Homework 2 Answers

Question 1

$$\underset{(z_1,z_2,q_3)}{Max}\{q_3 \mid q_3{}^4 \le \frac{256}{27} z_1 z_2{}^3 \mid p_1 z_1 + p_2 z_2 \le \bar{B}\} \ .$$

For any z output is maximized by choosing $q_3{}^4 = \frac{256}{27} z_1 z_2{}^3$.

If $q_3{}^4$ is maximized then q_3 is maximized so the problem becomes

$$\underset{(z_1,z_2,q_3)}{Max}\{q_3{}^4 = F(z) = \frac{256}{27} z_1 z_2{}^3 \mid p_1 z_1 + p_2 z_2 \le \bar{B}\}$$

Output is zero if z_1 or $z_2 = 0$. Therefore the solution $\bar{z} >> 0$.

Use the Lagrange method or note that the marginal gain per dollar must be equal at the maximum.

Therefore

$$\frac{1}{p_1}\frac{\partial F}{\partial z_1}(\bar{z}) = \frac{1}{p_2}\frac{\partial F}{\partial z_2}(\bar{z})$$

$$\frac{1}{p_1}\frac{256}{27} z_2{}^3) = \frac{1}{p_2}\frac{256}{27} 3 z_1{}^2$$

Therefore $p_2 z_2 = 3 p_1 z_1$ and so

$p_1 z_1 + p_2 z_2 = p_1 z_1 + 3 p_1 z_1 = \bar{C}$ and so $p_1 z_1 = \frac{\bar{C}}{4}$ and $p_2 z_2 = 3 p_1 z_1 = \frac{3\bar{C}}{4}$

Therefore

$z_1 = \frac{\bar{C}}{4 p_1}$ and $z_2 = \frac{3\bar{C}}{4 p_2}$

$$\bar{q}_3{}^4 = \frac{256}{27} z_1 z_2{}^3 = \frac{256}{27}(\frac{\bar{C}}{4 p_1})(\frac{3\bar{C}}{4 p_2})^3 = \frac{\bar{C}^4}{p_1 p_2{}^3}$$

And so

$$\bar{q}_3 = \frac{\bar{C}}{p_1^{1/4} p_2^{3/4}} \ .$$

We have shown that $\bar{q}_3$ can be produced at a cost of $\bar{C}$. Note that maximized output is a strictly increasing function of the budget. Therefore if the budget is smaller, then output is lower then $\bar{q}_3$.

Therefore $\bar{C}$ is the minimized cost of producing $\bar{q}$. So the cost function is

$$C(q_3) = p_1^{1/4} p_2^{3/4} q_3$$

Question 2

The FOC can be written as follows:

$$\frac{1}{p_1(z_1-4)} = \frac{1}{p_2 z_2} .$$

Hence $p_2 z_2 = p_1(z_1-4)$ (*)

The budget constraint is

$$p_1 z_1 + p_2 z_2 = \bar{C}$$

Substitute for $p_2 z_2$ and solve for $z_1 - 4$

Short-cut

$$p_1(z_1-4) + p_2 z_2 = \bar{C} - 4p_1 .$$

Appealing to (*)

$$2p_1(z_1-4) = 2p_2 z_2 = \bar{C} - 4p_1$$

End of short-cut.

$$\bar{q}_3^{\,3} = 4(z_1-4)z_2 = 4\left(\frac{\bar{C}-4p_1}{2p_1}\right)\left(\frac{\bar{C}-4p_1}{2p_2}\right) = \frac{(\bar{C}-4p_1)^2}{p_1 p_2}$$

Therefore

$$\bar{q}_3^{\,3/2} = \frac{\bar{C}-4p_1}{(p_1 p_2)^{1/2}}$$

Hence

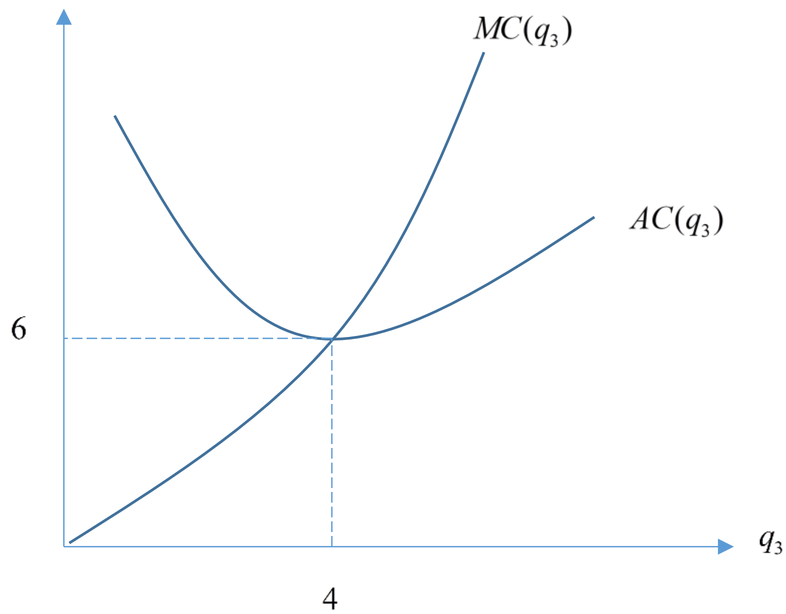
$$C = (p_1 p_2)^{1/2} q_3^{3/2} + 4p_1 .$$

Set $p_1 = p_2 = 2$

$$C = 2q_3^{3/2} + 8$$

$$AC = 2q_3^{1/2} + \frac{8}{q_3}\ ,\ MC = C'(q_3) = 3q_3^{1/2}$$

These are equal at $q_3 = 4$.



Answer to Q3

Let z_1^f be the input for firm f . Maximized output is $q_2^f = (z_1^f)^{1/2}$. Therefore industry output is

$q = \sum_{f=1}^{16} (z_1^j)^{1/2}$. Suppose z is allocated to the production of commodity 2. Then

$$\sum_{f=1}^{16} z_1^f \le z\ .$$

Maximized industry output is then

$$q = Max\{F(z) = \sum_{f=1}^{16} (z_1^f)^{12} \mid \sum_{f=1}^{16} z_1^f \le z\}\ .$$

This is just like a consumer problem with all the prices equal to 1. The maximand and the constraint are concave so the FOS are both necessary and sufficient for a maximum.

$$\frac{\partial F}{\partial z_j^f} = \frac{1}{2}(z_j^f)^{-1/2} \ .$$

FOC

Marginal payoff he r unit of the resource must be equalized. Therefore

$$\frac{\partial F}{\partial z_1^1} = ... = \frac{\partial F}{\partial z_{16}^1} .$$

It follows that $z_1^f = ... = z_{16}^f$.

Since the total allocation is z it follows that

$$z_1^f = ... = z_{16}^f = \frac{z}{16} .$$

Then

$$q_2^f = (z_2^f)^{1/2} = (\frac{1}{16}z)^{1/2} = \frac{1}{4}z^{1/2} \ .$$

Hence

$$q_2 = q_2^1 + ... + q_2^{16} = 4z^{1/2} \ .$$

Then

$x_1 = \omega_1 - z = 32 - z$ and $x_2 = q_2 = 4z^{1/2}$.

$$U(x) = \ln x_1 + 2\ln x_2 = \ln(32 - z) + 2\ln(4z^{1/2}) = \ln(32 - z) + 2\ln z + 2\ln 4 \ .$$

This is maximized at $z = 16$.

Then $x_1 = 16$ and $x_2 = 4z^{1/2} = 16$.

FOC (representative consumer)

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1} = \frac{1}{p_2}\frac{\partial U}{\partial x_2}$$

$$\frac{1}{p_1 x_1} = \frac{2}{p_2 x_2}$$

$$\frac{1}{p_1 16} = \frac{2}{p_2 16} .$$

Then $p_2 = 2$

Total profit is

$$\pi = p_2 q_2 - p_1 z = 2(16) - 16 = 16.$$

Profit per firm is 1.

Answer to question 4.

Arguing as in 3, $z_1^f = \ldots = z_{64}^f$.

Since the total allocation is z it follows that

$$z_1^f = \ldots = z_{64}^f = \frac{z}{64}.$$

Then

$$q_2^f = (z_2^f)^{1/2} = (\frac{1}{64} z)^{1/2} = \frac{1}{8} z^{1/2} \ .$$

Hence

$$q_2 = q_2^1 + \ldots + q_2^{64} = 8z^{1/2} \ .$$

Then

$x_1 = \omega_1 - z = 32 - z$ and $x_2 = q_2 = 8z^{1/2}$.

$$U(x) = \ln x_1 + 2\ln x_2 = \ln(32 - z) + 2\ln(8z^{1/2}) = \ln(32 - z) + 2\ln z + 2\ln 8 \ .$$

This is maximized at $z = 16$.

Then $x_1 = 16$ and $x_2 = 8z^{1/2} = 32$.

FOC (representative consumer)

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1} = \frac{1}{p_2}\frac{\partial U}{\partial x_2}$$

$$\frac{1}{p_1 x_1} = \frac{2}{p_2 x_2}$$

$$\frac{1}{p_1 16} = \frac{2}{p_2 32}.$$

Then $p_2 = 1$

Total profit is

$\pi = p_2 q_2 - p_1 z = 1(32) - 16 = 16$.

Profit per firm is 1/4.

Total profit is unchanged.

Intuitively. More firms and hence more competition will increase output and therefore push output price down and hence profit per firm.

This is correct.

Note that as the number of firms increases the profit per firm declines towards zero. But as long as the marginal cost of each firm is strictly increasing, then the industry marginal cost is strictly increasing.

The industry demand price function and industry marginal cost function are graphed below. SInce $p = MC$ for each firm this is true for the industry.

Thus the industry profit remains positive even in the limit as the number of firms grows very large.

Total cost is the area under the MC curve. Total revenue is $\bar{p}\bar{q}$. Thus the industry profit is the shaded area.