Firms and returns to scale

A. Increasing returns to scale and monopoly

Production plan: a vector of inputs and outputs, (z,q)

Production set of a firm: The set S^f of feasible plans for firm f

Efficient production plan

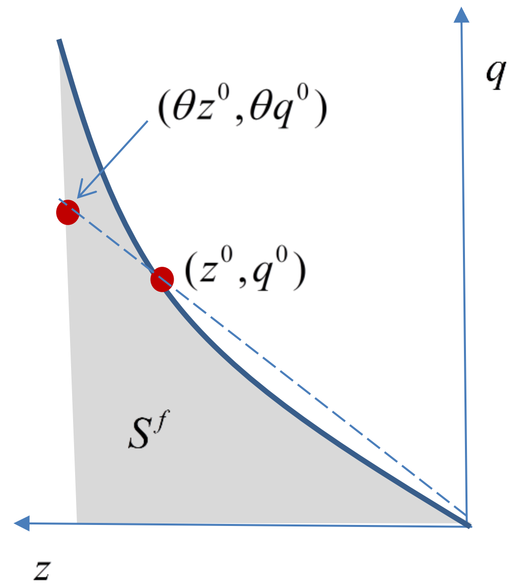
(z^0,q^0) is production efficient if no plan (z,q) satisfying $z \le z^0$ and $q > q^0$ is feasible.

IRS

The production set S^f exhibits increasing returns to scale

on $Q = \{q \mid \underline{q} \le q \le \overline{q}\}$ if for all q and θq in Q

$(z,q) \in S$ implies that $(\theta z, \theta q) \in \operatorname{int} S^f$.

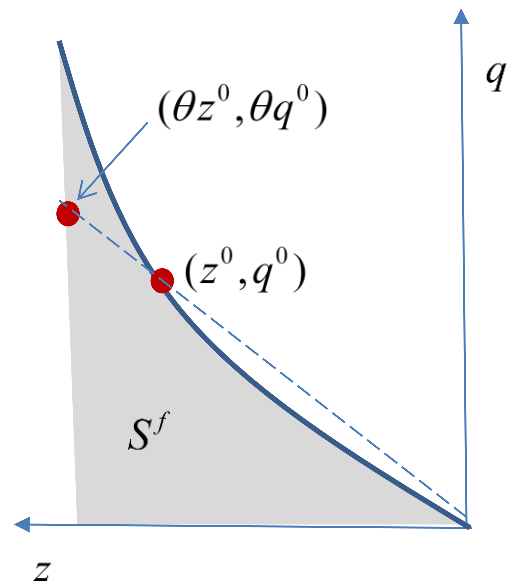


Increasing returns to scale

Cost Function of a firm

$$C(q,r) = \underset{z}{Min}\{r \cdot z \,|\, (z,q) \in S^f\}$$

The cheapest way to produce q units.



Increasing returns to scale

Proposition: If a firm exhibits IRS on $Q = \{q \,|\, \underline{q} \le q \le \overline{q}\}$, then for any q^0 and $\theta q^0 > q^0$ in this interval

$$C(\theta q^0) < \theta C(q^0)$$

Proof:

Since $(\theta z^0, \theta q^0)$ is feasible and costs $\theta C(q^0)$

It follows that $C(\theta q^0) \le r \cdot (\theta z^0) = \theta r \cdot z^0 = \theta C(q^0)$

Since θz^0 is not a boundary point we can lower some

Input and still produce the same output. Therefore

$$C(\theta q^0) < \theta r \cdot z^0 = \theta C(q^0)$$

Corollary: If a firm exhibits IRS everywhere, then the firm cannot be a price taker.

Proof:

$$\frac{R(\theta q)}{C(\theta q)} = \frac{p \cdot \theta q}{C(\theta q)} > \frac{p \cdot \theta q}{\theta C(q)} = \frac{\theta p \cdot q}{\theta C(q)} = \frac{p \cdot q}{C(q)} = \frac{R(q)}{C(q)}. \qquad (*)$$

**

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If q^0 is profitable so that $\frac{R(q^0)}{C(q^0)} \geq 1$ then $q^1 = \theta q^0$ with $\theta > 1$ is strictly profitable.

*

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If q^0 is profitable so that $\dfrac{R(q^0)}{C(q^0)} \geq 1$ then for any $\hat{\theta} > 1,\ q^1 = \hat{\theta} q^0$ is strictly profitable.

From (*) it follows that for all $\theta > 1$

$$\frac{R(\theta q^1)}{C(\theta q^1)} > \frac{R(q^1)}{C(q^1)} > 1$$

So it is always more profitable to expand

QED

Single output firm

If there is a single output, then for every z, the production efficient output is the maximum feasible output.

We write this as $q = F(z)$.

This is called the firm's "**production function**".

The production set is $S^f = \{(z,q) \mid q \leq F(z)\}$

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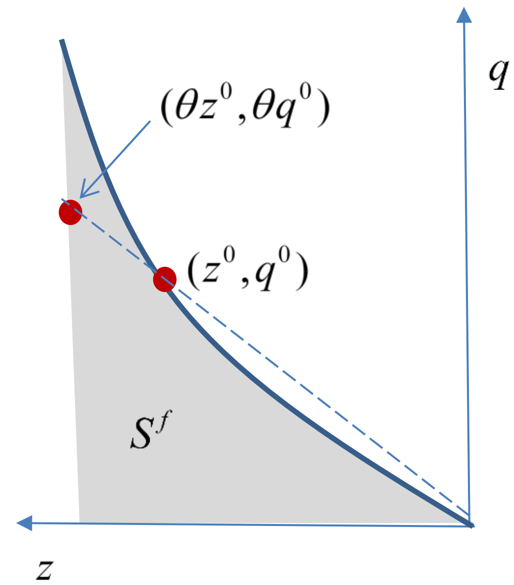

Increasing returns to scale

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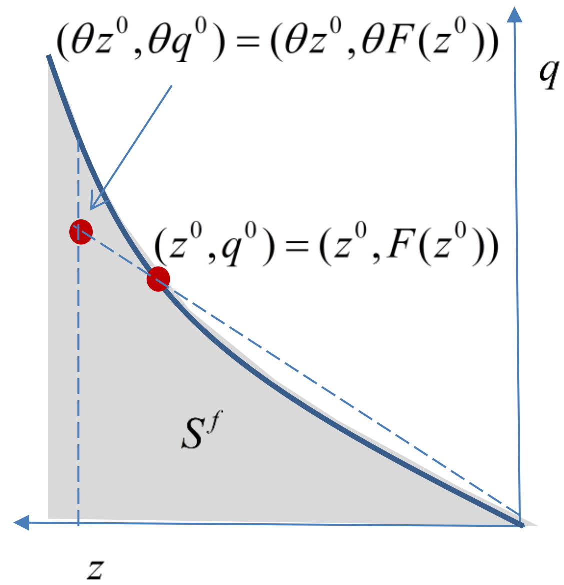


Increasing returns to scale

IRS

The production function exhibits increasing returns to scale

on $Q = \{q \mid \underline{q} \le q \le \overline{q}\}$ if, for any q^0 and $\theta q^0 > q^0$ in this interval,

if $q^0 = F(z^0)$ then $(\theta z^0, \theta q^0) = (\theta z^0, \theta F(z^0)) \in \operatorname{int} S$.

Therefore $\theta F(z^0)$ is not maximized output and so $F(\theta z^0) > \theta F(z^0)$.

Group exercise

$$q = F(z) = z_1^{1/2} + z_2^{1/2}$$

As the manager you are given a fixed budget $\bar{B}$, the input price vector is r . Your objective is to choose the input vector $\bar{z}$ that maximizes output.

(a) Show that $\bar{z}_1 = \dfrac{1}{r_1^2}\dfrac{\bar{B}}{(\frac{1}{r_1}+\frac{1}{r_2})}$ and solve for $\bar{z}_2$.

(b) Show that maximized output is $\bar{q} = (\frac{1}{r_1}+\frac{1}{r_2})^{1/2}\bar{B}^{1/2}$.

(c) What is the lowest cost of producing $\bar{q}$?

B. Natural monopoly

A firm that exhibits IRS for all output levels .

Consider the single product case. Then for all $\theta > 1$

$$AC(\theta q) = \frac{C(\theta q)}{\theta q} < \frac{\theta C(q)}{\theta q} = AC(q)$$

Smaller firms have higher average costs so it

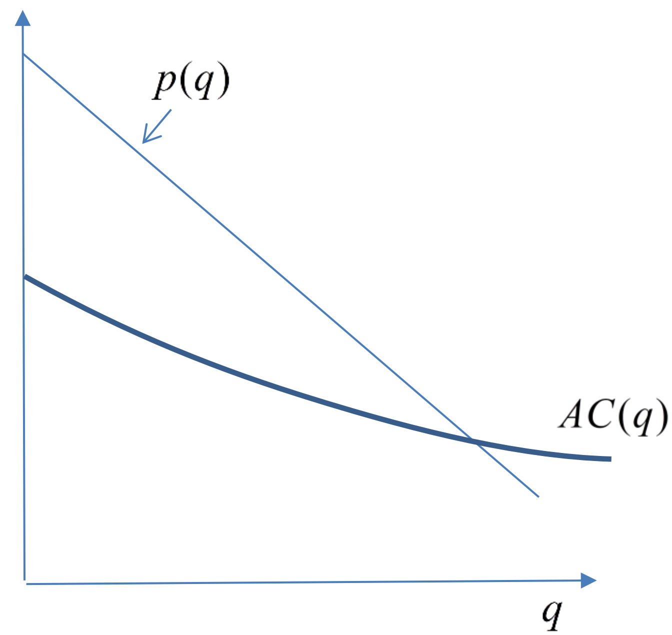
Is less costly for all the output to be produced by

a single firm.

Moreover a larger firm with lower average costs

can drive a firm out of business by lowering prices

and still remain profitable.



Decreasing average cost

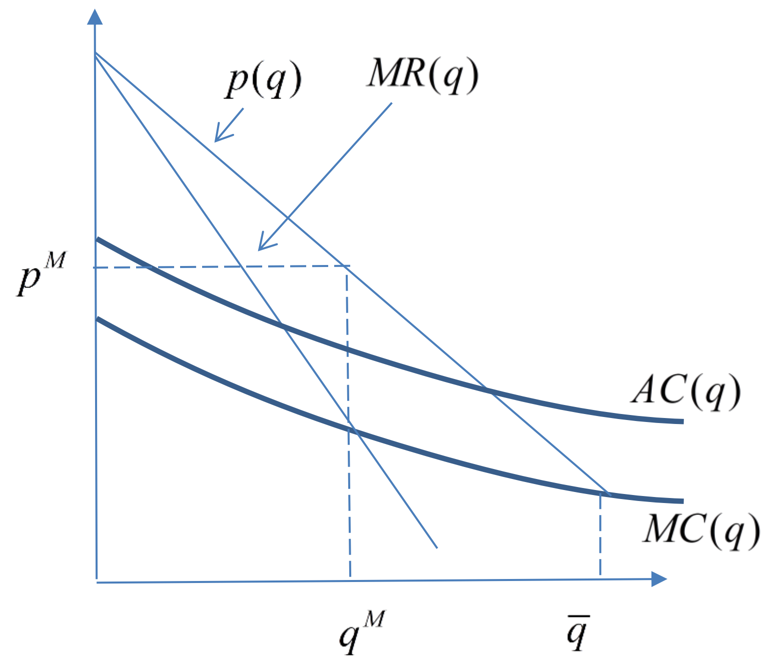
Invert the market demand curve and consider the demand price function $p(q)$, i.e. the price for which demand is q.

$$\pi(q) = R(q) - C(q) = qp(q) - C(q)$$

$$\pi'(q) = R'(q) - C'(q)$$

$$= p(q) + qp'(q) - C'(q)$$

$$< p(q) - C'(q)\ .$$



Monopoly output and price

Profit is maximized at q^M where

$$\pi'(q) = MR(q) - MC(q) = 0$$

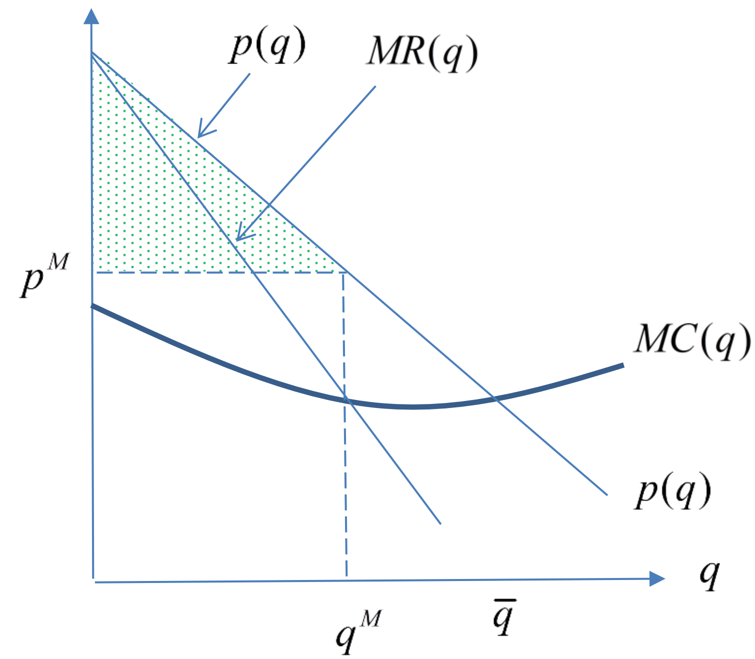
Note that

$$p^M - MC(q^M) > 0$$

Consumer surplus

The green dotted region to the left of the demand curve

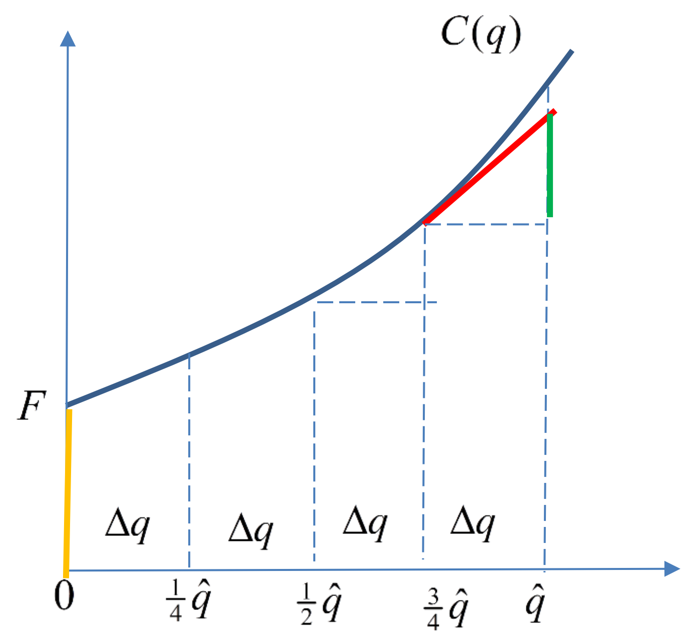
Is the consumer surplus (net benefit).



Consumer surplus

Total cost

Suppose that you know the fixed cost F.



You also know the marginal cost at

$q = \{0, \frac{1}{4}\hat{q}, \frac{1}{2}\hat{q}, \frac{3}{4}\hat{q}\}$

i.e. in increments of $\Delta q = \frac{1}{4}\hat{q}$.

So you know the slope of the cost curve at

each of these points.

Suppose output is $\frac{3}{4}\hat{q}$.

$MC(\frac{3}{4}\hat{q})$ is the slope of the red line.

$MC(\frac{3}{4}\hat{q})\Delta q$ is the length of the green line

It is an estimate of the increase in the firm's cost as output increases from $\frac{3}{4}\hat{q}$ to $\hat{q}$

$$MC(\tfrac{3}{4}\hat{q})\Delta q \approx C(\hat{q}) - C(\tfrac{3}{4}\hat{q})$$

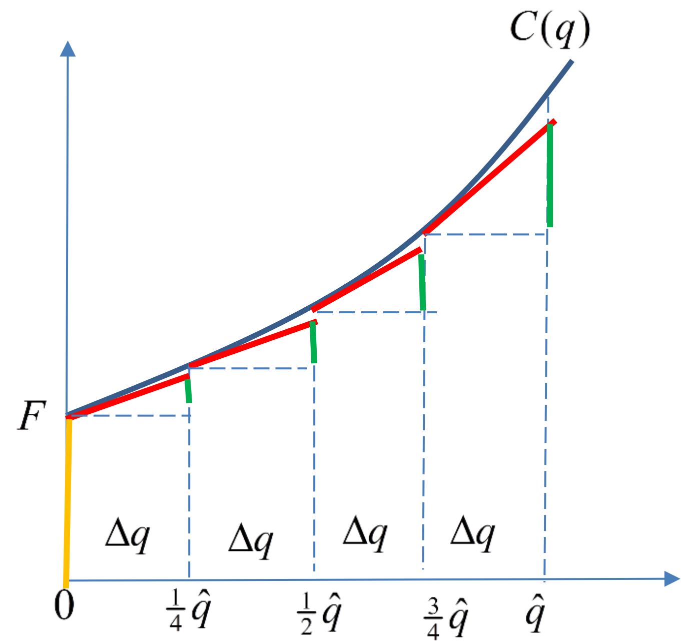
By the same argument,

$C(\tfrac{1}{4}\hat{q}) - C(0) \approx MC(0)\Delta q$

$C(\tfrac{1}{2}\hat{q}) - C(\tfrac{1}{4}\hat{q}) \approx MC(\tfrac{1}{4}\hat{q})\Delta q$

$C(\tfrac{3}{4}\hat{q}) - C(\tfrac{1}{2}\hat{q}) \approx MC(\tfrac{1}{2}\hat{q})\Delta q$

$C(\hat{q}) - C(\tfrac{3}{4}\hat{q}) \approx MC(\tfrac{3}{4}\hat{q})\Delta q$



Adding these together,

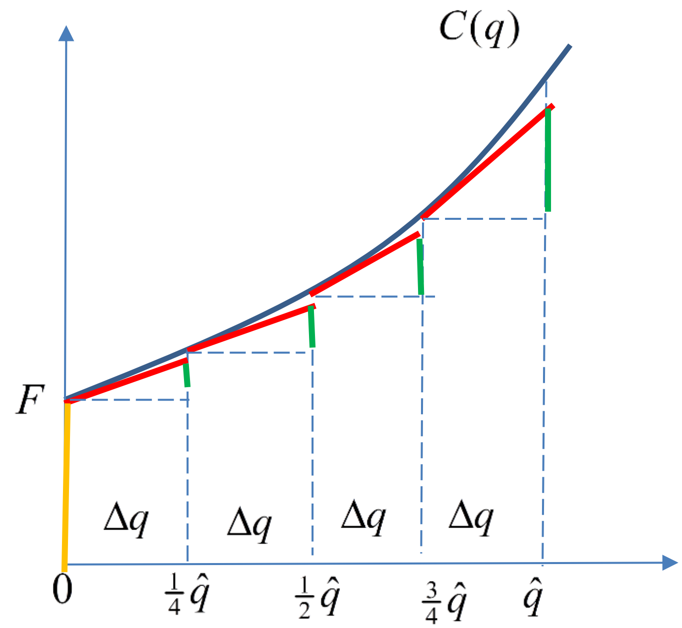
$C(\hat{q}) - C(0) \approx$

$MC(0)\Delta q + MC(\tfrac{1}{4}\hat{q})\Delta q + MC(\tfrac{1}{2}\hat{q})\Delta q + MC(\tfrac{3}{4}\hat{q})\Delta q$

$C(\hat{q}) - C(0)$ is the cost of increasing output from zero to $\hat{q}$,

i.e. the firm's Variable Cost $VC(\hat{q})$

Adding the fixed cost F we have an approximation of the total cost $C(\hat{q})$

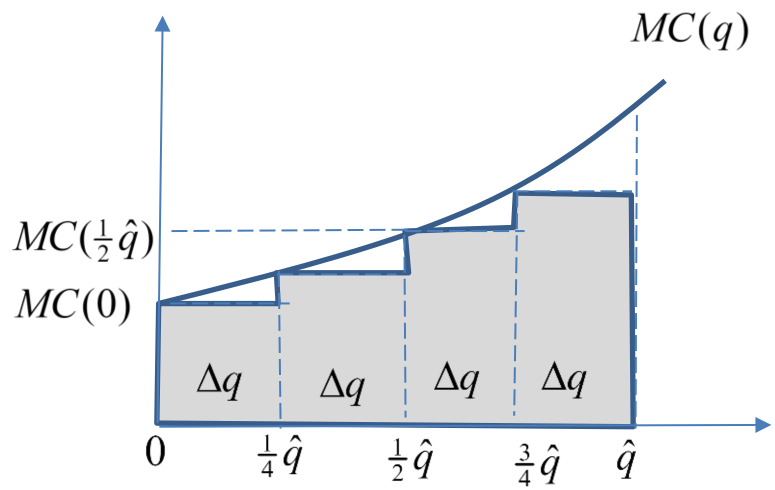


Variable cost

As we have seen, this can be approximated as follows:

$$VC(\hat{q}) \approx MC(0)\Delta q + MC(\tfrac{1}{4}\hat{q})\Delta q + MC(\tfrac{1}{2}\hat{q})\Delta q + MC(\tfrac{3}{4}\hat{q})\Delta q$$

This is also the shaded area under the firm's



Marginal cost curve

The smaller the increment in quantity Δq

the more accurate is the approximation.

In the limit it is the area under the cost curve.

$$VC'(q) = MC(q)$$

$$VC(\hat{q}) = \int_0^{\hat{q}} MC(q)dq$$

Welfare analysis

Consumer surplus CS

The green dotted region to the left of the demand curve

Is the consumer surplus (net benefit).

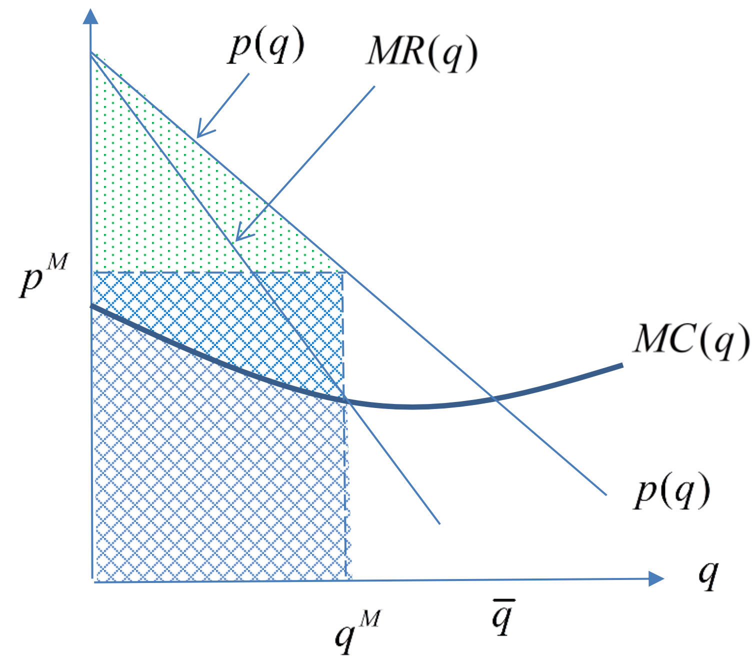
Total Benefit

The total cost to the consumers is $p^M q^M$.

This is the cross-hatched area.

Thus the sum of all the shaded areas is a

measure of the total benefit to the consumers



Total benefit $B(q)$

Social surplus $SS(q)$

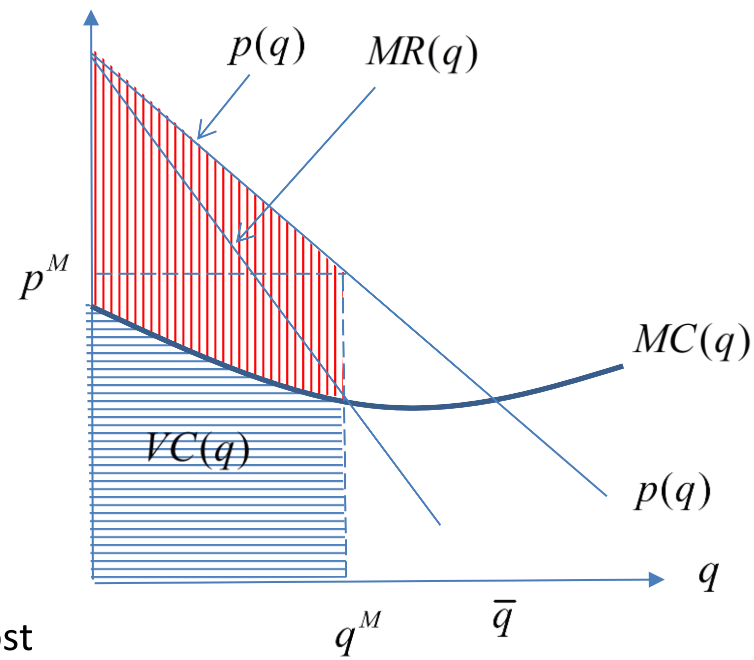
Since $\dfrac{d}{dq}TC = MC$,

$$TC(q^M) = TC(0) + \int_0^{q^M} MC(q)$$

$$= F + \int_0^{q^M} MC(q)$$

where F is the fixed cost.

Thus the blue horizontally lined region is the variable cost



Social surplus

The vertically lined region is the total benefit to

the consumers less the variable cost.

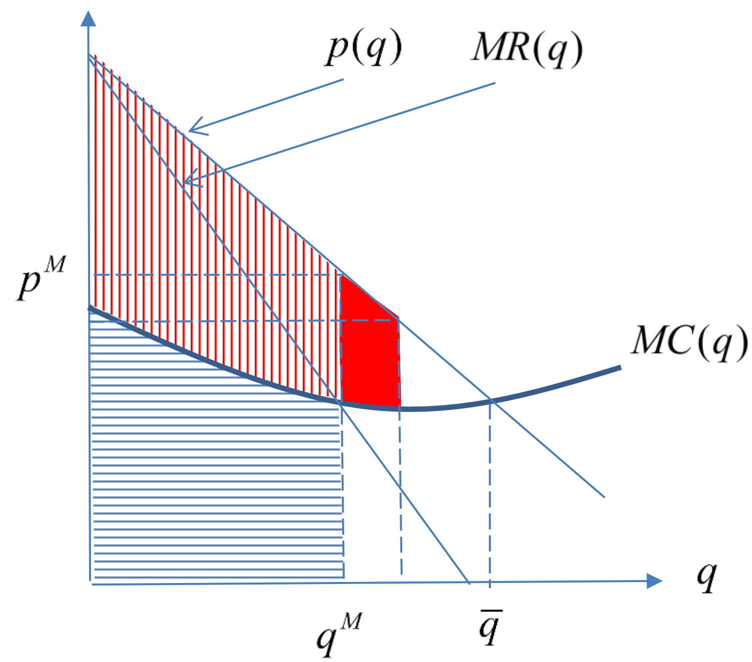
This is called the social surplus.

Society gains as long as the social surplus exceeds the fixed cost

As long as $p(q) > MC(q)$, social surplus rises as the price is reduced.

Thus a monopoly always undersupplies relative to the social optimum.

Hence there is always a potential gain to the regulation of monopoly



Undersupply

Markup over marginal cost

$$MR = \frac{d}{dq}qp(q) = p(q) + q\frac{dp}{dq} = p(q)(1 + \frac{q}{p}\frac{dp}{dq})$$

The second term is the inverse of the price elasticity. Therefore

$$MR(q) = p(q)(1 + \frac{1}{\varepsilon(q,p)}) \ .$$

*

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$$MR(q) = p(q)(1 + \frac{1}{\varepsilon(q,p)}) \ .$$

The monopoly chooses p^M so that

$$MR(q) = p(q)(1 + \frac{1}{\varepsilon(q,p)}) = MC(q) \, .$$

Therefore

$$p^M = \frac{MC(q)}{1 + \dfrac{1}{\varepsilon(q,p)}}$$

Elasticity is negative. A more negative ε denotes a more elastic demand curve.

Thus the markup is lower when demand is more elastic.

Class discussion: Market segmentation and price discrimination

C. Constant returns to scale CRS

The production set S exhibits constant returns to scale

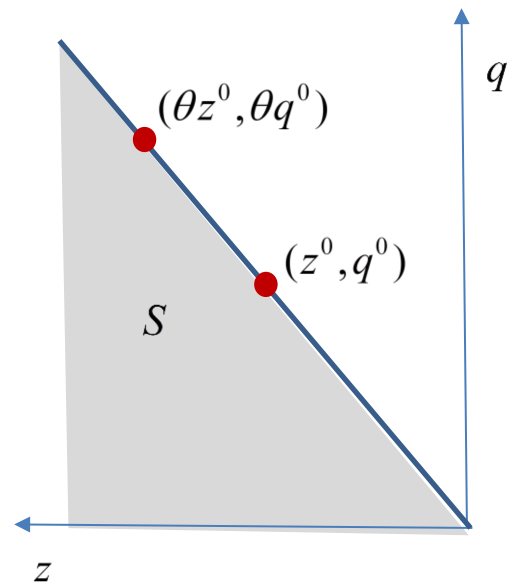
if for all $(z,q)\in S$ and all $\theta>0$, $(\theta z,\theta q)\in S$.

Proposition: If a firm exhibits CRS then for any q^0 and $\theta q^0>q^0$ in Q

$$C(\theta q^0)=\theta C(q^0)$$

Group Exercise: Prove this by showing that if z^0 is cost minimizing for q^0 then θz^0 is cost minimizing for θq^0 .

Remark: If a firm exhibits CRS everywhere, then equilibrium profit must be zero for a price taker.



Constant returns to scale

Single product firm with production function $F(z)$

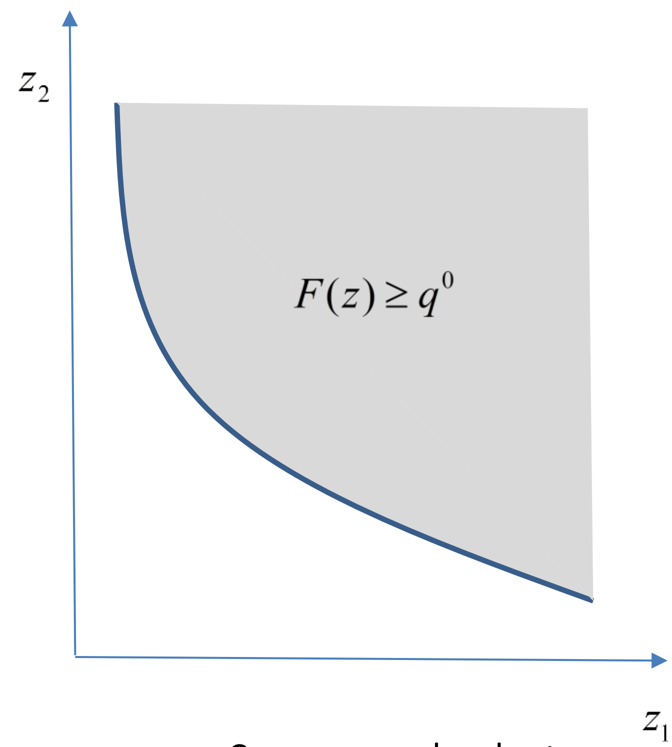
CRS

The production set S exhibits constant returns to scale if, for any q^0 and $\theta q^0 > q^0$, if $q^0 = F(z^0)$ then $F(\theta z^0) = \theta F(z^0)$.

Remark: It follows that $F(z)$ is a homothetic function

Proposition: Concavity of a CRS production function

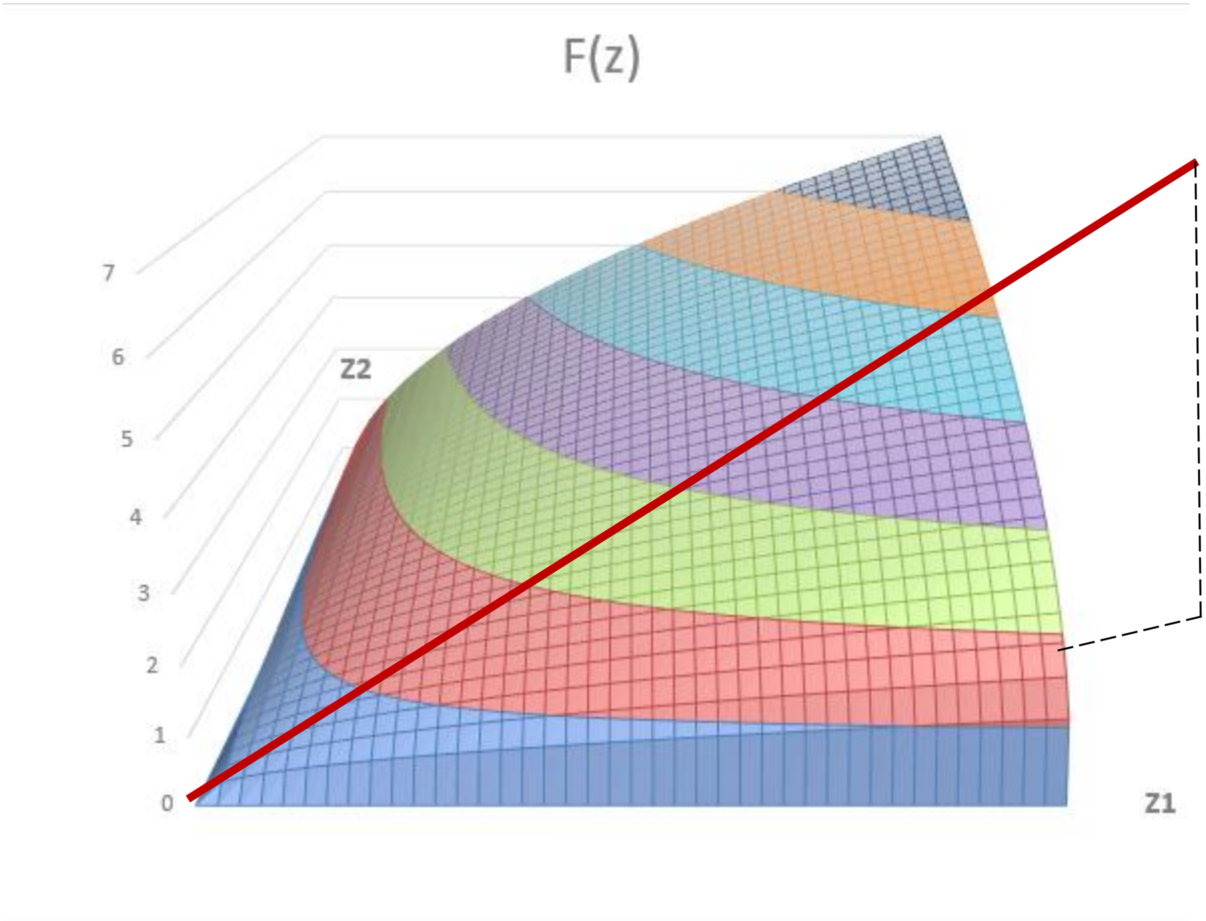
For a single product firm, if the production function $F(z)$ exhibits constant returns to scale and has convex superlevel sets, then $F(z)$ is a concave function.



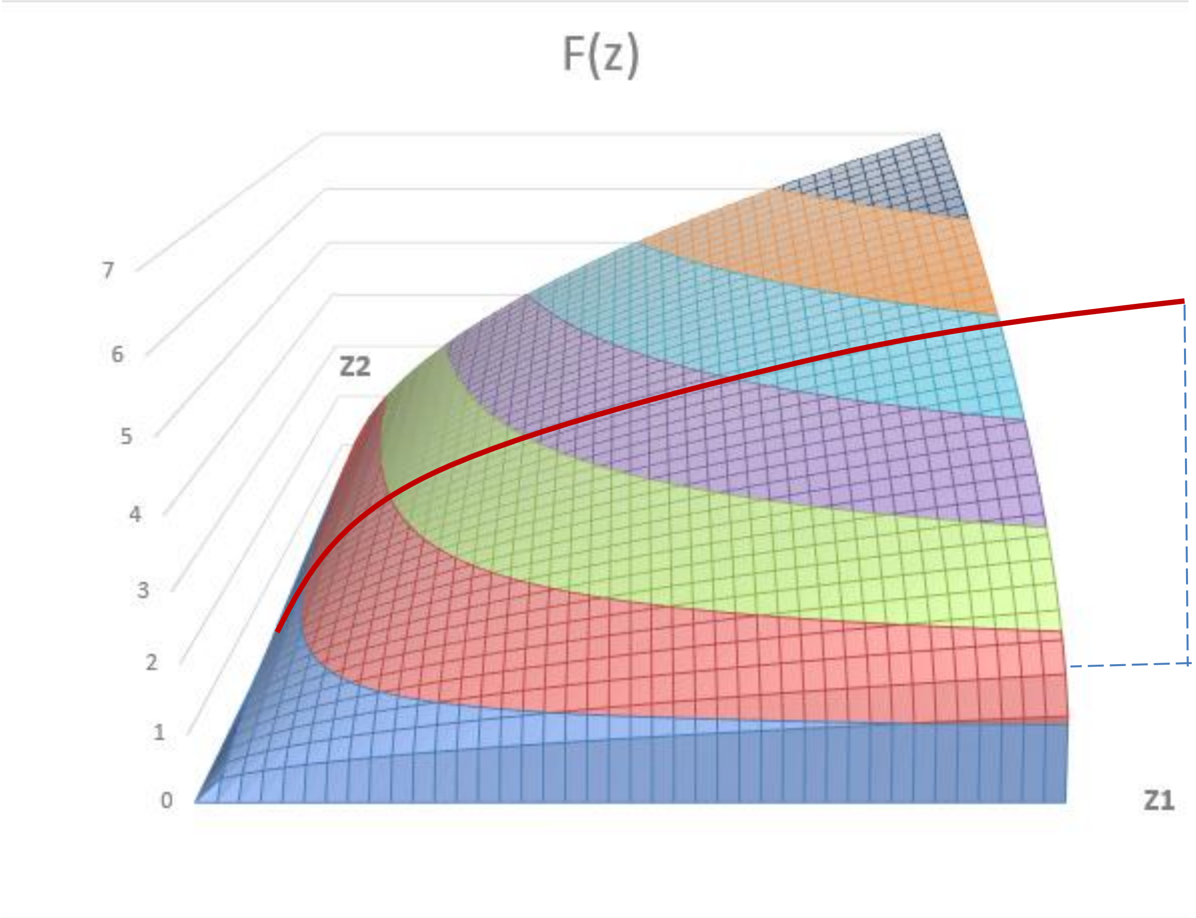
Convex superlevel set

(Those interested can read the derivation of this result in the Appendix.)

Example: Cobb-Douglas production function $F(z) = a_0 z_1^{a_1} z_2^{a_2}$ **where** $a_1 + a_2 = 1$



Diminishing marginal product of each input



Exercise: CRS production function

Show that $q = \dfrac{1}{(\frac{a_1}{z_1^{\,2}} + \frac{a_2}{z_2^{\,2}} + \frac{a_3}{z_3^{\,2}})^{1/2}})$ exhibits constant returns to scale.

Explain why $S = \{z \mid g(z) \equiv k - \frac{a_1}{z_1^{\,2}} - \frac{a_2}{z_2^{\,2}} - \frac{a_3}{z_3^{\,2}} \geq 0\}$ is a superlevel set.

Explain why the super level sets are convex.

Solve for the maximum output for any input price vector r.

Hence show that minimized total cost $C(q,r)$ is a linear function of q .

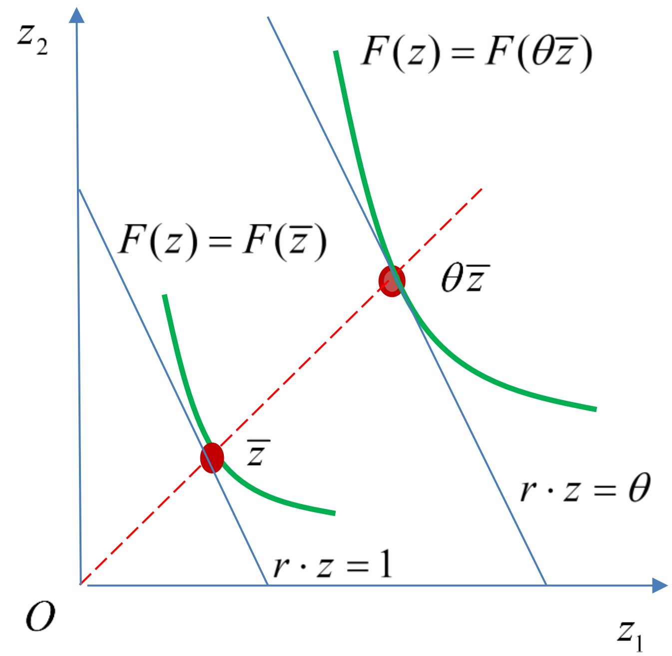
D. The two input two output constant returns to scale economy

Review:

A CRS function is $F(z)$ is homothetic since $F(\theta z)=\theta F(z)$.

Therefore if $\bar{z}$ solves $\underset{z}{Max}\{F(z)\,|\,r_1z_1+r_2z_2\le 1\}$

we have shown that $\theta\bar{z}$ solves $\underset{z}{Max}\{F(z)\,|\,r_1z_1+r_2z_2\le \theta\}$



$\dfrac{\dfrac{\partial F}{\partial z_1}(\bar{z})}{\dfrac{\partial F}{\partial z_2}(\bar{z})}$ is called the marginal rate of technical substitution

FOC: $MRTS(\bar{z})=\dfrac{r_1}{r_2}$ and $MRTS(\theta\bar{z})=\dfrac{r_1}{r_2}$

Thus the MRTS is constant along that ray through $\bar{z}$.

D. The two input two output constant returns to scale economy

The model

Commodities 1 and 2 inputs.

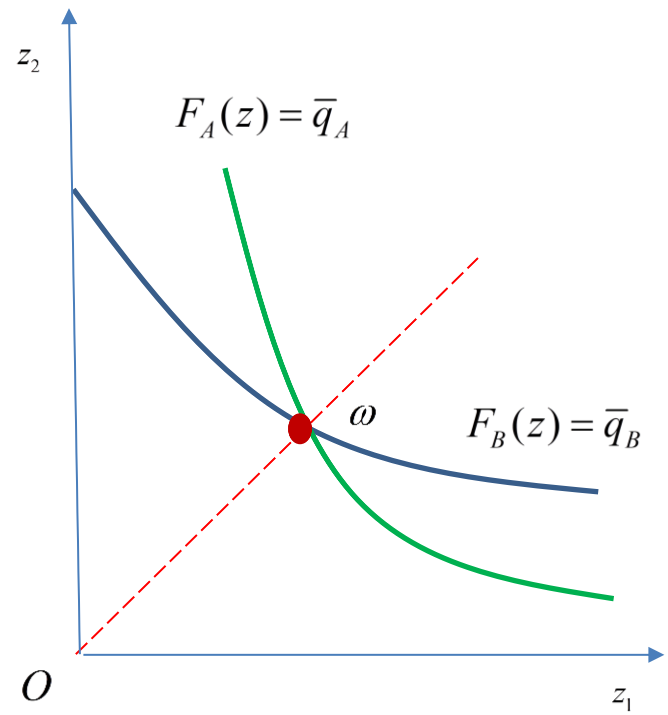
Commodities A and B are consumed goods.

CRS productions functions:

$q_A = F_A(z^A)$, $q_B = F_B(z^B)$.

Total supply of inputs is fixed:

$$z^A + z^B \leq \omega \ .$$



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Total supply of inputs is fixed:

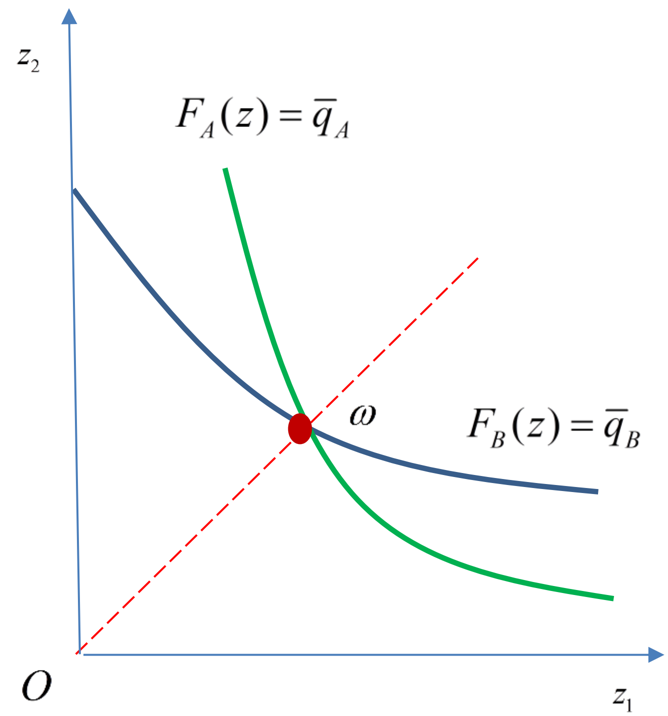
$$z^A + z^B \le \omega\ .$$

The superlevel sets are strictly convex.

Hence both production functions are concave.

At the aggregate endowment, the marginal rate of technical substitution is greater for commodity 1.

$$MRTS_A(\omega) = \frac{\frac{\partial F_A}{\partial z_1}}{\frac{\partial F_A}{\partial z_2}} = \frac{\frac{\partial F_B}{\partial z_1}}{\frac{\partial F_B}{\partial z_2}} > MRTS_B(\omega)\ .$$



Input intensity

At the aggregate endowment

$$MRTS_A(\omega) > MRTS_B(\omega)$$

Graphically the level set is steeper for

commodity A at the red marker.

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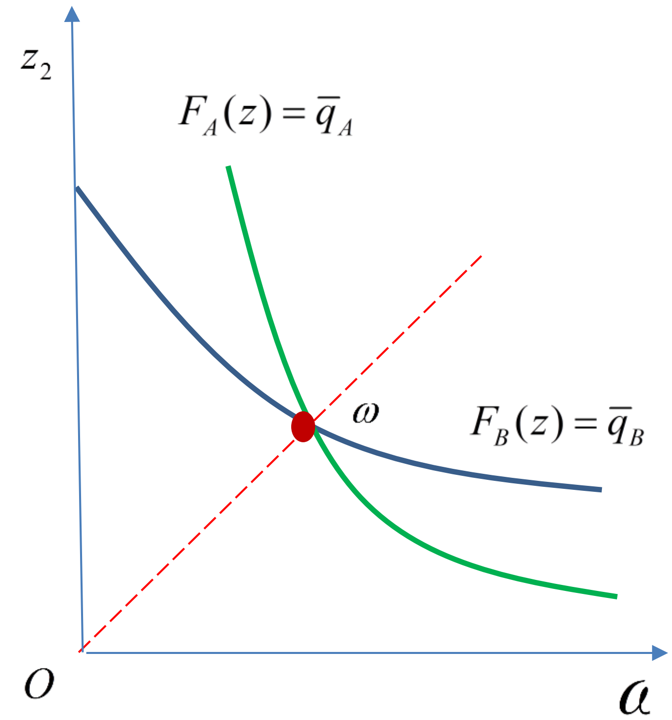

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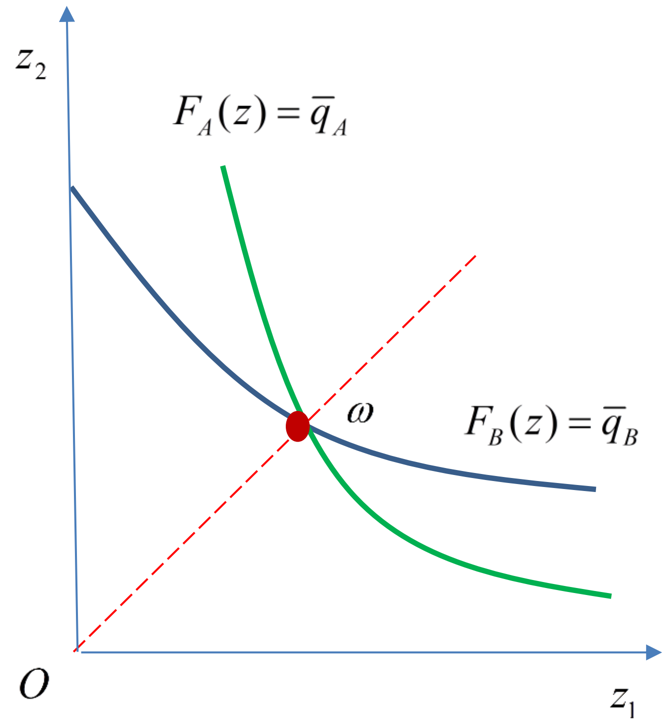


The production of commodity A is then

said to be more "input 1 intensive".

At the aggregate input endowment ,

the firms producing commodity A are willing

to give up more of input 2 in order to obtain more of input 1.

Production Efficient outputs

The output vector $\hat{q} = (\hat{q}_A, \hat{q}_B)$ is inefficient if it is possible to increase one output without decreasing the other output. When this is not possible the output vector is "production efficient".

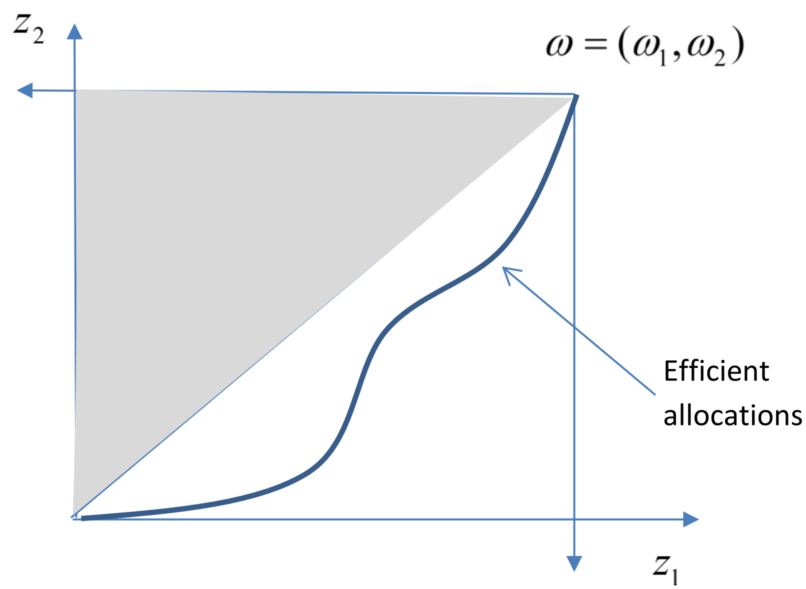
Class Exercise:

If commodity 1 is more input 1 intensive

explain why the efficient

input allocations must lie below the

diagonal of the Edgeworth Box



Proof:

We assumed that $MRTS_A(\omega) > MRTS_B(\omega)$

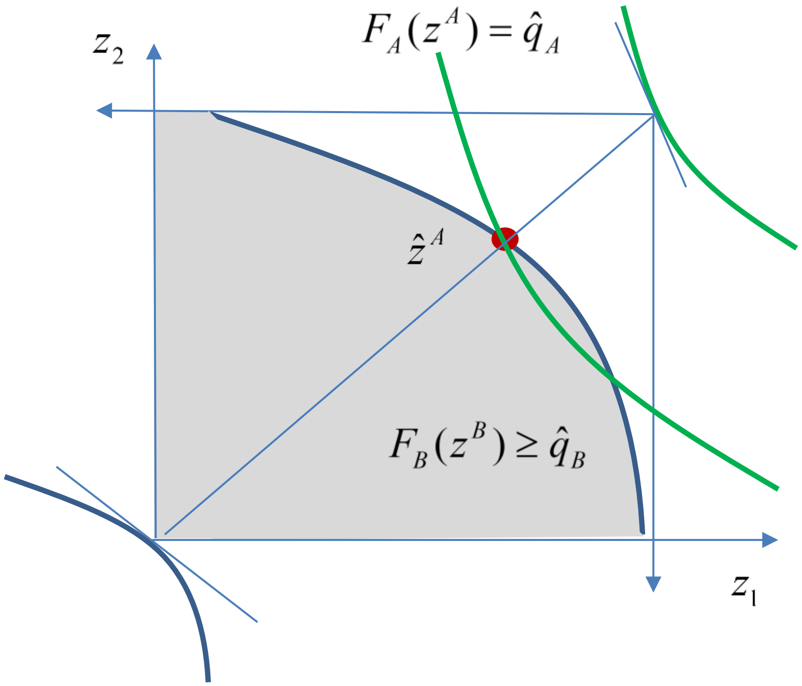
Note that if $\hat{z}^A = \theta\omega$ then $\hat{z}^B = \omega - \theta^A = (1-\theta)\omega$.

Since CRS functions are homothetic, the MRS

are constant along a ray.

Therefore

$$MRTS_A(\hat{z}^A) = MRTS_A(\omega) > MRTS_B(\omega) = MRTS_B(\hat{z}^B).$$



Characterization of efficient allocations

At the PE allocation $\hat{z}^A$

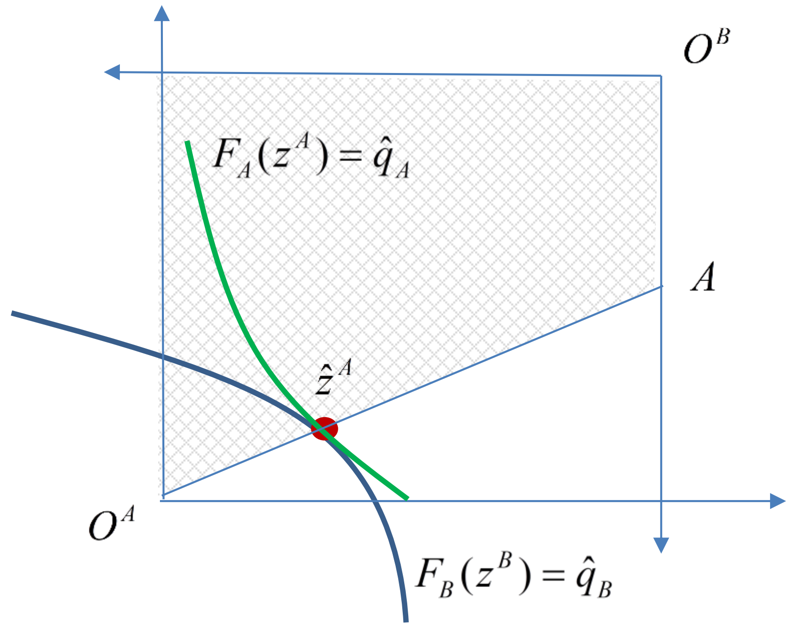
$$MRTS(\hat{z}^A) = MRTS(\hat{z}^B)$$

Step 1: Along the line $O^A A$

$$MRTS(z^A) = MRTS(\hat{z}^A)$$

Therefore in the cross-hatched region **above** this line

$$MRTS(z^A) > MRTS(\hat{z}^A)$$



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Characterization of efficient allocations

At the efficient allocation $\hat{z}^A$

$$MRTS(\hat{z}^A) = MRTS(\hat{z}^B)$$

Step 1: Along the line $O^A A$

$$MRTS(z^A) = MRTS(\hat{z}^A)$$

Therefore in the cross-hatched region **above** this line

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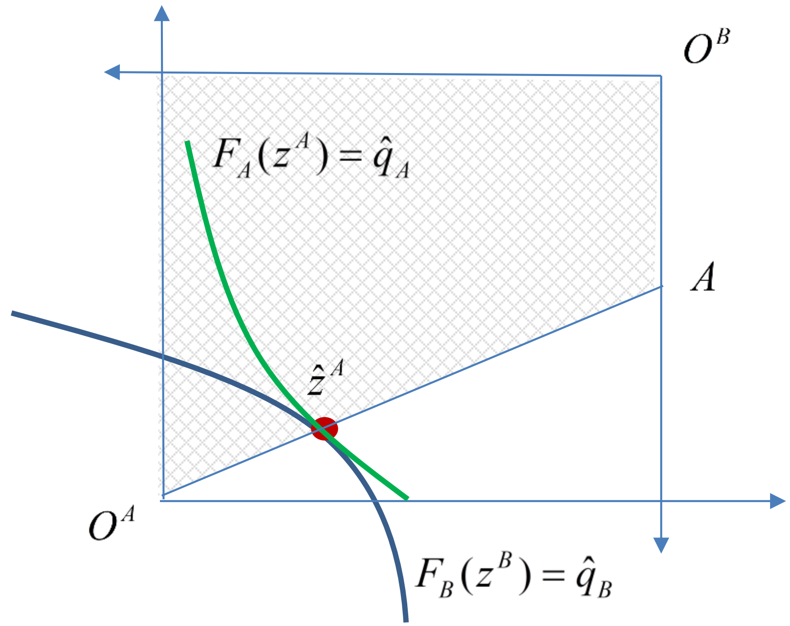


Step 2: Along the line $O^B B$

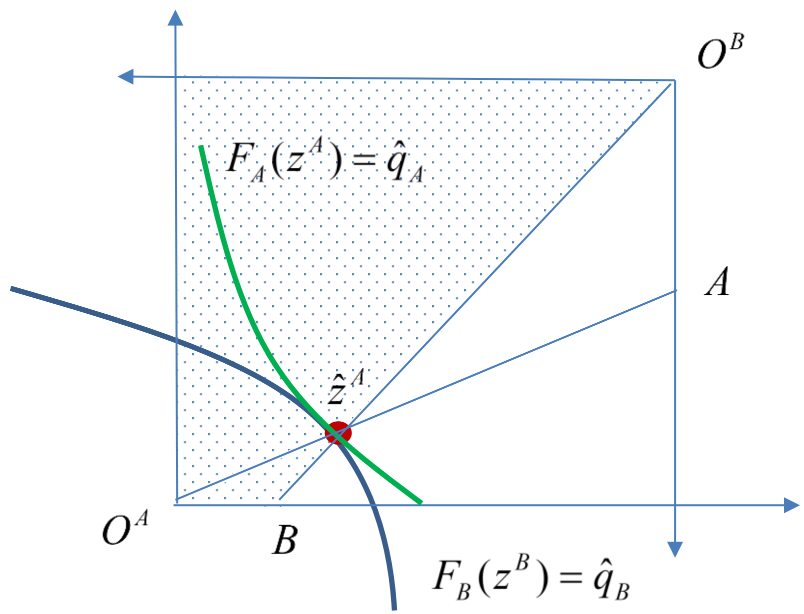
$$MRTS(z^B) = MRTS(\hat{z}^B)$$

Therefore in the dotted region **above** this line

$$MRTS(z^B) < MRTS(\hat{z}^B).$$

It follows that in the intersection of these two sets

$$MRTS(z^B) < MRTS(\hat{z}^B) = MRTS(\hat{z}^A) < MRTS(z^A)$$



Then no allocation in the vertically lined area

above the lines $O^A A$ and $O^B B$

is an efficient allocation.

By an almost identical argument,

no allocation in the shaded area horizontally

lined region below the lines $O^A A$ and $O^B B$

is an efficient allocation either.

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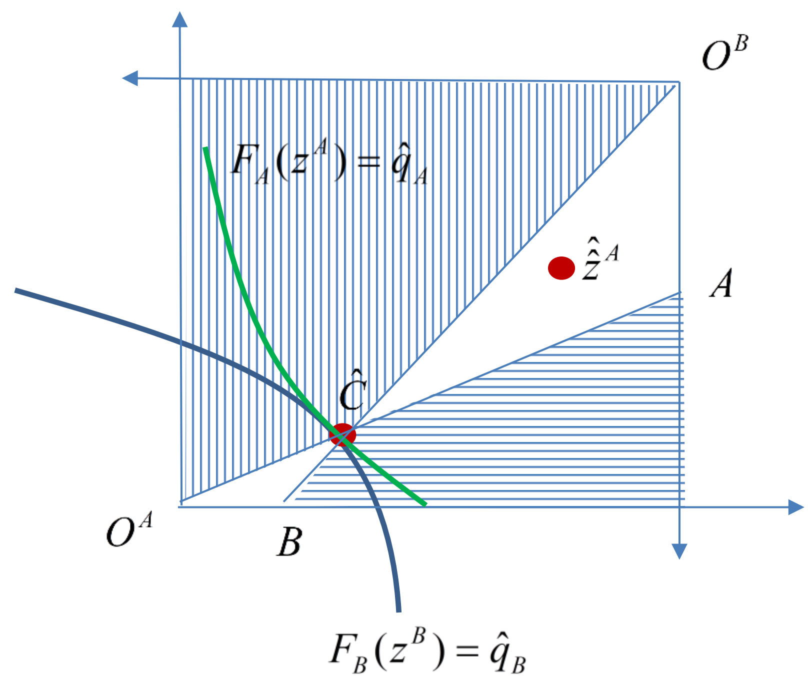

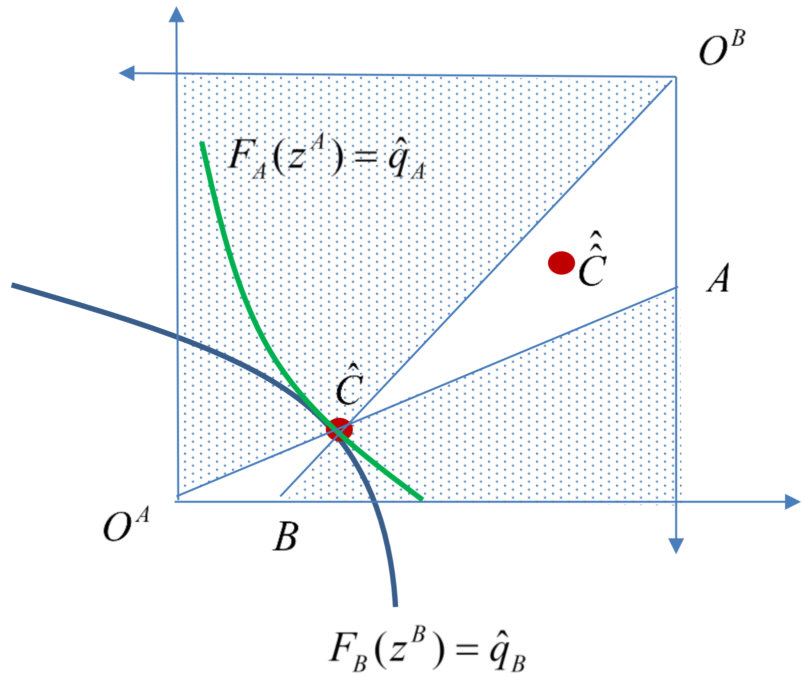

Then no allocation in the dotted area

above the lines $O^A A$ and $O^B B$

is an efficient allocation.

By an almost identical argument,

no allocation in the dotted area

below the lines $O^A A$ and $O^B B$

is an efficient allocation either.

Then any efficient $\hat{\hat{z}}^A$ with higher output of commodity A

(the point $\hat{\hat{C}}$ in the figure) lies in the triangle $A\hat{C}O^B$.

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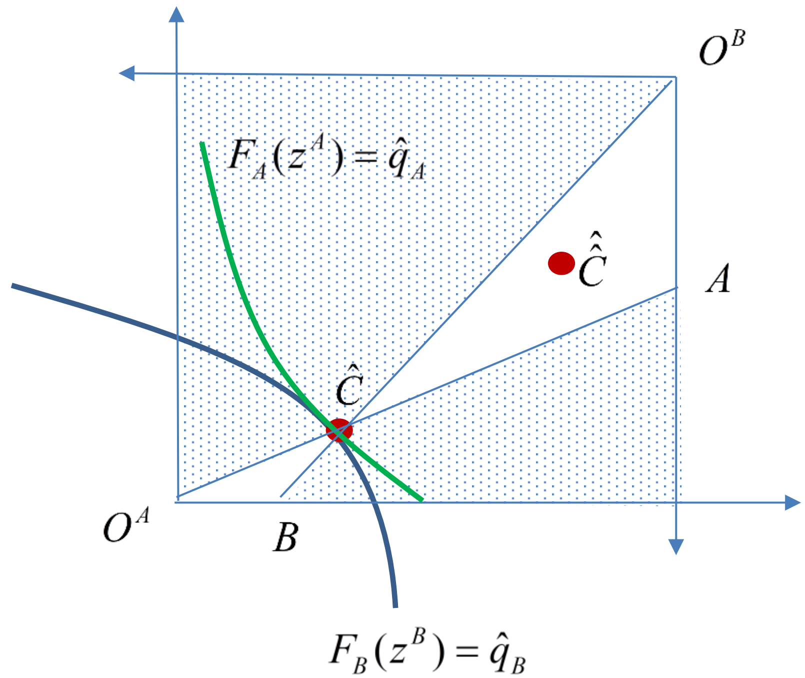

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Then any efficient $\hat{\hat{z}}^A$ with higher output of commodity A

(the point $\hat{\hat{C}}$ in the figure) lies in the triangle $A\hat{C}O^B$.

Since $MRTS^A(z)$ is constant along a ray,

Two implications: (i) $\dfrac{\hat{\hat{z}}_2}{\hat{\hat{z}}_1} > \dfrac{\hat{z}_2}{\hat{z}_1}$ (ii) $MRTS^A(\hat{\hat{z}}) > MRTS(\hat{z})$

Key result: For efficient allocations, $MRTS^A = MRTS^B$ rises as q_A rises.

Proposition: If output of the input 1 intensive input rises then the input rice ratio $\frac{r_1}{r_2}$ must rise

<u>Proof</u>: In the equilibrium $MRTS_A = MRTS_B = \frac{r_1}{r_2}$ so this follows directly from the previous result.

(Thus the owners of the input 1 intensive input benefit more.)

E. Opening an economy to trade

Suppose the country opens its borders to trade and as a result the price of commodity A (now exported) rises relative to the price of commodity B (facing import competition).

We normalize by setting $\bar{p}_B = \bar{\bar{p}}_B = 1$.

Proposition: If the price of commodity A rises then the equilibrium output of commodity A must rise

Closed economy

Let $(\bar{z}^A, \bar{q}_A),\ (\hat{z}^B, \bar{q}_B)$ be the aggregate equilibrium demand for inputs and equilibrium supply of the two outputs when prices are $\bar{p}$ and $\bar{r}$. Note that for equilibrium total input use is $\bar{z}^A + \bar{z}^B = \omega$

Open economy

Let $(\bar{\bar{z}}^A, \bar{\bar{q}}_A),\ (\bar{\bar{z}}^B, \bar{\bar{q}}_B)$ be the aggregate equilibrium demand for inputs and equilibrium supply of the two outputs when prices are $\bar{\bar{p}}$ and $\bar{\bar{r}}$. Note that for equilibrium total input use is $\bar{\bar{z}}^A + \bar{\bar{z}}^B = \omega$

In the closed economy $(\bar{z}^A, \bar{q}_A)$, $(\bar{z}^B, \bar{q}_B)$ is profit-maximizing.

Thus it is revenue maximizing over the set of feasible alternative outputs that have the same total cost. One such alternative is $(\bar{\bar{z}}^A, \bar{\bar{q}}_A)$, $(\bar{\bar{z}}^B, \bar{\bar{q}}_B)$.

(Remember that $\bar{z}^A + \bar{z}^B = \bar{\bar{z}}^A + \bar{\bar{z}}^B = \omega$)

Therefore $\bar{p}_A\bar{q}_A + \bar{p}_B\bar{q}_B \geq \bar{p}_A\bar{\bar{q}}_A + \bar{p}_B\bar{\bar{q}}_B$ i.e. $\bar{p}\cdot\bar{q} \geq \bar{p}\cdot\bar{\bar{q}}$

Hence $\bar{p}\cdot(\bar{\bar{q}} - \bar{q}) \leq 0$.

**

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Therefore $\bar{p}_A \bar{q}_A + \bar{p}_B \bar{q}_B \geq \bar{p}_A \bar{\bar{q}}_A + \bar{p}_B \bar{\bar{q}}_B$ i.e. $\bar{p} \cdot \bar{q} \geq \bar{p} \cdot \bar{\bar{q}}$

Hence $\bar{p} \cdot (\bar{\bar{q}} - \bar{q}) \leq 0$.

In the open economy $(\bar{\bar{z}}^A, \bar{\bar{q}}_A)$, $(\bar{\bar{z}}^B, \bar{\bar{q}}_B)$ is profit-maximizing. Thus by the same argument

$\bar{\bar{p}} \cdot \bar{\bar{q}} \geq \bar{\bar{p}} \cdot \bar{q}$

Hence $\bar{\bar{p}} \cdot (\bar{\bar{q}} - \bar{q}) \geq 0$.

Combining these results,

$\bar{\bar{p}} \cdot (\bar{\bar{q}} - \bar{q}) - \bar{p} \cdot (\bar{\bar{q}} - \bar{q}) \geq 0$ i.e. $(\bar{\bar{p}} - \bar{p}) \cdot (\bar{\bar{q}} - \bar{q}) = \Delta p \cdot \Delta q \geq 0$

*

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Combining these results,

$\bar{\bar{p}} \cdot (\bar{\bar{q}} - \bar{q}) - \bar{p} \cdot (\bar{\bar{q}} - \bar{q}) \geq 0$ i.e. $(\bar{\bar{p}} - \bar{p}) \cdot (\bar{\bar{q}} - \bar{q}) = \Delta p \cdot \Delta q \geq 0$

Since we have normalized by setting $\bar{p}_B = \bar{\bar{p}}_B = 1$ it follows that

$\Delta p_A \Delta q_A + \Delta p_B \Delta q_B = \Delta p_A \Delta q_A \geq 0$

and hence that $\Delta q_A \geq 0$.

Since output of commodity 1 rises it follows from the previous result that $\dfrac{r_1}{r_2}$ must rise.

Thus the owners of the input 1 intensive input gain relatively more.

Proposition: The price of input 1 rises and the price of input 2 falls.

Proof: Under CRS: $C_A(q,r) = qC_A(1,r)$ and $C_B(q,r_1,r_2) = qC_B(1,r_1,r_2)$.

Therefore

$$MC_A(q_A,r) = C_A(1,r) \quad \text{and} \quad MC_B(q_B,r) = C_B(1,r)$$

In equilibrium price = MC.

Therefore (i) $C_A(1,r_1,r_2) = p_A$ and (ii) $C_B(1,r_1,r_2) = p_B$.

If p_A rises it follows from (i) that

(a) at least one input price must rise.

It follows from (ii) that

(b) the other input price must fall.

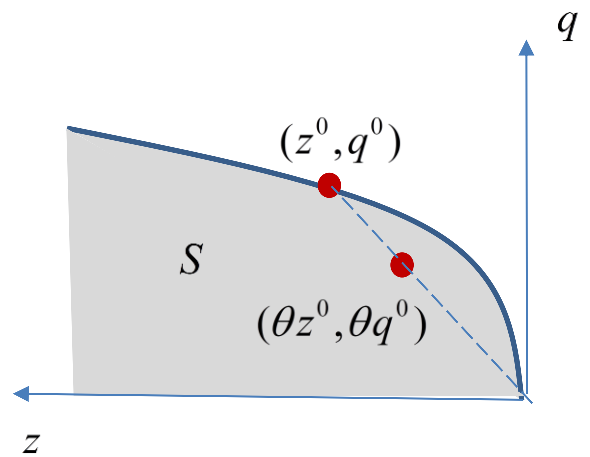
Suppose r_2 rises. We showed that the ratio $\frac{r_1}{r_2}$ must rise therefore r_1 must rise as well. But this contradicts (b). Thus opening the economy to exports of commodity A raises the price of input 1 and lowers the price of input 2.

F. Decreasing returns to scale

The production set S exhibits decreasing returns to scale

on $Q = \{q \mid \underline{q} \leq q \leq \overline{q}\}$ if, for any q^0 and $\theta q^0 < q^0$ in Q

if (z^0, q^0) is in S then $(\theta z^0, \theta q^0) \in \operatorname{int} S$.



Decreasing returns to scale

Single product firm
DRS
The production set S exhibits decreasing returns to scale on

the interval $\underline{q} < q < \overline{q}$ if, for any q^0 and $\theta q^0 < q^0$ in this interval,

if $q^0 = F(z^0)$ then $F(\theta z^0) > \theta F(z^0)$.

Proposition: $AC(q)$ increases under DRS

Proof: Check on your own that you can prove this (See the discussion of IRS).

In many industries, technology exhibits IRS at low output levels and DRS at high output levels. Thus at low output levels AC falls and at high output levels AC rises.

Let $\bar{q}$ be the output where AC is minimized.

Then $MC(q) < AC(q)$ for $q < \bar{q}$

and $MC(q) > AC(q)$ for $q > \bar{q}$

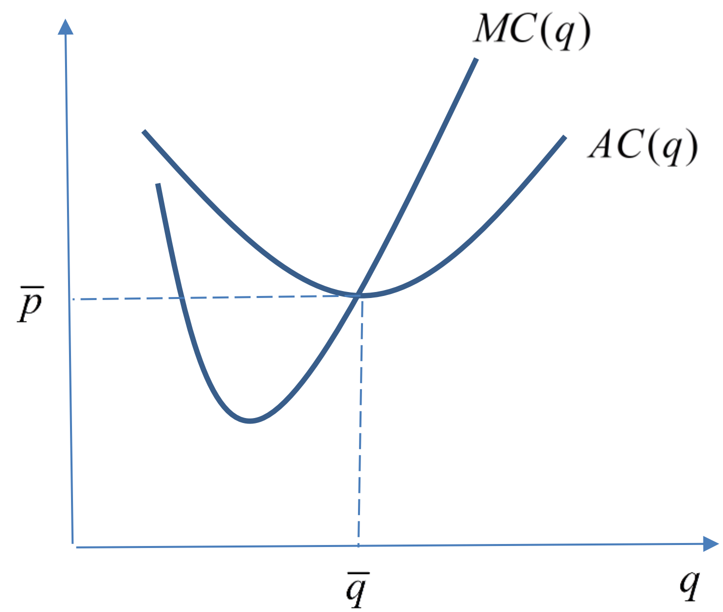
Proof:

$TC(q) = qAC(q)$

$MC(q) = TC'(q) = AC(q) + qAC'(q)$

Thus marginal cost exceeds AC if AC is increasing

and marginal cost is below AC if AC is decreasing.



Equilibrium with free entry of identical firms

At any price $\hat{p} > \underline{p}$ the profit maximizing

output is $\hat{q} > \underline{q}$.

All the firms are profitable

Thus entry continues until the price is pushed

down very close to $\underline{p}$.

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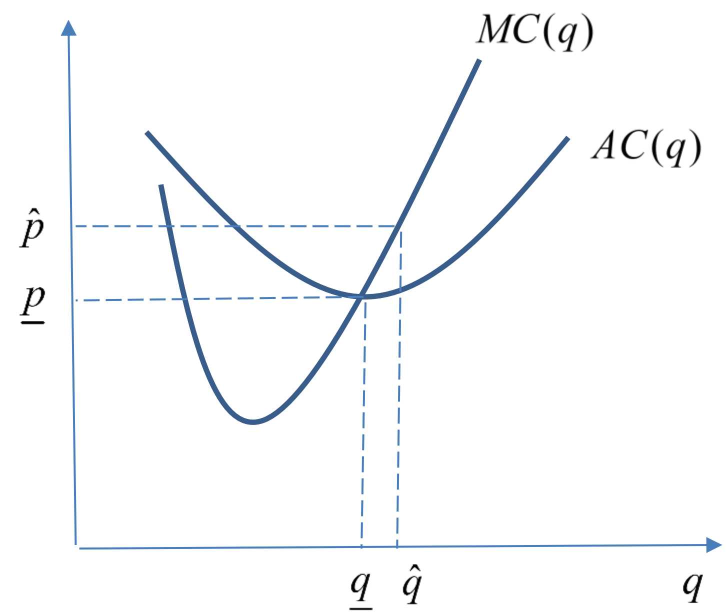


Equilibrium with free entry of identical firms

At any price $\hat{p} > \underline{p}$ the profit maximizing

output is $\hat{q} > \underline{q}$.

All the firms are profitable

Thus entry continues until the price is pushed

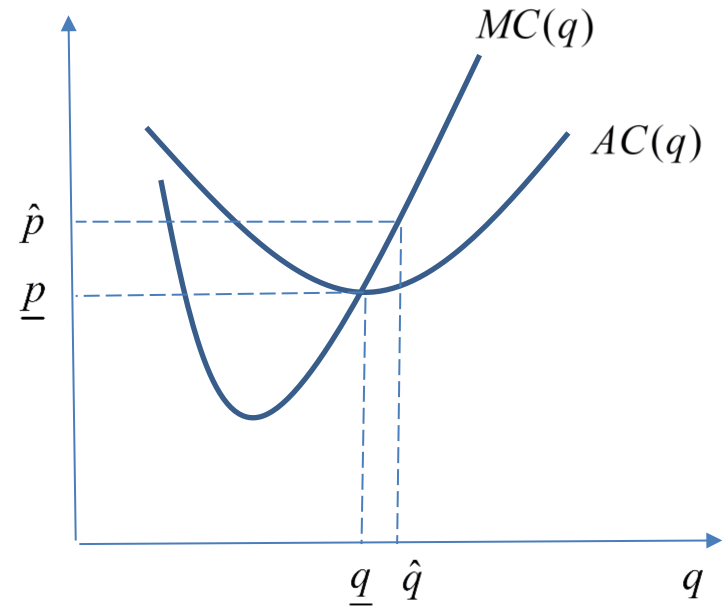
down very close to $\underline{p}$.

If $\underline{q}$ is small relative to market demand then to

a first approximation the equilibrium price is $\underline{p}$.

Thus demand has no effect on the equilibrium output

price.



At the industry level expansion is by entry so there are constant returns to scale

G. Joint Costs

An electricity company has an interest cost $c_o = 20$ per day for each unit of turbine capacity. For simplicity we define a unit of capacity as megawatt. It faces day-time and night-time demand price functions as given below.

The operating cost of running a each unit of turbine capacity is 10 in the day time and 10 at night.

$p_1 = 200 - q_1,\ \ p_2 = 100 - q_2$

Formulating the problem

Solving the problem

Understanding the solution (developing the economic insight)

Try a simple approach.

Each unit of turbine capacity costs 10+10 to run each day plus has an interest cost of 20 so MC=40

With q units sold the sum of the demand prices is $300-2q$ so revenue is $TR=q(300-2q)$ hence $MR=300-4q$. Equate MR and MC. $q^*=65$.

Try a simple approach.

Each unit of turbine capacity costs 10+10 to run each day plus has an interest cost of 20 so MC=40

With q units sold the sum of the demand prices is $300-2q$ so revenue is $TR=q(300-2q)$ hence $MR=300-4q$. Equate *MR* and *MC*. $q^*=65$.

Lets take a look on the margin in the day and night time

$TR_1=q_1(200-q_1)$ then $MR_1=200-2q_1$.

$TR_2=q_2(200-q_2)$ then $MR_2=100-2q_2$.

$MR_2<0$!!!!!!!!!!

Now what?

Key is to realize that there are really three variables. Production in each of the two periods q_1 and q_2 and the plant capacity q_0.

$$C(q) = 20q_0 + 10q_1 + 10q_2$$

$$R(q) = R_1(q_1) + R_2(q_2) = (200 - q_1)q_1 + (100 - q_2)q_2$$

Constraints

$$h_1(q) = q_0 - q_1 \geq 0 \quad h_2(q) = q_0 - q_2 \geq 0 .$$

**

Key is to realize that there are really three variables. Production in each of the two periods q_1 and q_2 and the plant capacity q_0.

$C(q) = c \cdot q$ where $c = (c_0, c_1, c_2) = (20,10,10)$

$R(q) = R_1(q_1) + R_2(q_2) = (200 - q_1)q_1 + (100 - q_2)q_2$

Constraints: $h_1(q) = q_0 - q_1 \geq 0$ $h_2(q) = q_0 - q_2 \geq 0$.

$\mathcal{L} = R_1(q_1) + R_2(q_2) - c_0 q_0 - c_1 q_1 - c_2 q_2 + \lambda_1(q_0 - q_1) + \lambda_2(q_0 - q_2)$

*

Key is to realize that there are really three variables. Production in each of the two periods q_1 and q_2 and the plant capacity q_0.

$C(q) = c \cdot q$ where $c = (c_0, c_1, c_2) = (20, 10, 10)$

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Constraints: $h_1(q) = q_0 - q_1 \geq 0$ $h_2(q) = q_0 - q_2 \geq 0$.

$\mathcal{L} = R_1(q_1) + R_2(q_2) - c_0 q_0 - c_1 q_1 - c_2 q_2 + \lambda_1(q_0 - q_1) + \lambda_2(q_0 - q_2)$

Kuhn-Tucker conditions

$\dfrac{\partial \mathcal{L}}{\partial q_j} = MR_j(q_j) - c_j - \lambda_j \leq 0$, with equality if $\bar{q}_j > 0$, j=1,2

$\dfrac{\partial \mathcal{L}}{\partial q_0} = -c_0 + \lambda_1 + \lambda_2 \leq 0$, with equality if $\bar{q}_0 > 0$

$\dfrac{\partial \mathcal{L}}{\partial \lambda_i} = q_0 - q_i \geq 0$, with equality if $\lambda_i > 0$, i=1,2

Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial q_1} = 200 - 2q_1 - 10 - \lambda_1 \le 0,\text{ with equality if } \bar{q}_1 > 0$$

$$\frac{\partial \mathcal{L}}{\partial q_2} = 100 - 2q_2 - 10 - \lambda_2 \le 0,\text{ with equality if } \bar{q}_1 > 0$$

$$\frac{\partial \mathcal{L}}{\partial q_0} = -20 + \lambda_1 + \lambda_2 \le 0,\text{ with equality if } \bar{q}_0 > 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_1} = q_0 - q_1 \ge 0,\text{ with equality if } \lambda_1 > 0.$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = q_0 - q_2 \ge 0,\text{ with equality if } \lambda_2 > 0.$$

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Kuhn-Tucker conditions

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$$\frac{\partial \mathcal{L}}{\partial \lambda_2} = q_0 - q_2 \ge 0, \text{ with equality if } \lambda_2 > 0.$$

Solve by trial and error. **Class exercise: Why will this work?**

(i) Suppose $\bar{q}_0 > 0$ and both shadow prices are positive. Then $q_0 = q_1 = q_2$.

(ii) Solve and you will find that one of the shadow prices is negative. But this is impossible.

(iii) Inspired guess. That shadow price must be zero. Then the positive shadow price must be 20.

Appendix:

Proposition: Convex production sets and decreasing returns to scale

Let S be a production set. If $S_+ = \{(z,q) \in S \mid (z,q) > 0\}$ is strictly convex, then the production set exhibits decreasing returns to scale.

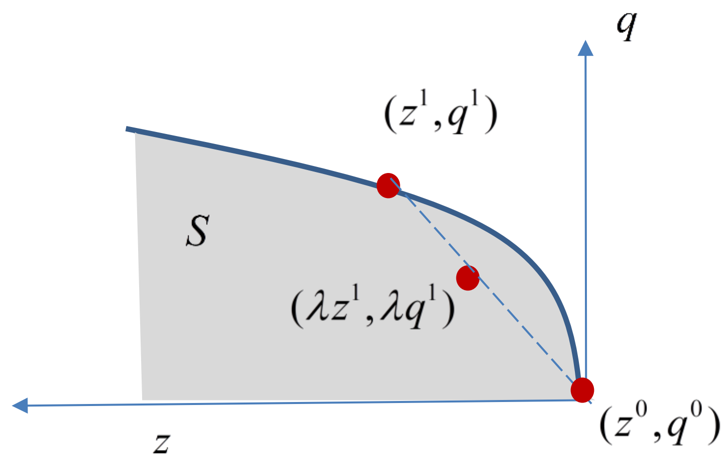
Proof:

Consider $(z^0, q^0) = (0,0)$ and $(z^1, q^1) >> 0$.

Since S_+ is strictly convex,

$$(z^\lambda, q^\lambda) = ((1-\lambda)z^0 + \lambda z^1, (1-\lambda)q^0 + \lambda q^1) = (\lambda z^1, \lambda q^1)$$

is in the interior of S_+ and hence in the interior of S



Local decreasing returns

Cost functions with convex production sets

Proposition: If a production set S is a convex set then

the cost function $C(q,r)$ is a convex function of the output vector q.

Proof: We need to show that for any q^0 and q^1

$$C(q^\lambda) \le (1-\lambda)C(q^0) + \lambda C(q^1)$$

Let z^0 be cost minimizing for q^0 and let z^1 be cost minimizing for q^1 . Then

$C(q^0) = r \cdot z^0$ and $C(q^1) = r \cdot z^1$.

Therefore

$$(1-\lambda)C(q^0) + \lambda C(q^1) = (1-\lambda)r \cdot z^0 + \lambda r \cdot z^1 = r \cdot [(1-\lambda)q^0 + \lambda q^1] = r \cdot z^\lambda \qquad (*)$$

Since S is convex, $(z^\lambda, q^\lambda) \in S$. Then it is feasible to produce q^λ at a cost of $r \cdot z^\lambda$. It follows that the minimized cost is (weakly smaller) i.e.

$$C(q^\lambda) \le r \cdot z^\lambda$$

Appealing to (*) it follows that

$$C(q^\lambda) \le (1-\lambda)C(q^0) + \lambda C(q^1)$$

QED

Profit maximization by a price taking firm

Since $C(q,r)$ is a convex function of q , $-C(q,r)$ is concave and so the firm's profit

$$\pi(q,p,r) = p \cdot q - C(q,r)$$

is a concave function of output.

It follows that the FOC are both necessary and sufficient for a maximum.

Technical Lemma: Super-additivity of a CRS production function

For a single product firm, if the production function $F(z)$ exhibits constant returns to scale and has convex superlevel sets, then for any input vectors x, y

$$F(x+y) \geq F(x) + F(y)$$

For some $\theta > 0$, $F(x) = \theta F(y) = F(\theta y)$.

First note that

$$F(y) = \frac{1}{\theta} F(x) \qquad (*).$$

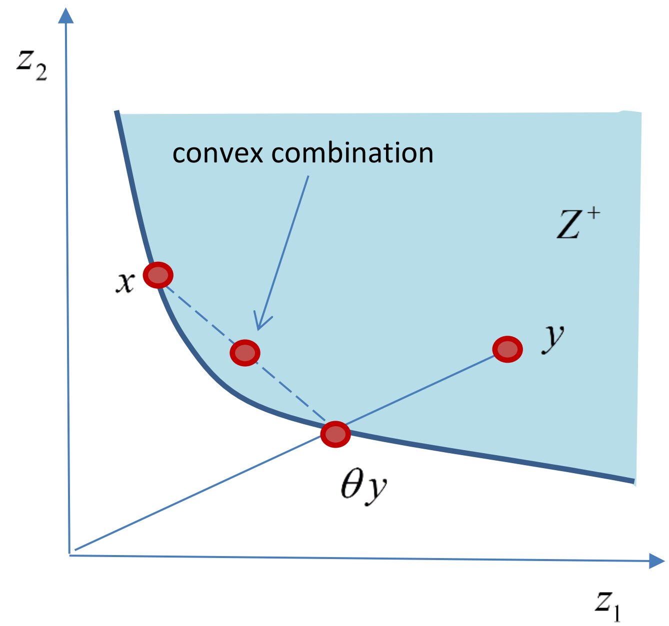
Consider the superlevel set

$$Z^+ = \{z \mid F(z) \geq F(x)\}$$

Since θy is in Z^+, all convex combinations of x and θy are in Z^+.

In particular, the convex combination

$$(\frac{\theta}{1+\theta})x + (\frac{1}{1+\theta})\theta y = \frac{\theta}{1+\theta}(x+y) \text{ is in } Z^+.$$



Therefore

$$F\left(\frac{\theta}{1+\theta}(x+y)\right) \geq F(x) \ .$$

Appealing to CRS

$$F\left(\frac{\theta}{1+\theta}(x+y)\right) = \frac{\theta}{1+\theta}F(x+y)$$

Therefore

$$\frac{\theta}{1+\theta}F(x+y) \geq F(x).$$

Therefore

$$F(x+y) \geq \left(1+\frac{1}{\theta}\right)F(x) = F(x) + \frac{1}{\theta}F(x)$$

From the previous slide,

$$F(y) = \frac{1}{\theta}F(x) \qquad (*)$$

Therefore

$$F(x+y) \geq F(x) + F(y) \ . \qquad \text{QED}$$

Proposition: Concavity of a CRS production function

For a single product firm, if the production function $F(z)$ exhibits constant returns to scale and has convex superlevel sets, then $F(z)$ is a concave function.

Proof: Define $y=(1-\lambda)z^0$ and $z=\lambda z^1$. Then from the super-additivity proposition,

$$F(z^{\lambda})\geq F((1-\lambda)z^0)+F(\lambda z^1)\ .$$

Since F exhibit CRS it follows that

$$F(z^{\lambda})\geq F((1-\lambda)z^0)+F(\lambda z^1)=(1-\lambda)F(z^0)+\lambda F(z^1)$$