Econ 401A Microeconomic Theory

Answers

Answer to 1

(a) $\pi = R - C = f(x) - \lambda g(x)$

FOC

Marginal profit is zero at the maximum.

$$\frac{\partial \pi}{\partial x_j} = \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) = 0, \quad j = 1,...,n$$

(b)

$p_1(x_1) = 180 - x_1$ and $p_2(x_2) = 320 - 4x_2$.

$R = 100x_1 - x_1^2 + 320x_2 - 4x_2^2$

$\pi(x) = f(x) - \lambda g(x) = R - \lambda(2x_1^2 + \frac{3}{2}x_2^2)$.

$\frac{\partial \pi}{\partial x_1}(x) = 180 - 2x_1 - 4\lambda x_1 = 0$ for an interior maximum. Then $\bar{x}_1 = \frac{180}{2+4\lambda}$

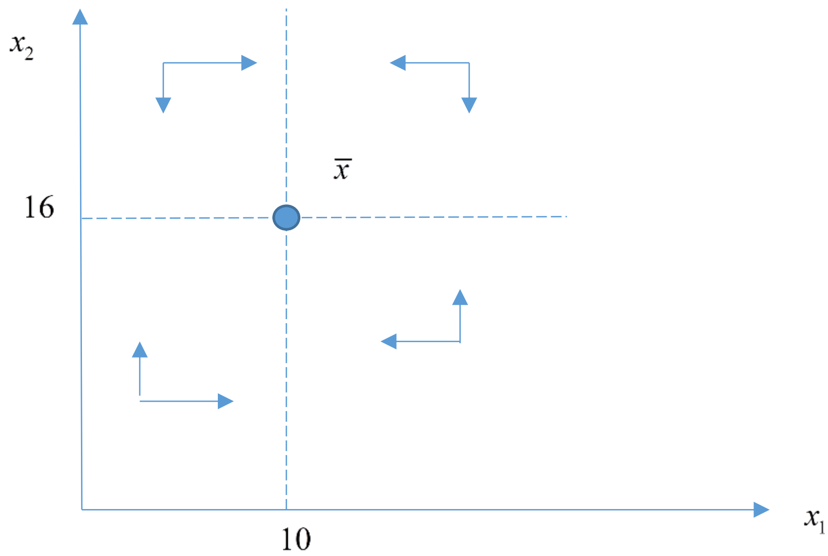
$\frac{\partial \pi}{\partial x_2}(x) = 320 - 8x_2 - 3\lambda x_3 = 0$ for an interior maximum. Then $\bar{x}_2 = \frac{320}{8+3\lambda}$.

(c) Total demand for the input is therefore

$$b(\lambda) = g(x(\lambda)) = 2(\frac{180}{2+4\lambda})^2 + \frac{3}{2}(\frac{320}{8+3\lambda})^2$$

$b(4) = 2(10)^2 + \frac{3}{2}(16)^2 = 584$.

(d)



Answer to 2

If the price is 4 the firm will choose $\bar{b} = 584$. Thus at this price the firm will not wish to purchases or sell any additional units.

Answer to 3

At this price the firm would choose $\bar{b} = 584$ so

if $\lambda = 4$,

$$f(\bar{x}) - \lambda g(\bar{x}) \geq f(x) - \lambda g(x) \text{ for all } x \neq \bar{x}$$

Rearranging this inequality

$$f(\bar{x}) - f(x) \geq \lambda(g(\bar{x}) - \lambda g(x)) \text{ for all } x \neq \bar{x}$$

It follows that if x must satisfy the constraint

$g(x) \leq \bar{b} = g(\bar{x})$

Then $g(\bar{x}) - g(x) = \bar{b} - g(x) \geq 0$

And so

$f(\bar{x}) - f(x) \geq \lambda(g(\bar{x}) - \lambda g(x)) = \lambda(\bar{b} - g(x)) \geq 0$.

Therefore $f(\bar{x}) \geq f(x)$ for all x satisfying the constraint $g(x) \leq \bar{b}$.

General remark

The above three exercises provide another way of thinking about constrained maximization with a single resource constraint.

$$\underset{x\geq 0}{Max}\{f(x)\,|\,\bar{b}-g(x)\geq 0\}$$

Step 1: Imagine that b is a resource that can be bought and sold at the price λ and write down the First Order Necessary Conditions (FOC) for the "relaxed problem" where b is a variable. Under this interpretation $f(x)-\lambda b = f(x)-\lambda g(x)$ is the profit and so the FOC,

$$\frac{\partial \mathfrak{L}}{\partial x_j}\leq 0 \text{ with equality if } \bar{x}_j > 0 \text{ , } j=1,...,n$$

are simply the FOC for unconstrained profit-maximization.

Step 2: Do this for different prices and so map out the demand price function $b(\lambda)$. Then choose the shadow price $\bar{\lambda}$ so that $\bar{b}=b(\bar{\lambda})$. Let $\bar{x}(\bar{\lambda})$ be the optimized x .

Step 3: Note that at the price $\bar{\lambda}$, the maximization with no resource constraint solves the optimization problem with the constraint. Therefore $\bar{x}(\bar{\lambda})$ must be the solution of the resource constrained problem

Answer to 4

$R_1(q_1)=56q_1-q_1^{\,2}$, $R_2(q_2)=104q_2-2q_2^{\,2}$, $C(q)=q_1^{\,2}+4q_1q_2+4q_2^{\,2}$.

$$\pi = R-C = 56q_1-q_1^{\,2}+104q_2-2q_2^{\,2}-q_1^{\,2}-4q_1q_2-4q_2^{\,2}$$

$$\frac{\partial \pi}{\partial q_1}=MR_1-MC_1=56-2q_1-2q_1-4q_2=0 \text{ if } \bar{q}_1>0$$

$$\frac{\partial \pi}{\partial q_2}=MR_2-MC_2=104-4q_1-4q_2-8q_2=0 \text{ if } \bar{q}_1>0$$

$$q_1=m_1(q_2)=(56-4q_2)/4$$

$$q_2=m_2(q_1)=(104-4q_1)/12$$

Solving these two equations,

$$\bar{q}=(8,6) \text{ .}$$

The profit-maximizing lines are exactly the same as in question 1. Thus $\bar{q}$ is the solution.

Answer to 5

$$\mathfrak{L} = \ln(10 + x_1) + 2\ln x_2 + 3\ln x_3 + \lambda(I - p_1x_1 - x_2 - x_3)$$

Note that x_2 and x_3 must be strictly positive since $\ln x$ decreases without bound as x approaches zero.

If $x_1 > 0$

$$\frac{\partial\mathfrak{L}}{\partial x_1} = \frac{1}{10 + x_1} - \lambda p_1 = 0. \quad \text{Hence } \frac{\partial\mathfrak{L}}{\partial x_1} = \frac{1}{p_1(10 + x_1)} = \lambda.$$

$$\frac{\partial\mathfrak{L}}{\partial x_2} = \frac{2}{x_2} - \lambda p_2 = 0. \quad \text{Hence } \frac{\partial\mathfrak{L}}{\partial x_2} = \frac{2}{p_2x_2} = \lambda.$$

$$\frac{\partial\mathfrak{L}}{\partial x_3} = \frac{3}{x_3} - \lambda p_3 = 0. \quad \text{Hence } \frac{\partial\mathfrak{L}}{\partial x_3} = \frac{3}{p_3x_3} = \lambda$$

Appealing to the Ratio Rule

$$\frac{1}{p_1(10 + x_1)} = \frac{2}{p_2x_2} = \frac{3}{p_3x_3} = \lambda = \frac{6}{10p_1 + p \cdot x} = \frac{6}{10p_1 + I}$$

$$p_1x_1 + 10p_1 = \tfrac{1}{6}(10p_1 + I)$$

$$p_1x_1 = \tfrac{1}{6}(10p_1 + I) - 10p_1 = \tfrac{1}{6}(I - 50p_1).$$

Thus the assumption that $\bar{x}_1 > 0$ is correct if and only if $I > 50p_1$.

Suppose that $I < 50p_1$.

The solution above is incorrect as it implies that $\bar{x}_1 < 0$.

Thus the solution must be a corner solution.

A neat diagram at this point depicting a corner solution would be enough.

Here is a formal solution. Read it only if you don't like diagrams.

It seems reasonable to assume that since the non-negativity constraint is violated the actual solution must be $\bar{x}_1 = 0$.

FOC

$$\frac{\partial \mathfrak{L}}{\partial x_1} = \frac{1}{10+x_1} - \lambda p_1 = \frac{1}{10} - \lambda p_1 \le 0 . \quad . \qquad (1)$$

$$\frac{\partial \mathfrak{L}}{\partial x_2} = \frac{2}{x_2} - \lambda p_2 = 0 . \qquad \text{Hence } \frac{2}{p_2 x_2} = \lambda . \qquad (2)$$

$$\frac{\partial \mathfrak{L}}{\partial x_3} = \frac{3}{x_3} - \lambda p_3 = 0 . \qquad \text{Hence } \frac{3}{p_3 x_3} = \lambda \qquad (3)$$

$$\frac{2}{p_2 x_2} = \lambda = \frac{3}{p_2 x_2}$$

I use the Ratio Rule as a short-cut.

$$\frac{2}{p_2 x_2} = \frac{3}{p_2 x_2} = \frac{5}{p_2 x_2 + p_3 x_3} = \frac{5}{I}$$

Then

$$\lambda = \frac{5}{I}$$

(Also $x_2 = \frac{2}{5}\frac{I}{p_2}$ and $x_3 = \frac{3}{5}\frac{I}{p_3}$) .

Therefore

$$\lambda p_1 = \frac{5p_1}{I} \ge \frac{1}{10} \text{ since } I < 50 p_1$$

Thus inequality (1) is satisfied.

Answer to Exercise 6: Numerical analysis of consumer choice

$U(x) = 8x_1^{1/2} + 3x_2^{2/3} \quad p = (1,1)$.

$$\frac{\partial U}{\partial x_1} = 4x_1^{-1/2} \text{ , } \frac{\partial U}{\partial x_2} = 2x_1^{-1/3}$$

$$MRS = \frac{4x_1^{-1/2}}{2x_1^{-1/3}} = 2(\frac{x_2}{x_1})^{1/3}\frac{1}{x_1^{1/2}} \ .$$

Note that along ANY ray the MRS declines as x increases. Thus the consumer becomes less willing to substitute x_1 for x_2 in his consumption bundle as his income rises. So he consumes an ever lower ratio x_1 / x_1 .