r}}=\frac{(1+r)^2}{\frac{x_3}{(1+r)^2}}=\frac{1+(1+r)+(1+r)^2}{W}$$

Then solve for $\bar{x}$.

2. Walrasian Equilibrium with production

(a) $U(x)=3\ln x_1+2\ln x_2+\ln x_3$

$$U(\theta x)=3\ln\theta x_1+2\ln\theta x_2+\ln\theta x_3=6\ln\theta+3\ln x_1+2\ln x_2+\ln x_3=6\ln\theta+U(x)\ .$$

Thus if $U(y)\geq U(x)$ then $U(\theta y)\geq U(\theta x)$

(b) $x_1=\omega_1-z_1=80-z_1$, $x_1=\omega_2-z_2=6-z_2$ $x_3=q=z_1^{1/4}z_2^{1/2}$

$$U(x)=3\ln(80-z_1)+2\ln(6-z_2)+\ln z_1^{1/2}z_2^{2/3}$$

$$=3\ln(80-z_1)+2\ln(6-z_2)+\tfrac{1}{3}\ln z_1+\tfrac{2}{3}\ln z_2$$

FOC

$\frac{\partial U}{\partial z_1}=-\frac{3}{80-z_1}+\frac{\frac{1}{3}}{z_1}=0$ Therefore $\frac{3}{80-z_1}=\frac{\frac{1}{3}}{z_1}=\frac{\frac{10}{3}}{80}=\frac{\frac{1}{3}}{8}$ and so $\bar{z}_1=8$

$$\frac{\partial U}{\partial z_2} = -\frac{1}{4-z_2} + \frac{\frac{2}{3}}{z_2} = 0$$. Therefore $\frac{\frac{2}{3}}{z_2} = \frac{2}{4-z_2} = \frac{\frac{8}{3}}{4} = \frac{\frac{2}{3}}{1}$ and so $\bar{z}_2 = 1$

Then $x_1 = 72,\ x_2 = 3$. Also $q_3 = 24(2) = 48$ so $x_3 = 48$

Profit

$$\pi = p_3 z_1^{1/3} z_2^{2/3} - p_1 z_1 - p_2 z_2 \quad \text{with } p_3 = 1$$

$$\frac{\partial \pi}{\partial z_1} = \tfrac{1}{3} 24 z_1^{-2/3} z_2^{2/3} - p_1 = \tfrac{1}{3}\frac{24}{4} - p_1 = 0. \text{ So } p_1 = 2$$

$$\frac{\partial \pi}{\partial z_2} = \tfrac{2}{3} 24 z_1^{1/3} z_2^{-1/3} - p_2 = \tfrac{2}{3} 48 - p_2 = 0. \text{ So } p_2 = 32$$

For the consumer

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \frac{MU_3}{p_3}$$

$$\frac{3}{p_1 x_1} = \frac{2}{p_2 x_2} = \frac{1}{p_3 x_3},$$

$\bar{x} = (72, 3, 48)$ so we require that

$$\frac{3}{(2)72} = \frac{2}{(32)3} = \frac{1}{(1)48}.$$

This is satisfied.

3. State claims equilibrium prices and asset prices

(a) $U(x,\pi) = \tfrac{1}{3} x_1^{1/2} + \tfrac{1}{3} x_2^{1/2} + \tfrac{1}{3} x_3^{1/2}$

$$U(\theta x,\pi) = \tfrac{1}{3}(\theta x_1)^{1/2} + \tfrac{1}{3}(\theta x_2)^{1/2} + \tfrac{1}{3}(\theta x_3)^{1/2} = \theta(\tfrac{1}{3} x_1^{1/2} + \tfrac{1}{3} x_2^{1/2} + \tfrac{1}{3} x_3^{1/2}) = \theta U(x,\pi)$$

(b) FOC

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \frac{MU_3}{p_3}$$

$$\frac{\tfrac{1}{6}\bar{x}_1^{-1/2}}{p_1} = \frac{\tfrac{1}{6}\bar{x}_2^{-1/2}}{p_2} = \frac{\tfrac{1}{6}\bar{x}_3^{-1/2}}{p_3},$$

For the representative consumer $\bar{x} = \omega = (64, 25, 9)$. Therefore

$$\frac{(64)^{-1/2}}{p_1}=\frac{(25)^{-1/2}}{p_2}=\frac{(9)^{-1/2}}{p_3}$$ i.e. $8p_1=5p_2=3p_3$

Then $p=(1,\frac{8}{5},\frac{8}{3})$ is a WE price vector

(c) The value of asset a is $p\cdot z^a=(1,\frac{8}{5},\frac{8}{3})\cdot(40,10,3)=64$

The value of asset b is $p\cdot z^b=(1,\frac{8}{5},\frac{8}{3})\cdot(24,15,6)=64$

(d) If you own all of both assets you own the entire endowment (your optimal consumption)

If you own assets worth 1% of the market portfolio you want to consumer 1%

of ω and you can achieve this by buying 1% of each asset.

(e) From the answer to (d), you simply buy a share of the market portfolio (a mutual fund)

4. Production and Cost

After spending the fixed cost the net budget is $B-18$. Then

$$r_1z_1+r_2z_2=4z_1+8z_2\le B-18$$

$$Max\{q=z_1^{1/4}z_2^{1/2}\,|\,4z_1+8z_2\le B-18\}$$

$$Max\{\tfrac{1}{4}\ln z_1+\tfrac{1}{2}\ln z_2\,|\,4z_1+8z_2\le B-18\}$$

$$\frac{\frac{1}{4}}{4z_1}=\frac{\frac{1}{2}}{8z_2}=\frac{\frac{3}{4}}{4z_1+8z_2}=\frac{\frac{3}{4}}{B-18}\,.$$

From the first equation, $z_1=z_2$.

Therefore $z_1=\dfrac{B-18}{12}$

Therefore

$$q(B)=z_1^{1/4}z_2^{1/2}=z_1^{3/4}=(\frac{B-18}{12})^{3/4}.$$

Suppose the firm wants to produce $\hat{q}$. Choose $\hat{B}$ so that $\hat{q}=q(\hat{B})$. Since $q(B)$ is strictly increasing, with any smaller budget $\hat{q}$ is not feasible. Thus $\hat{B}$ is the smallest budget.

$$q=(\frac{C-64}{12})^{3/4}$$

Therefore

$$C = 64 + 12q^{4/3}$$

$$AC = \frac{64+12q^{4/3}}{q} \;. \quad MC = 16q^{1/3} \;.$$

At the minimum $AC = MC$ and so $4q^{4/3} = 64$. Then $\underline{q} = 8$ and so $p = MC = 32$.

5. Quantity competition

(a) $p_1 - c = 60 - 2q_1 - q_2$.

Then profit is

$$U_1(q_1, q_2) = (p_1 - c)q_1 = 60q_1 - 2q_1^{\,2} - q_2 q_1$$

$\frac{\partial U_1}{\partial q_1} = 60 - 4q_1 - q_2 = 0$ and so the best response is $q_1^b = 15 - \frac{1}{4}q_2$.

By the same argument

$$q_2^b = 15 - \tfrac{1}{4}q_1 \;.$$

Mutual best responses

$q_1^b = 15 - \frac{1}{4}q_2^b$ and $q_1^b = 15 - \frac{1}{4}q_2^b$.

Guess $q_1^b = q_2^b$ and solve $\bar{q} = (12,12)$

(c) $p_1 - c = 60 - 2q_1 - q_2 - q_3$.

$$U_1(q_1, q_2) = (p_1 - c)q_1 = 60q_1 - 2q_1^{\,2} - q_2 q_1 - q_3 q_1$$

$\frac{\partial U_1}{\partial q_1} = 60 - 4q_1 - q_2 - q_3 = 0$ and so the best response is $q_1^b = 15 - \frac{1}{4}(q_2 + q_3)$.

Appealing to symmetry

$q_1^b = 15 - \frac{1}{4}(q_2^b + q_3^b) = 15 - \frac{1}{4}(q_1^b + q_1^b) = 15 - \frac{1}{2}q_1^b$. Then $\bar{q} = (10,10,10)$.

(d) $U_1(q_1, q_2^b) = (p_1 - c)q_1 = 60q_1 - 2q_1^{\,2} - q_2^b q_1 = 60q_1 - 2q_1^{\,2} - (15 - \frac{1}{4}q_1)q_1$

$\frac{\partial U_1}{\partial q_1} = 60 - 4q_1 - 15 + \frac{1}{2}q_1 = 45 - \frac{7}{2}q_1 = 0$. Then $q_1^* = 12\frac{6}{7}$.

(e) When firm 1 raises its output it understands that form 2 will respond by lowering its output and that this will raise p_1. So the marginal revenue is higher.

6. All pay auction

$F(\theta) = \theta - 1$

$U(\theta) = W(\theta)\theta - B(\theta) = F(\theta)\theta - B(\theta)$.

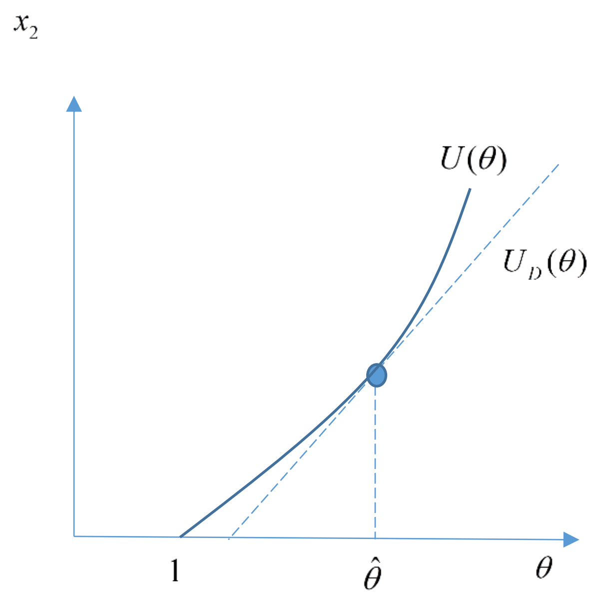
Method presented in course

Consider a deviation from the best response in which a buyer always bids $B(\hat{\theta})$. His payoff is

$U_D(\theta) = F(\hat{\theta})\theta - B(\hat{\theta})$.

Note that this is linear with slope $F(\hat{\theta})$.

The graphs of the two functions are depicted.



It cannot be better to deviate so the two curves have the same slope at $\hat{\theta}$. So $U'(\hat{\theta}) = F(\hat{\theta})$.

This argument holds for all $\hat{\theta}$

Alternative method.

Appeal to the Envelope Theorem as follows:

Any bid b can be written as $b = B(x)$ for some x . If the buyer bids $b = B(x)$ her win probability is $W(x) = \Pr\{\theta_2 \le x\} = F(x)$.

Her expected payoff is therefore

$$u(\theta, x) = W(x)(\theta - B(x)) \quad .$$

For $B(\theta)$ to be a best response, this takes on it maximum at $x = \theta$.

$$U(\theta) = \underset{x}{Max}\{W(x)(\theta - B(x))\} \; .$$

By the Envelope Theorem it follows that $U'(\theta) = \frac{\partial}{\partial\theta}u(\theta, x)\Big|_{x=\theta} = W(\theta)$.

Remark: Note that the class method is actually a proof of the Envelope Theorem for the auction case.

(b) Therefore

$$U'(\theta)=F(\theta)=\theta-1$$

Then

$$U(\theta)=\tfrac{1}{2}(\theta-1)^2+K$$

The lowest type has a zero probability of winning so $U(1)=0$. Then $K=0$

$$U(\underline{\theta})=\tfrac{1}{2}(\underline{\theta}-1)^2+K\ .$$

Therefore

$$(\theta-1)\theta-B(\theta)=\tfrac{1}{2}(\theta-1)^2$$

$$B(\theta)=(\theta-1)\theta-\tfrac{1}{2}(\theta-1)^2=(\theta-1)(\theta-\tfrac{1}{2}(\theta-1))=\tfrac{1}{2}(\theta-1)(1+\theta)$$

(c) $U(\underline{\theta})=(\underline{\theta}-1)\underline{\theta}-B(\underline{\theta})=(\tfrac{3}{2}-1)\tfrac{3}{2}-B(\underline{\theta})=\tfrac{3}{4}-B(\underline{\theta})=0$.

Therefore $B(\underline{\theta})=\tfrac{3}{4}$.

(d) $$U(\theta)=\tfrac{1}{2}(\theta-1)^2+K$$

$U(\underline{\theta})=\tfrac{1}{2}(\underline{\theta}-1)^2+K$ so $K=-\tfrac{1}{8}$.

$$U(\theta)=\tfrac{1}{2}(\theta-1)^2-\tfrac{1}{8}$$

$$U(\theta)=W(\theta)\theta-B(\theta)=F(\theta)\theta-B(\theta)=(\theta-1)\theta-B(\theta).$$

Then

$$(\theta-1)\theta-B(\theta)=U(\theta)=\tfrac{1}{2}(\theta-1)^2-\tfrac{1}{8}$$

7. Auction with multiple units for sale

(a) A bit sketchy but OK answer.

If I win I pay L, the lowest of the other bids. This is independent of my bid. I want to win if my value is higher than L and lose if my value is lower than L.

If I bid my value θ I win if and only if $\theta-L\geq 0$.

A perfect answer

(i) If I bid $b<\theta$ the only time it matters is if the maximum of the other bids, m is between, i.e. $b<m<\theta$. If I had bid θ I would have won and made a profit of $\theta-m$

(ii) If I bid $b>\theta$ the only time it matters is if the maximum of the other bids, m is between, i.e. $\theta<m<b$. If I had bid θ I would have not won. By bidding b my loss is $\theta-m$

(b) The argument above is not based on the strategies of the other players so yes.

$$\Pr\{\theta_i \le \theta\} = F(\theta) = \theta .$$

(c) If $B(\theta)$ is strictly increasing I lose to another buyer if his bid is higher than mine i.e. if his value is higher than mine. This probability is $1-\Pr\{\theta_i \le \theta\} = 1-F(\theta) = 1-\theta$.

I win an item unless I lose to all three other buyers so my loss probability is $(1-\theta)^3$. Then my win probability is

$$W(\theta) = 1-(1-\theta)^3 .$$

(d) My expected payoff is

$$U(\theta) = W(\theta)(\theta - B(\theta)) . \qquad (*)$$

By the argument in (6)

$$U'(\theta) = W(\theta) .$$

A buyer with value 0 has a zero probability of winning so $U(0)=0$.

Then

$$U(\theta) = \int_0^\theta W(x)dx .$$

(e) Solve for $U(\theta)$ and substitute into (*).

(f) In both cases $U(\theta) = \int_0^\theta W(x)dx$ and the win probability is the same. So the buyers are indifferent. Then the seller must be as well.