Practice questions for the Final

Note: The first question was designed with the last lecture in mind. It is much too long to be a question on the final.

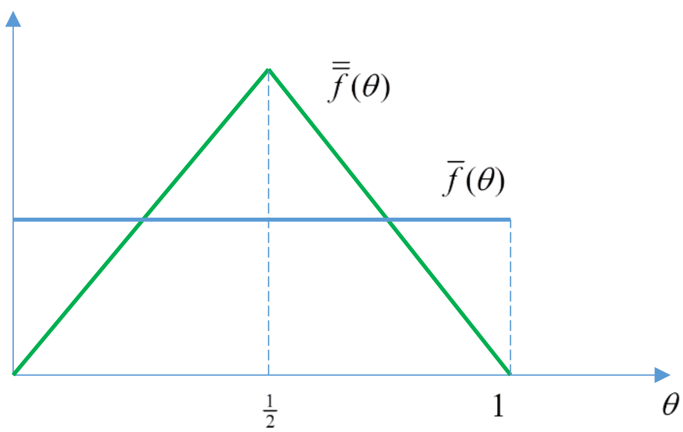
1. Comparing equilibrium bids for two distributions with the same mean

We will discuss a more general version of this in the last lecture.

Values are continuously distributed on $[0,1]$. There are two buyers. In the first auction values are uniformly distributed so the density function is $\bar{f}(\theta)=1$ and c.d.f. is $\bar{F}(\theta)=\theta$. In the second auction the density function is triangular.

$$\bar{\bar{f}}(\theta)=\begin{cases}4\theta, & \theta\le\frac{1}{2}\\ 4(1-\theta), & \theta\ge\frac{1}{2}\end{cases}.$$

The two density functions are depicted below. Since they are symmetric about $\theta=\frac{1}{2}$, the mean value in both cases, $\mathbb{E}[\theta]=\frac{1}{2}$.



Integrating the p.d.f. yields the following c.d.f.

$$\bar{\bar{F}}(\theta)=\begin{cases}2\theta^2, & \theta\le\frac{1}{2}\\ 1-2(1-\theta)^2, & \theta\ge\frac{1}{2}\end{cases}$$

(a) Show that the uniform distribution has more probability mass in both tails, i.e.

(i) $\bar{F}(\frac{1}{2})=\bar{\bar{F}}(\frac{1}{2})=\frac{1}{2}$ (ii) $\bar{F}(\theta)>\bar{\bar{F}}(\theta)$ on $[0,\frac{1}{2}]$ and (iii) that $1-\bar{F}(\theta)>1-\bar{\bar{F}}(\theta)$ on $(\frac{1}{2},1)$.

(b) Show that the equilibrium payoffs, $\bar{U}(1)$ and $\bar{\bar{U}}(1)$ of a buyer with the maximum value are the same in the two auctions.

(c) Hence explain why the maximum bids in the two auctions, $\bar{B}(1)$ and $\bar{\bar{B}}(1)$ are the same.

(d) Use (a) to explain why (i) $\bar{U}'(\theta) > \bar{\bar{U}}'(\theta)$ on $[0,\frac{1}{2}]$ and (ii) $\bar{U}'(\theta) < \bar{\bar{U}}'(\theta)$ on $[\frac{1}{2},1]$.

(e) Hence explain why $\bar{U}(\theta) > \bar{\bar{U}}(\theta)$ on $[0,\frac{1}{2}]$.

(f) Since $\bar{U}(1) = \bar{\bar{U}}(1)$ explain why $\bar{U}(\theta) > \bar{\bar{U}}(\theta)$ on $[\frac{1}{2},1]$.

(g) Show that $\bar{\bar{B}}(\theta) > \bar{B}(\theta)$ if $0 < \theta \le \frac{1}{2}$

Remark: In class we showed that for $\bar{\bar{B}}(\theta) > \bar{B}(\theta)$ if $\frac{1}{2} < \theta < 1$. Thus in this case

$$\bar{\bar{B}}(\theta) > \bar{B}(\theta) \text{ if } 0 < \theta < 1$$

2. Sealed high-bid auction with uniformly distributed values.

Each buyer's value is uniformly distributed on $[0,1]$ in millions of dollars. There are two bidders.

(a) Show that the equilibrium payoff is $B(\theta_i) = (\frac{I-1}{I})\theta_i$.

(b) Suppose instead that values are uniformly distributed on $[\alpha, \alpha+1]$. Is the new equilibrium bid function

$$B(\theta_i) = \alpha + (\frac{I-1}{I})(\theta_i - \alpha) = \frac{1}{I}\alpha + (\frac{I-1}{I})\theta_i$$

3. Sealed high-bid auction with a reserve price and uniformly distributed values.

Each buyer's value is uniformly distributed on $[0,1]$ in millions of dollars. There are two bidders.

(a) Use the formula for the marginal revenue of increasing the reserve price $\underline{\theta}$ to show that the profit-maximizing seller will set a reserve price of $\underline{\theta} = \frac{1}{2}$.

(b) Solve for the equilibrium payoff function $U(\theta)$ if $\underline{\theta} = \frac{1}{2}$.

(c) Hence solve for the equilibrium bid function.

(d) Confirm that $B'(\underline{\theta}) = 0$.

(e) Solve for the profit-maximizing reserve price and equilibrium bid function if there are three buyers.

(f) Is it again the case that $B'(\underline{\theta}) = 0$.

4. Sealed high-bid auction with a reserve price and two buyers

Buyer's values are continuously distributed on $[\alpha, \beta]$.

(a) Explain why the equilibrium marginal payoff with two buyers is $W(\theta) = F(\theta)$.

(b) Use this and the fact that $U(\theta) = W(\theta)(\theta - B(\theta))$ to show that if the reserve price is $\underline{\theta}$,

then $B'(\underline{\theta}) = 0$.

Remark: For the two buyer case the slope of the equilibrium bid function typically remains low for all values.

5. Really sad loser auction

A single item is for sale. Values are uniformly distributed on $[0,1]$ so $F(\theta_i) = \theta_i$. The high bidder wins the item and has his bid refunded. The low bidder loses his bid.

(a) If there are two buyers, show that the equilibrium bid function is $B(\theta) = \dfrac{\frac{1}{2}\theta^2}{1-\theta}$

(b) Solve for the equilibrium bid function if there are three buyers.

6. All pay auction

In the literature an auction in which the seller keeps all of the bids is called an all pay auction. Suppose that each buyer's value is in the interval $\Theta = [0,1]$. The c.d.f. is $F(\theta) = 2\theta - \theta^2$.

Solve for the equilibrium bid function in the all pay auction.

7. Two items for sale and three buyers

Two identical items are for sale. Values are uniformly distributed on $[0,1]$ so $F(\theta_i) = \theta_i$. The two highest bids are the winning bids. Each buyer wants only one unit. Each winner pays his own bid. You should assume that the equilibrium bid function is strictly increasing.

(a) What is the equilibrium probability that a buyer with value θ loses? Use this to show that the equilibrium win probability is $W(\theta) = 2\theta - \theta^2$.

(b) Hence solve for the equilibrium payoff $U(\theta)$.

(c) Use this to solve for the equilibrium bid function.

Answers

Exercise 1

(a)

(i) $\bar{F}(\frac{1}{2})=\frac{1}{2}$, $\bar{\bar{F}}(\frac{1}{2})=2(\frac{1}{2})^2=\frac{1}{2}$

(ii) $\bar{F}(\theta)-\bar{\bar{F}}(\theta)=\theta-2\theta^2=2\theta(\frac{1}{2}-\theta)>0$ for θ in $(0,\frac{1}{2})$.

(iii) $\bar{F}(\theta)-\bar{\bar{F}}(\theta)=(1-\bar{\bar{F}}(\theta))-(1-\bar{F}(\theta))$

$$=2(1-\theta)^2-(1-\theta)=(1-\theta)[2(1-\theta)-1]$$

$$=2(1-\theta)(\tfrac{1}{2}-\theta)<0 \text{ for } \theta \text{ in } (\tfrac{1}{2},1) \text{ .}$$

(b) Direct method

$$\bar{U}'(\theta)=W(\theta)=F(\theta)=\theta \text{ .}$$

Therefore

$$\bar{U}(\theta)=\bar{U}(0)+\int_0^\theta \theta d\theta=\tfrac{1}{2}\theta^2 \text{ since } \bar{U}(0)=0 \text{.}$$

On $[0,\frac{1}{2}]$ $\bar{\bar{F}}(\theta)=2\theta^2$.

Therefore

$$\bar{\bar{U}}(\theta)=\bar{\bar{U}}(0)+\int_0^\theta 2\theta^2 d\theta=\tfrac{2}{3}\theta^3 \text{ since } \bar{\bar{U}}(0)=0 \text{.}$$

Therefore $\bar{\bar{U}}(\frac{1}{2})=\frac{1}{12}$

On $[\frac{1}{2},1]$ $\bar{\bar{F}}(\theta)=1-2(1-\theta)^2$.

Therefore

$$\bar{\bar{U}}(1)=\bar{\bar{U}}(\tfrac{1}{2})+\int_{\frac{1}{2}}^1 (1-2(1-\theta)^2)d\theta=\tfrac{1}{12}+\int_{\frac{1}{2}}^1 d\theta-2\int_{\frac{1}{2}}^1 (1-\theta)^2)d\theta$$

$$\int_{\frac{1}{2}}^1 d\theta=\theta \Big|_{\frac{1}{2}}^1=1-\tfrac{1}{2}=\tfrac{1}{2} \text{,}$$

$$\int_{\frac{1}{2}}^1 (1-\theta)^2 d\theta=-\tfrac{1}{3}(1-\theta)^3\Big|_{\frac{1}{2}}^1=-\tfrac{1}{3}(1-\tfrac{1}{2})^3=-\tfrac{1}{12}$$

Therefore $\bar{\bar{U}}(1) = \frac{1}{2}$.

(b) General method.

Since the distributions are symmetric the mean for both is $\mu = \frac{1}{2}$.

In class we showed that

$$\mu = \beta - \int_{\alpha}^{\beta} F(\theta)d\theta = 1 - \int_{0}^{1} F(\theta)d\theta \ .$$

$$U'(\theta) = W(\theta) = F(\theta) = \theta \, .$$

Therefore

$$U(1) = U(0) + \int_{\alpha}^{1} F(\theta)d\theta = 1 - \mu = \tfrac{1}{2} \ .$$

Thus the equilibrium payoff is the same for a buyer with the maximum value.

(c) $U(\theta) = F(\theta)(\theta - B(\theta))$

Therefore

$$U(1) = F(1)(1 - B(1)) = 1 - B(1)$$

From (b) it follows that $B(1) = \frac{1}{2}$.

(d)

$\bar{F}(\theta) > \bar{\bar{F}}(\theta)$ for $0 < \theta < \frac{1}{2}$. Therefore

$$\bar{U}'(\theta) = \bar{F}(\theta) > \bar{\bar{F}}(\theta) = \bar{\bar{U}}'(\theta) \quad \text{for } 0 < \theta < \tfrac{1}{2} \ . \quad (*)$$

$\bar{F}(\theta) < \bar{\bar{F}}(\theta)$ for $\frac{1}{2} < \theta < 1$. Therefore

$$\bar{U}'(\theta) = \bar{F}(\theta) < \bar{\bar{F}}(\theta) = \bar{\bar{U}}'(\theta) \quad \text{for } \tfrac{1}{2} < \theta < 1 \, .$$

(e) From (*) $\bar{U}(\theta)$ rises more quickly than $\bar{\bar{U}}(\theta)$ when $0 < \theta < \frac{1}{2}$ .Since the payoffs are zero for a zero value buyer it follows that

$$\bar{U}(\theta) > \bar{\bar{U}}(\theta) \text{ for } 0 < \theta < \tfrac{1}{2} \ .$$

Also from (b) $\bar{U}(1) = \bar{\bar{U}}(1) = 1 - \mu$.

(f) From (**), $\bar{U}(\theta)$ rises less quickly than $\bar{\bar{U}}(\theta)$ when $\frac{1}{2} < \theta < 1$. Since the payoffs are equal for a buyer with value 1 it follows that

$$\bar{U}(\theta) > \bar{\bar{U}}(\theta) \text{ for } \tfrac{1}{2} < \theta < 1 .$$

(g)

From (b)

On $[0, \frac{1}{2}]$ $\bar{\bar{F}}(\theta) = 2\theta^2$.

Therefore

$$\bar{\bar{U}}(\theta) = \tfrac{2}{3}\theta^3$$

Also

$$\bar{\bar{U}}(\theta) = \bar{\bar{F}}(\theta)(\theta - \bar{\bar{B}}(\theta)) = 2\theta(\theta - \bar{\bar{B}}(\theta)) = 2\theta^3 - 2\theta^2 \bar{\bar{B}}(\theta)$$

Therefore

$$2\theta^2 \bar{\bar{B}}(\theta) = \tfrac{4}{3}\theta^3 \text{ and so } \bar{\bar{B}}(\theta) = \tfrac{2}{3}\theta$$

In the uniform case $\bar{B}(\theta) = \frac{1}{2}\theta$.

7. Two items for sale and three buyers

(a) You win unless your bid is lowest. Assuming that the equilibrium bid function, $B(\theta)$, is strictly increasing, it follows that you lose only if your value is the lowest.

An opposing buyer has a higher value than buyer 1 with value θ_1 with probability $1 - F(\theta_1)$. Thus both have higher values with probability $(1 - F(\theta_1))$. This is the prbabiolity that youlosae.

Thus the equilibrium win probability is

$$W(\theta_1) = 1 - (1 - F(\theta_1))^2 .$$

For the uniform case it follows that

$$W(\theta_1) = 2\theta_1 - \theta_1^2 .$$

(b) You pay your bid $B(\theta_1)$ if you win one of the items. Therefore

$$U(\theta_1) = W(\theta_1)(\theta_1 - B(\theta_1)) .$$

It follows that if we can solve for the equilibrium payoff we can solve for the equilibrium bid function.

But from the equivalence theorem

$$U'(\theta) = W(\theta)$$

Also $U(0) = 0$.

Thus the solution is obtained in the usual way.

$$U(\theta_1) = \int_0^{\theta_1} U'(\theta)d\theta = \int_0^{\theta_1} W(\theta)d\theta = \int_0^{\theta_1} (2\theta - \theta^2)d\theta = \theta_1^2 - \tfrac{1}{3}\theta_1^3$$

Finally

$$U(\theta_1) = W(\theta_1)(\theta_1 - B(\theta_1)) = (2\theta_1 - \theta_1^2)(\theta_1 - B(\theta_1))$$

Equate these and solve for $B(\theta_1)$.