Answers to Final Practice questions

1 and 7 only

Exercise 1

(a)

(i) $\bar{F}(\frac{1}{2}) = \frac{1}{2}$, $\bar{\bar{F}}(\frac{1}{2}) = 2(\frac{1}{2})^2 = \frac{1}{2}$

(ii) $\bar{F}(\theta) - \bar{\bar{F}}(\theta) = \theta - 2\theta^2 = 2\theta(\frac{1}{2} - \theta) > 0$ for θ in $(0, \frac{1}{2})$.

(iii) $\bar{F}(\theta) - \bar{\bar{F}}(\theta) = (1 - \bar{\bar{F}}(\theta)) - (1 - \bar{F}(\theta))$

$$= 2(1-\theta)^2 - (1-\theta) = (1-\theta)[2(1-\theta) - 1]$$

$$= 2(1-\theta)(\tfrac{1}{2} - \theta) < 0 \text{ for } \theta \text{ in } (\tfrac{1}{2}, 1) \; .$$

(b) Direct method

$$\bar{U}'(\theta) = W(\theta) = F(\theta) = \theta \; .$$

Therefore

$$\bar{U}(\theta) = \bar{U}(0) + \int_0^\theta \theta d\theta = \tfrac{1}{2}\theta^2 \text{ since } \bar{U}(0) = 0 .$$

On $[0, \frac{1}{2}]$ $\bar{\bar{F}}(\theta) = 2\theta^2$.

Therefore

$$\bar{\bar{U}}(\theta) = \bar{\bar{U}}(0) + \int_0^\theta 2\theta^2 d\theta = \tfrac{2}{3}\theta^3 \text{ since } \bar{\bar{U}}(0) = 0 .$$

Therefore $\bar{\bar{U}}(\frac{1}{2}) = \frac{1}{12}$

On $[\frac{1}{2}, 1]$ $\bar{\bar{F}}(\theta) = 1 - 2(1-\theta)^2$.

Therefore

$$\bar{\bar{U}}(1) = \bar{\bar{U}}(\tfrac{1}{2}) + \int_{\frac{1}{2}}^1 (1 - 2(1-\theta)^2) d\theta = \tfrac{1}{12} + \int_{\frac{1}{2}}^1 d\theta - 2\int_{\frac{1}{2}}^1 (1-\theta)^2) d\theta$$

$$\int_{\frac{1}{2}}^1 d\theta = \theta \Big|_{\frac{1}{2}}^1 = 1 - \tfrac{1}{2} = \tfrac{1}{2} ,$$

$$\int_{\frac{1}{2}}^{1} (1-\theta)^2 d\theta = -\tfrac{1}{3}(1-\theta)^3\Big|_{\frac{1}{2}}^{1} = -\tfrac{1}{3}(1-\tfrac{1}{2})^3 = -\tfrac{1}{12}$$

Therefore $\bar{\bar{U}}(1) = \tfrac{1}{2}$.

(b) General method.

Since the distributions are symmetric the mean for both is $\mu = \tfrac{1}{2}$.

In class we showed that

$$\mu = \beta - \int_{\alpha}^{\beta} F(\theta)d\theta = 1 - \int_{0}^{1} F(\theta)d\theta \ .$$

$$U'(\theta) = W(\theta) = F(\theta) = \theta \, .$$

Therefore

$$U(1) = U(0) + \int_{\alpha}^{1} F(\theta)d\theta = 1 - \mu = \tfrac{1}{2} \ .$$

Thus the equilibrium payoff is the same for a buyer with the maximum value.

(c) $U(\theta) = F(\theta)(\theta - B(\theta))$

Therefore

$$U(1) = F(1)(1 - B(1)) = 1 - B(1)$$

From (b) it follows that $B(1) = \tfrac{1}{2}$.

(d)

$\bar{F}(\theta) > \bar{\bar{F}}(\theta)$ for $0 < \theta < \tfrac{1}{2}$. Therefore

$$\bar{U}'(\theta) = \bar{F}(\theta) > \bar{\bar{F}}(\theta) = \bar{\bar{U}}'(\theta) \quad \text{for } 0 < \theta < \tfrac{1}{2} \ . \quad (*)$$

$\bar{F}(\theta) < \bar{\bar{F}}(\theta)$ for $\tfrac{1}{2} < \theta < 1$. Therefore

$$\bar{U}'(\theta) = \bar{F}(\theta) < \bar{\bar{F}}(\theta) = \bar{\bar{U}}'(\theta) \quad \text{for } \tfrac{1}{2} < \theta < 1 .$$

(e) From (*) $\bar{U}(\theta)$ rises more quickly than $\bar{\bar{U}}(\theta)$ when $0 < \theta < \tfrac{1}{2}$ .Since the payoffs are zero for a zero value buyer it follows that

$$\bar{U}(\theta) > \bar{\bar{U}}(\theta) \text{ for } 0 < \theta < \tfrac{1}{2} \ .$$

Also from (b) $\bar{U}(1) = \bar{\bar{U}}(1) = 1-\mu$.

(f) From (**), $\bar{U}(\theta)$ rises less quickly than $\bar{\bar{U}}(\theta)$ when $\frac{1}{2} < \theta < 1$. Since the payoffs are equal for a buyer with value 1 it follows that

$$\bar{U}(\theta) > \bar{\bar{U}}(\theta) \text{ for } \tfrac{1}{2} < \theta < 1 \ .$$

(g)

From (b)

On $[0, \frac{1}{2}]$ $\bar{\bar{F}}(\theta) = 2\theta^2$.

Therefore

$$\bar{\bar{U}}(\theta) = \tfrac{2}{3}\theta^3$$

Also

$$\bar{\bar{U}}(\theta) = \bar{\bar{F}}(\theta)(\theta - \bar{\bar{B}}(\theta)) = 2\theta(\theta - \bar{\bar{B}}(\theta)) = 2\theta^3 - 2\theta^2\bar{\bar{B}}(\theta)$$

Therefore

$$2\theta^2\bar{\bar{B}}(\theta) = \tfrac{4}{3}\theta^3 \text{ and so } \bar{\bar{B}}(\theta) = \tfrac{2}{3}\theta$$

In the uniform case $\bar{B}(\theta) = \frac{1}{2}\theta$.

7. Two items for sale and three buyers

(a) You win unless your bid is lowest. Assuming that the equilibrium bid function, $B(\theta)$, is strictly increasing, it follows that you lose only if your value is the lowest.

An opposing buyer has a higher value than buyer 1 with value θ_1 with probability $1-F(\theta_1)$. Thus both have higher values with probability $(1-F(\theta_1))$. This is the prbabiolity that youlosae.

Thus the equilibrium win probability is

$$W(\theta_1) = 1-(1-F(\theta_1))^2 \ .$$

For the uniform case it follows that

$$W(\theta_1) = 2\theta_1 - \theta_1^2 \ .$$

(b) You pay your bid $B(\theta_1)$ if you win one of the items. Therefore

$$U(\theta_1) = W(\theta_1)(\theta_1 - B(\theta_1)) \ .$$

It follows that if we can solve for the equilibrium payoff we can solve for the equilibrium bid function.

But from the equivalence theorem

$$U'(\theta) = W(\theta)$$

Also $U(0) = 0$.

Thus the solution is obtained in the usual way.

$$U(\theta_1) = \int_0^{\theta_1} U'(\theta)d\theta = \int_0^{\theta_1} W(\theta)d\theta = \int_0^{\theta_1} (2\theta - \theta^2)d\theta = \theta_1^{\ 2} - \tfrac{1}{3}\theta_1^{\ 3}$$

Finally

$$U(\theta_1) = W(\theta_1)(\theta_1 - B(\theta_1)) = (2\theta_1 - \theta_1^{\ 2})(\theta_1 - B(\theta_1))$$

Equate these and solve for $B(\theta_1)$.