Homework 1 Answers

1. **Consumer choice**

Remark: Note that

$$u \equiv \ln U = \ln x_1 + \ln(a_2 + x_2) \; .$$

Maximizing $u(x)$ is equivalent to maximizing $U(x)$. Maximizing $u(x)$ is simpler. Also (and this is the key point) the sum of two concave functions it is concave. Therefore the necessary conditions are also sufficient.

Remark: If $\bar{x} >> 0$ there are three methods for getting the FOC. (i) Lagrange method (ii) Equate MRS(x) and price ratio (iii) note that at a maximum with $x >> 0$ the marginal utility per dollar must be equal.

All three lead to the same conclusion:

(a) FOC

$$\frac{1}{p_1}\frac{\partial u}{\partial x_1} = \frac{1}{p_2}\frac{\partial u}{\partial x_2} . \qquad \frac{1}{p_1}\frac{\partial u}{\partial x_1} = \frac{1}{p_1 x_1} , \; \frac{1}{p_2}\frac{\partial u}{\partial x_2} = \frac{1}{p_2 a_2 + p_2 x_2}$$

Therefore

$$\frac{1}{p_1 x_1} = \frac{1}{p_2 a_2 + p_2 x_2}$$

Hence

$p_2 a_2 + p_2 x_2 = p_1 x_1$ and so $p_2 a_2 + 2 p_2 x_2 = p_1 x_1 + p_2 x_2 = I$.

Then

$$\bar{x}_2 = \frac{1}{2p_2}(I - p_2 a_2) \; .$$

Also $p_1 x_1 = p_2 a_2 + p_2 x_2$. Therefore

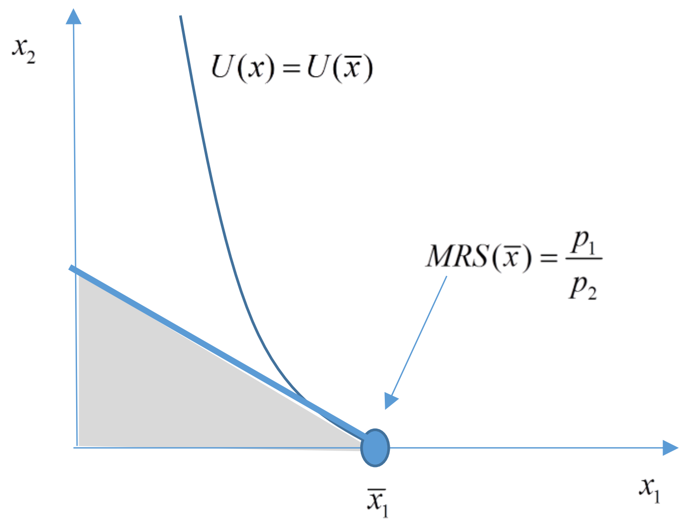
$$\bar{x}_1 = \frac{1}{2p_1}(I + p_2 a_2)$$

This is the solution for all $\bar{x}_2 > 0$, that is if $I - p_2 a_2 > 0$.

It is therefore the solution also in the limit if $I - p_2 a_2 \geq 0$.

(b) since the problem is a concave problem the necessary conditions are sufficient for a maximum.

(c)



(d) There is no solution with $\bar{x}_2 > 0$ so $\bar{x}_2 = 0$. Then $p_1\bar{x}_1 = I$ and so $\bar{x}_1 = \frac{I}{p_1}$.

To show that this is the solution we need to show that

$$\frac{1}{p_1}\frac{\partial u}{\partial x_1} > \frac{1}{p_2}\frac{\partial u}{\partial x_2} \quad \text{at } \bar{x} = (\frac{I}{p_1}, 0) \ ,$$

This is easily checked.

(e) From (a)

FOC

$$\frac{1}{p_1}MU_1 = \frac{1}{p_1 x_1} = \frac{1}{p_2(a_2 + x_2)} = \frac{1}{p_2}MU_2$$

Appealing to the Ratio Rule,

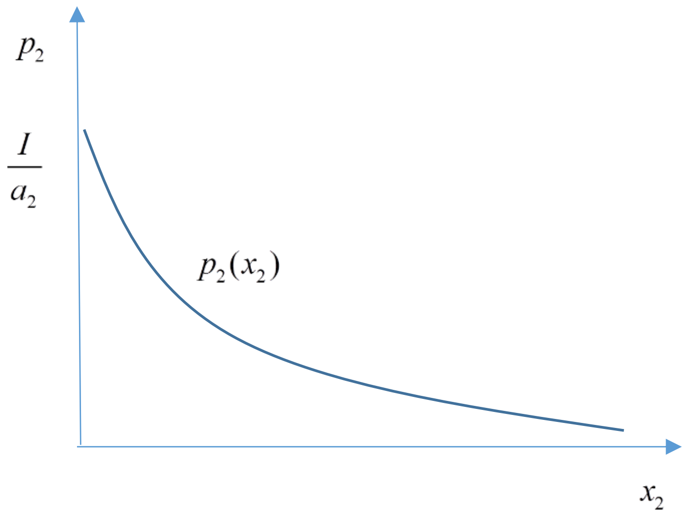
$$\frac{1}{p_1 x_1} = \frac{1}{p_2(a_2 + x_2)} = \frac{2}{p_2 a_2 + I} .$$

Then

$$p_1\bar{x}_1 = \frac{1}{2}(I + p_2 a_2) \quad \text{and} \quad p_2 x_2 = \tfrac{1}{2}(I + p_2 a_2) - p_2 a_2 \ .$$

We can then solve for $\bar{x}_1$ and $\bar{x}_2$

(f) $x_2 = \frac{1}{2}(\frac{I}{p_2} - a_2)$. Therefore $p_2 = \frac{I}{2x_2 + a_2}$



(g) $x_2 = \tfrac{1}{2}(\frac{I}{p_2} - a_2)$.

Therefore

$$\frac{\partial x_2}{\partial p_2} = -\frac{I}{2{p_2}^2}$$

So

$$\mathcal{E}(x_2, p_2) = \frac{p_2}{x_2}\frac{\partial x_2}{\partial p_2} = -\frac{p_2}{\tfrac{1}{2}(\frac{I}{p_2} - a_2)}\frac{I}{2{p_2}^2} = -\frac{I}{I - p_2 a_2}$$

As p_2 rises $\mathcal{E}(x_2, p_2)$ falls from $-\infty$ to -1.

2. Profit maximization

$$\pi(x, p) = 12x_1 + 12x_2 - C(x) = 12x_1 + 12x_2 - x_1^2 - 4x_1x_2 - {x_2}^2$$

Remark: This was not such an easy question as the profit function is not a concave function. Why is this? The cost function can be rewritten as follows:

$$C(x) = (x_1 + 2x_2)^2 - 3{x_2}^2 .$$

Then

$$\pi(x, p) = 12x_1 + 12x_2 - (x_1 + 2x_2)^2 + 3{x_2}^2$$

The first three terms are concave functions but the third is not.

The key point is that unless you can show that the function to be maximized is concave, x^* satisfying the FOC may not be a maximizer.

It can easily be confirmed that $x=(0,0)$ is not a maximizer so there are three possibilities (i) $x^* >> 0$ (ii) $x^* =(x_1^*,0)$ (iii) $x^* =(0,x_2^*)$.

(a) Marginal profit

$$\frac{\partial\pi}{\partial x_1}(x,p)=12-2x_1-4x_2\ ,\ \frac{\partial\pi}{\partial x_2}(x,p)=12-4x_1-2x_2$$

These are equal if $x_2 = x_1$.

(b) When I started with $x_2 = x_1$ I found that Solver give me the "solution" $x^* =(2,2)$.

(c) Starting off the line $x_2 = x_1$, I got a corner solution with a higher profit. Given the symmetry of the problem, the profit at each corner is the same and is greater than the profit at $x=(2,2)$.

Remark: Looking at the surface graph you may be able to see that $x=(2,2)$ is a saddle point. It is a maximum if you are constrained to stay on the line $x_2 = x_1$. But in other directions profit rises.

(d) Marginal profit

$$\frac{\partial\pi}{\partial x_1}(x,p)=15-2x_1-4x_2\ ,\ \frac{\partial\pi}{\partial x_2}(x,p)=12-4x_1-2x_2$$

Equating these,

$$x_2 = x_1 + \tfrac{3}{2}$$

(e) In this case I found that Solver would go to one of the corner solutions for all starting values of x .

So the program worked except in the symmetric case.

Technical Remark: I now believe that if the marginal profits are the same then the program increases x_1 and x_2 by the same amount. This simple mistake would presumably not occur for more sophisticated software.

(f) First look for a solution $x^* >> 0$

Setting marginal profits equal to zero,

$$2x_1 + 4x_2 = 15 \qquad \times 2 \qquad 4x_1 + 8x_2 = 30$$

$$4x_1 + 2x_2 = 12 \qquad \times 1 \qquad 4x_1 + 2x_2 = 12$$

Subtracting the second equation from the first, $6x_2 = 18$ so $x_2 = 3$. Then $x_1 = 3/2$.

Next look for a corner solution with $x_2^* = 0$ and $x_1^* > 0$

FOC

$$\frac{\partial \pi}{\partial x_1}(x,p) = 15 - 2x_1 - 4x_2 = 15 - 2x_1 = 0$$

$$\frac{\partial \pi}{\partial x_2}(x,p) = 12 - 4x_1 - 2x_2 = 12 - 4x_1 \le 0 .$$

Both are satisfied at $x_1^* = 7.5$

Next look for a corner solution with $x_1^* = 0$ and $x_2^* > 0$

FOC

$$\frac{\partial \pi}{\partial x_1}(x,p) = 15 - 2x_1 - 4x_2 = 15 - 4x_2 \le 0$$

$$\frac{\partial \pi}{\partial x_2}(x,p) = 12 - 4x_1 - 2x_2 = 12 - 2x_2 = 0 .$$

Both are satisfied at $x_2^* = 6$.

Comparing the profit at the three values of x that satisfy the FOC, the global maximum is $x^* = (7.5, 0)$.

Regulatory constraint

Remark: In the class discussion the constraint was $x_1 + x_2 \le M$. In the homework there was a typo so the constraint was different. I will consider the constraint $x_1 + x_2 \le 4$. If you considered a different constraint it will be graded accordingly.

Using Solver you can solve the problem numerically. Let $\bar{x}$ be the numerical solution. You can then write down the FOC and check that these conditions are all satisfied.

(h) For completeness I provide a full numerical solution.

The constraint must be binding since the unconstrained solution, $x^* = (7.5, 0)$ does not satisfy this constraint.

The Lagrangian is

$$L = \pi(q, p) + \lambda(4 - x_1 - x_2) .$$

FOC

$$\frac{\partial L}{\partial x_1} = \frac{\partial \pi}{\partial x_1} - \lambda = 15 - 2x_1 \ - 4x_2 \leq 0 \text{ with equality if } x_1^* > 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial \pi}{\partial x_2} - \lambda = 12 - 4x_1 \ - 2x_2 \leq 0 \text{ with equality if } x_2^* > 0$$

$$\frac{\partial L}{\partial \lambda} = 4 - x_1 - x_2 = 0 \ .$$

Case (i) $x >> 0$. Then all 3 FOC are satisfied with equality. Eliminating λ from the first two equations, $x_2 = x_1 + \frac{3}{2}$. Substituting for x_2 in the constraint you can solve for x^*

Case (ii) $x_1 > 0 = x_2$

$$\frac{\partial L}{\partial x_1} = \frac{\partial \pi}{\partial x_1} - \lambda = 15 - 2x_1 \ - 4x_2 = 15 - 2x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial \pi}{\partial x_2} - \lambda = 12 - 4x_1 \ - 2x_2 = 12 - 4x_1 \ - \lambda \leq 0$$

$$\frac{\partial L}{\partial \lambda} = 4 - x_1 - x_2 = 4 - x_1 = 0 \ .$$

Solution $x_1 = 4, \lambda = 3$

Case (iii) $x_2 > 0 = x_1$

Marginal revenue is lower in case (iii) so this cannot be optimal.

The final step is to compare the profit in Case (i) and Case (ii). It is higher in Case (ii).

3. Concave functions

(a) The second derivatives are all positive.

(b) If f and g are concave, then for any x^0, x^1

$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1)$$

$$g(x^\lambda) \geq (1-\lambda) g(x^0) + \lambda g(x^1)$$

Adding these inequalities

$$f(x^\lambda) + g(x^\lambda) \geq (1-\lambda)(f(x^0) + g(x^0)) + \lambda(f(x^1) + g(x^1))$$

Then for $h = f + g$

$$h(x^\lambda) \ge (1-\lambda)h(x^0) + \lambda h(x^1)$$

Use this again to add a third function. Use repeatedly for the sum of m functions.

(c) In all cases the second derivatives of each function are negative thus functions are concave and so the sum is concave.

(d) From the third definition

$f(x) \le f(\bar{x}) + f'(\bar{x})(x-\bar{x})$ for all $x \in \mathbb{R}_+$.

Thus if $f'(\bar{x}) = 0$,

$f(x) \le f(\bar{x})$ for all $x \in \mathbb{R}_+$.

Consider $\bar{x} = 0$. Then

$f(x) \le f(0) + f'(0)x$ for all $x \in \mathbb{R}_+$.

Therefore if $f'(0) < 0$ then

$f(x) < f(0)$ for all $x \in \mathbb{R}_{++}$.