Homework 1

Due in class on Tuesday October 4

1. Consumer choice

A consumer has a utility function $U(x) = x_1(a_2 + x_2)$. Her budget set is

$$B = \{x \mid x \geq 0, \;\; p \cdot x \leq I\} \; .$$

Suppose that $\bar{x}$ solves the constrained maximization problem $\underset{x\geq 0}{Max}\{U(x) \mid x \in B\}$

(a) If $p_2 a_2 = I$ show that $\bar{x} = (I / p_1, 0)$ by comparing the marginal utility per dollar for each commodity, i.e.

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1} \;\; \text{and} \;\; \frac{1}{p_2}\frac{\partial U}{\partial x_2}$$

(b) Explain why this is the unique solution.

(c) Depict (i) the budget set (ii) $\bar{x}$ and (iii) the level set of $U(x) = U(\bar{x})$.

(d) If $p_2 a_2 < I$ find some vector $\bar{x}$ that solves the maximization problem. Is this the unique solution?

(d) If $p_2 a_2 > I$ solve for $\bar{x}$.

(e) Draw the consumer's demand price function for commodity 2 (i.e. the market clearing price $p_2(x_2)$ for each feasible x_2. Indicate the intercepts.

(f) How does the elasticity of demand vary as p_2 changes?

2. Profit maximization

Consider the following problem discussed in the first class.

$$\pi(p, x) = p \cdot x - C(x) \;\; \text{where} \;\; p = (12,12) \;\; , \; C(x) = x_1^{\,2} + 4x_1x_2 + x_2^{\,2} \; .$$

To solve this numerically you will need to use Solver. Initially you may ignore the regulatory constraint. To learn about Solver, look at Lecture 0. Thinking on the margin. Download the spread-sheet Homework1.xlsx to your computer and look at Sheet1. The price and quantity arrays are columns. Yellow cells are data cells. Green cells are variables. Revenue is the sumproduct of these two arrays. You can see the formula by looking at the cell for profit $f(x)$. To compute the cost I created two square arrays

$$\left[c_{ij}\right] = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } \left[x_i x_j\right] = \begin{bmatrix} x_1^{\ 2} & x_1 x_2 \\ x_2 x_1 & x_2^{\ 2} \end{bmatrix}$$

The first is a data array, the second is a calculated array (color coded grey).

Then total cost is the sumproduct of these two arrays, $TC = [c_{ij}]\bullet[x_i x_j]$.

(a) Show that $\dfrac{\partial \pi}{\partial x_1}$, the marginal profit from increasing x_1 is the same as $\dfrac{\partial \pi}{\partial x_2}$, the marginal profit from increasing x_2 if $x_1 = x_2$.

(b) Pick some starting x satisfying this condition then click on Solver and run it. The answer should be $\bar{x} = (2,2)$.

(c) Pick any starting point (i) with $\dfrac{\partial \pi}{\partial x_1} > \dfrac{\partial \pi}{\partial x_2}$ (ii) with $\dfrac{\partial \pi}{\partial x_1} < \dfrac{\partial \pi}{\partial x_2}$. Now Solver correctly solves for a local maximum but not the same maximum.

Henceforth assume that $p = (15,12)$.

(d) Solve for the values of x such that $\dfrac{\partial \pi}{\partial x_1} = \dfrac{\partial \pi}{\partial x_2}$.

(e) Repeat (c) for the new data. Confirm numerically that there are again two local maxima but only one is the global maximum.

(f) Analyze the problem theoretically and show that there are three values of x that satisfy the necessary conditions for a maximum.

Now add the regulatory constraint $M - x_1 + x_2 \geq 0$. (See Sheet 3)

(g) Solve the problem numerically if the maximum M = 4.

(h) Also solve the problem analytically.

3. Concave functions

(a) Using the notes below show that (i) $f(x) = \ln x$, (ii) $f(x) = -\dfrac{1}{x^{\beta}}$, $\beta > 0$ (iii) $f(x) = x^{\alpha}, 0 < \alpha \leq 1$ are all concave functions on $\mathbb{R}_+$ (the positive real numbers)

(b) Using the third definition of a concave function, prove that the sum of two concave functions is concave. Explain how to use this result to show that the sum of (i) 3 (ii) n concave functions is concave

(c) Hence indicate which of the following functions is concave on $\mathbb{R}^2_{++}$ (i.e. $x >> 0$) where $a = (a_1, a_2) >> 0$ and $(0,0) << (\alpha_1, \alpha_2) \leq (1,1)$

(i) $f(x) = a_1 \ln x_1 + a_2 \ln x_2$ (ii) $f(x) = a_1 x_1^{\alpha_1} + a_2 x_2^{\alpha_2}$ (iii) $f(x) = -x_1^{-1} - x_2^{-2}$ (iv) $f(x) = -x_1^{2} - x_2^{5}$

(d) Consider the following problem

$\underset{x}{Max}\{f(x) \mid x \in \mathbb{R}_+\}$ where f is a concave function

Suppose that $f'(\bar{x}) = 0$ for some $\bar{x} \in \mathbb{R}_+$. Using one of the definitions below indicate why $\bar{x}$ solves the problem. If $f'(0) < 0$, does it also follow that $\bar{x} = 0$ solves the problem?

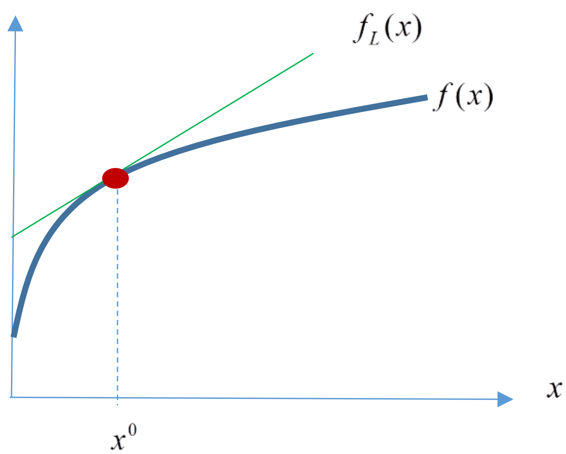
Mathematical definitions

Definition: Linear approximation of the function f at x^0

$$f_L(x) \equiv f(x^0) + f'(x^0)(x - x^0) \ .$$

Note that the linear approximation has the same value at x^0 and the same first derivative (the slope.)

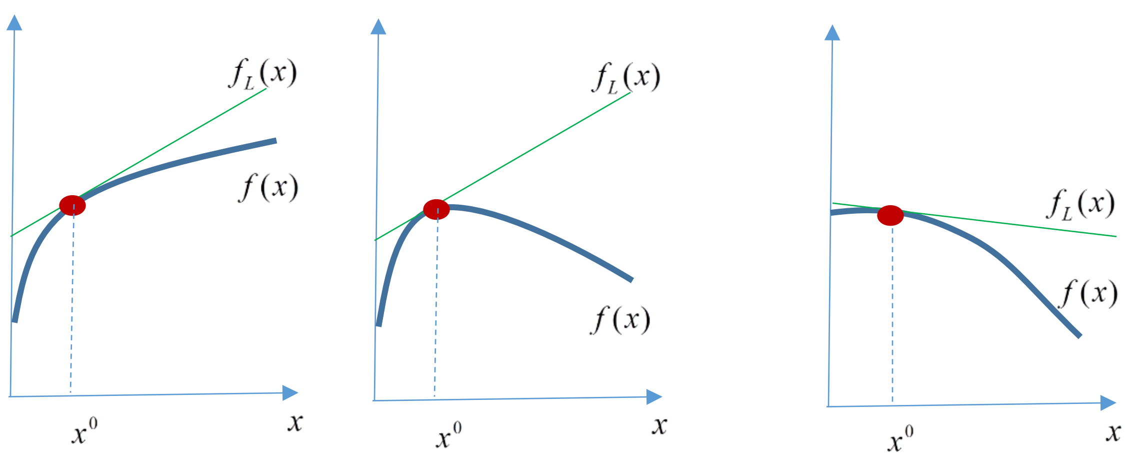
In the figure $f_L(x)$ is a line tangent to the graph of the function.



Definition 1: Concave Function

A differentiable function f defined on an interval X is concave if $f'(x)$, the derivative of $f(x)$ is decreasing.

The three types of differentiable concave function are depicted below.



Note that in each case the linear approximations at any point x^0 lie above the graph of the function.

While we will not formally prove it, the following is an equivalent definition.

Definition 2: Concave Function

A differentiable function f defined on an interval X is concave if , for any $x^0 \in X$

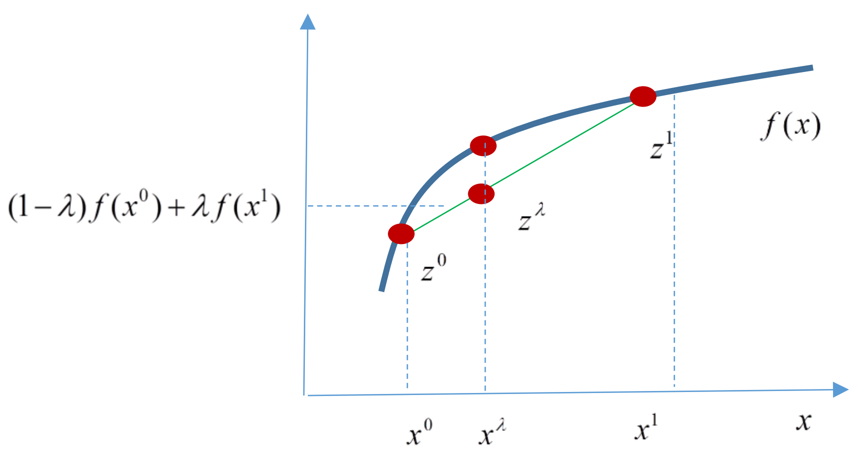
$f(x) \leq f(x^0) + f'(x^0)(x - x^0)$ $f'(x)$, the derivative of x is decreasing.

Consider any two vectors z^0 and z^1 . The set of weighted average of these two vectors can be written as follows.

$$z^\lambda = (1-\lambda)z^0 + \lambda z^1 \ , \ 0 < \lambda < 1$$

Such averages where the weighs are both strictly positive and add to 1 are called the convex combinations of z^0 and z^1 .

Consider the two points $z^0 = (x^0, f(x^0))$ and $z^1 = (x^1, f(x^1))$ depicted below.



We can think of these as vectors. The convex combination is

$$(1-\lambda)(x^0, f(x^0)) + \lambda(x^1, f(x^1)) = (x^\lambda, (1-\lambda)f(x^0) + \lambda f(x^1)) \ .$$

In the figure, such convex combination is on the line segment joining the two points.

For a concave function, it is clear from the figure that this line segment must lie below the graph of the function. i.e.

$$f(x^\lambda) \geq (1-\lambda)f(x^0) + \lambda f(x^1) \ .$$

In fact this is another equivalent definition.

Definition 3: Concave Function

A function f defined on an interval X is concave if , for any $x^0, x^1 \in X$ and any convex combination x^λ ,

$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1)$$