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Econ 401A: Microeconomic Theory

Homework 2 Answers

1. Constant returns to scale production function

(a) If $(z_1, z_2, q) \in S$ then

$$16z_1^{\ 3}z_2 - q^4 \geq 0 \ . \qquad (*)$$

We need to show that $(\theta z_1, \theta z_2, \theta q) \in S$

$$16(\theta z_1)^3(\theta z_2) - (\theta q)^4 = \theta^4 z_1^{\ 3} z_2 - \theta^4 q^4 = \theta^4 [z_1^{\ 3} z_2 - q^4] \ .$$

Appealing to (*) it follows that $(\theta z_1, \theta z_2, \theta q) \in S$

(b) The production function of the firm is $q = 2z_1^{3/4} z_2^{1/4}$.

Maximum output is

$$q(B) = \underset{z}{Max}\{2z_1^{3/4} z_2^{1/4} \mid z_1 + z_2 \leq B\} \ .$$

We instead maximize the logarithm

$$\underset{z}{Max}\{\ln 2 + \tfrac{3}{4}\ln z_1 + \tfrac{1}{4}\ln z_2 \mid r_1 z_1 + r_2 z_2 \leq B\}$$

FOC

$$\frac{\frac{3}{4}}{r_1 z_1} = \frac{\frac{1}{4}}{r_2 z_2} = \frac{1}{r_1 z_1 + r_2 z_2} = \frac{1}{B}$$

Therefore $z_1 = \dfrac{3B}{4r_1}$ and $z_2 = \dfrac{B}{4r_2}$

and so

$$q = 2(z_1)^{3/4}(z_2)^{1/2} = (\frac{3B}{4r_1})^{3/4}(\frac{B}{4r_2})^{1/4} = (\frac{3}{4r_1})^{3/4}(\frac{1}{4r_2})^{1/4} B \ .$$

(c) With a budget of $\hat{B}$ the maximum output is

$$\hat{q} = (\frac{3}{4r_1})^{3/4}(\frac{1}{4r_2})^{1/4}\hat{B} \ . \qquad (*)$$

Thus the minimum total cost of producing $\hat{q}$ is less than or equal to

$$\hat{B} = (\frac{4r_1}{3})^{3/4}(\frac{4r_2}{3})^{1/4}\hat{q} \ .$$

But for any budget $B < \hat{B}$ it follows from $(*)$ that it is impossible to produce $\hat{q}$.

So the minimum total cost of producing $\hat{q}$ is $TC(\hat{q}) = \hat{B} = (\frac{4r_1}{3})^{3/4}(\frac{4r_2}{3})^{1/4}\hat{q}$

(d) $AC(q) = MC(q) = (\frac{4r_1}{3})^{3/4}(\frac{4r_2}{3})^{1/4}$

(e) For a price taking firm , if $p > MC(q) = (\frac{4r_1}{3})^{3/4}(\frac{4r_2}{3})^{1/4}$ there is no profit maximizing output as it is always more profitable to increases output since $MR = p > MC$. If $p < MC(q) = (\frac{4r_1}{3})^{3/4}(\frac{4r_2}{3})^{1/4}$ the profit maximizing output is zero. Therefore if the firm's profit-maximizing output $\bar{q} > 0$, it must be the case that

$$p = MC(q) = (\frac{4r_1}{3})^{3/4}(\frac{4r_2}{3})^{1/4}$$

Remark: It is important to understand why this result holds for any constant returns to scale production function.

Observation 1: A CRS production function is homothetic.

To see this, suppose $F(z^0) \geq F(z^1)$. Appealing to CRS $F(\theta z^0) = \theta F(z^0)$ and $F(\theta z^1) = \theta F(z^1)$. Therefore

$$F(\theta z^0) = \theta F(z^0) \geq \theta F(z^1) = F(\theta z^0)$$

Observation 2: Given homotheticity, we showed that if z^* solves the budget problem

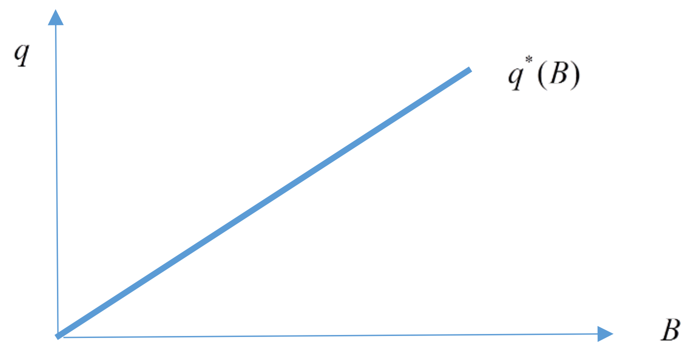
$$q^* = \underset{z}{Max}\{F(z) \,|\, p \cdot z \leq B\}$$

Then θz^* solves the budget problem $q^{**} = \underset{z}{Max}\{F(z) \mid p \cdot z \le \theta B\}$.

Then

$$q^{**} = F(\theta z^*) = \theta F(z^*) = \theta q^* .$$

Thus maximized output rises linearly with the budget. This is depicted below.



Appealing to observation 1 it follows that minimum total cost $C(q,r)$ is proportional to output. Hence MC and AC are constant.

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2. Three commodity economy

(a) $x_1 = \frac{64}{3} - z_1$, $x_2 = 32 - z_2$, $x_3 = q = 2z_1^{3/4} z_2^{1/4}$

Simplify by maximizing $u = \ln U = \ln x_1 + \ln x_2 + 4\ln x_3$

$$= \ln(\frac{64}{3} - z_1) + \ln(32 - z_2) + 4[\ln 2z_1^{3/4} z_2^{1/4}]$$

$$= \ln(\frac{64}{3} - z_1) + \ln(32 - z_2) + 4\ln 2 + 3\ln z_1 + \ln z_2 .$$

FOC: $\frac{1}{\frac{64}{3} - z_1} = \frac{3}{z_1} = \frac{4}{\frac{64}{3}}$, $\frac{1}{32 - z_2} = \frac{1}{z_2} = \frac{2}{32}$ (Note that I have use the ratio rule twice.)

Therefore

$z_1^* = z_2^* = 16$ and so $q^* = 2(16)^{3/4}(16)^{1/4} = 32$.

(b) From line 1 in (a), $x^* = (\omega_1 - z_1^*, \omega_2 - z_2^*, q^*) = (\frac{16}{3}, 16, 32)$

(c) This is the same production function as in question 1 so the average cost is

$$AC(r) = MC(q) = (\frac{4p_1}{3})^{3/4}(\frac{4p_2}{3})^{1/4}$$

As discussed above, for positive production $p_3 = AC(p_1, p_2)$. Therefore

$$\text{Total Revenue} = p_3 q = qAC(q) = \text{Total cost}$$

(d) $\pi = p_3 q - p_1 z_1 - p_2 z_2 = p_3 2 z_1^{3/4} z_2^{1/4} - p_1 z_1 - p_2 z_2$

FOC

$$\frac{\partial \pi}{\partial z_1}(z) = p_3 \frac{\partial F}{\partial z_1}(z) - p_1 = 2p_3 \frac{3}{4} z_1^{-1/4} z_2^{1/4} - p_1 \qquad (*)$$

and

$$\frac{\partial \pi}{\partial z_2}(z) = p_3 \frac{\partial F}{\partial z_2}(z) - p_2 = 2p_3 \frac{1}{4} z_1^{3/4} z_2^{-3/4} - p_2 . \qquad (**)$$

These must both be zero at $z^* = (16,16)$.

Therefore

$$\frac{\partial \pi}{\partial z_1}(z^*) = p_3 \frac{6}{4} - p_1 = 0 \text{ and } \frac{\partial \pi}{\partial z_2}(z^*) = \frac{p_3}{2} - p_2 = 0 .$$

The supporting price vector is therefore

$$(p_1, p_2, p_3) = (\tfrac{3}{2}, \tfrac{1}{2}, 1) .$$

(e) Note that $x^* = (\frac{16}{3}, 16, 32)$ and the value of the endowment is $p \cdot \omega = 48$.

In this exercise maximum profit is $\Pi = p_3 q^* - p_1 z_1^* - p_2 z_2^* = 1(32) - \tfrac{3}{2}(16) - \tfrac{1}{2}(16) = 0$. However the following remark is completely general.

General Remark: The level set for maximum profit is

$$p_3 q - p_1 z_1 - p_2 z_2 = p_3 q^* - p_1 z_1^* - p_2 z_2^* = \Pi \ .$$

The sublevel set for maximum profit is

$$p_3 q - p_1 z_1 - p_2 z_2 \le p_3 q^* - p_1 z_1^* - p_2 z_2^* = \Pi$$

Note that $q = x_3$, $z_1 = \omega_1 - x_1$ and $z_2 = \omega_2 - x_2$. Therefore the sublevel set for maximum profit can be rewritten as follows:

$$p_3 x_3 - p_1(\omega_1 - x_1) - p_2(\omega_2 - x_2) \le \Pi \ .$$

Equivalently,

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \le \Pi + p_1 \omega_1 + p_2 \omega_2$$

Using vector notation

$$p \cdot x \le \Pi^* + p \cdot \omega \ .$$

The right hand side is the dividend from the firms and the value of the aggregate endowment. Thus the sublevel set for maximized profit is also the budget set of the representative consumer.

The remaining step is to check that the FOC for the following consumer problem are satisfied at $x^* >> 0$

$$Max\{U(x) \mid p \cdot x \le \Pi + p \cdot \omega\} \ .$$

FOC: Marginal utility per dollar must be equal

$$\frac{1}{p_1} MU_1(x^*) = \frac{1}{p_2} MU_2(x^*) = \frac{1}{p_3} MU_3(x^*)$$

It is readily confirmed that these conditions hold.

3. Elasticity of substitution in a two commodity economy

$$M(p,\bar{u}) = \underset{x}{Min}\{p \cdot x \,|\, U(x) \geq \bar{u}\}$$

(a) Minimizing $p \cdot x$ is the same as maximizing $-p \cdot x$. So we consider the following problem.

$$-M(p,\bar{u}) = \underset{x}{Max}\{-p \cdot x \,|\, U(x) \geq \bar{u}\}$$

Lagrangian

$$L = -p \cdot x + \lambda(U(x) - \bar{u}) \ .$$

To make it clear that we are considering a fixed utility problem (so income is compensated) we write the solution as x^c . With $x^c >> 0$ the FOC are

$$\frac{\partial L}{\partial x_1} = -p_1 + \lambda \frac{\partial U}{\partial x_1}(x^c) = 0$$

$$\frac{\partial L}{\partial x_2} = -p_2 + \lambda \frac{\partial U}{\partial x_2}(x^c) = 0$$

From these two equations

$$MRS(x^*) = \frac{\frac{\partial U}{\partial x_1}(x^c)}{\frac{\partial U}{\partial x_2}(x^c)} = \frac{p_1}{p_2} \ .$$

(b) $U(x_1, x_2) = a_1 x_1^{1/2} + a_2 x_2^{1/2}$. Then

$$MRS(x^*) = \frac{\frac{1}{2} a_1 x_1^{-1/2}}{\frac{1}{2} a_2 x_2^{-1/2}} = \frac{a_1}{a_2}(\frac{x_2}{x_1})^{1/2} = \frac{p_1}{p_2}$$

Therefore

$$\frac{x_2^c}{x_1^c} = (\frac{a_2}{a_1})^2 (\frac{p_1}{p_2})^2$$

(c) $\sigma = \mathcal{E}(\frac{x_2^{\,c}}{x_1^{\,c}}, p_1) = p_1 \frac{\partial}{\partial p_1} \ln \frac{x_2^c}{x_1^c} = p_1 \frac{\partial}{\partial p_1} \ln (\frac{a_2}{a_1})^2 (\frac{p_1}{p_2})^2$

$$= p_1 \frac{\partial}{\partial p_1} \ln (\frac{a_2}{a_1})^2 (\frac{p_1}{p_2})^2$$

$$= p_1 \frac{\partial}{\partial p_1} [\ln (\frac{a_2}{a_1})^2 + 2\ln p_1 - 2\ln p_2] = 2 .$$

(d) (i)

$$MRS(x^*) = \frac{\frac{1}{3} a_1 x_1^{-2/3}}{\frac{1}{3} a_2 x_2^{-2/3}} = \frac{a_1}{a_2} (\frac{x_2}{x_1})^{2/3} = \frac{p_1}{p_2}$$

Therefore

$$\frac{x_2^c}{x_1^c} = (\frac{a_2}{a_1})^{3/2} (\frac{p_1}{p_2})^{3/2}$$

Arguing as in (c) $\sigma = \mathcal{E}(\frac{x_2^{\,c}}{x_1^{\,c}}, p_1) = \frac{3}{2}$

(d) (ii) I prefer to simplify by finding a more convenient representation of the consumer's preferences. The inverse of $U(x)$ is $a_1 x_1^{-1} + a_2 x_2^{-1}$. Thus has the same level sets but is a decreasing function. Then I choose the following representation

$$u(x) = -a_1 x_1^{-1} - a_2 x_2^{-1}$$

$$MRS(x^*) = \frac{a_1 x_1^{-2}}{a_2 x_2^{-2}} = \frac{a_1}{a_2} (\frac{x_2}{x_1})^2 = \frac{p_1}{p_2}$$

Therefore

$$\frac{x_2^c}{x_1^c} = (\frac{a_2}{a_1})^{1/2} (\frac{p_1}{p_2})^{1/2}$$

Arguing as in (c) $\sigma = \mathcal{E}(\frac{x_2^{\,c}}{x_1^{\,c}}, p_1) = \frac{1}{2}$.

(e) The three utility functions are all sums of concave functions so are concave. Then the superlevel sets are convex sets.

(i) Level set $U(x)=U(1)$. $U(x)=a_1x_1^{1/2}+a_2x_2^{1/2}=a_1+a_2$.

Consider $x_1=0$. Then $a_2x_2^{1/2}=a_1+a_2$ and so $x_2=(1+\frac{a_1}{a_2})^2$.

Similarly if $x_2=0$ then $x_1=(1+\frac{a_2}{a_1})^2$

(ii) Level set $U(x)=U(1)$. $U(x)=a_1x_1^{1/3}+a_2x_2^{1/3}=a_1+a_2$.

Consider $x_1=0$. Then $a_2x_2^{1/3}=a_1+a_2$ and so $x_2=(1+\frac{a_1}{a_2})^3$.

(iii) Level set $u(x)=u(1)$. $u(x)=-a_1x_1^{-1}-a_2x_2^{-1}=-a_1-a_2$.

Note that any x in which either x_1 or x_2 is very small cannot be on the level set since one of the terms in the utility functions has a limit of $-\infty$.

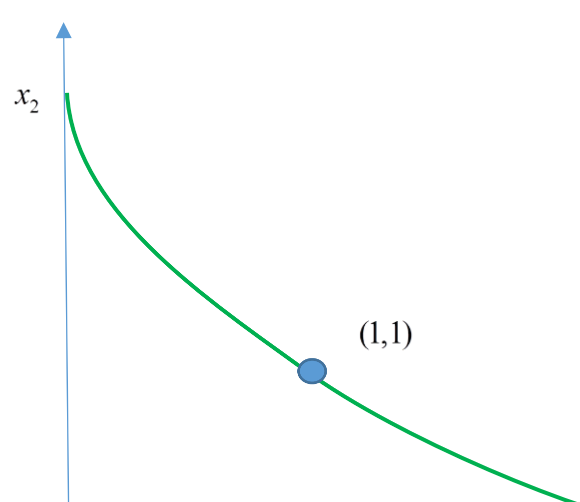
Suppose $x_2\to\infty$. Then $x_2^{-1}\to 0$ and so

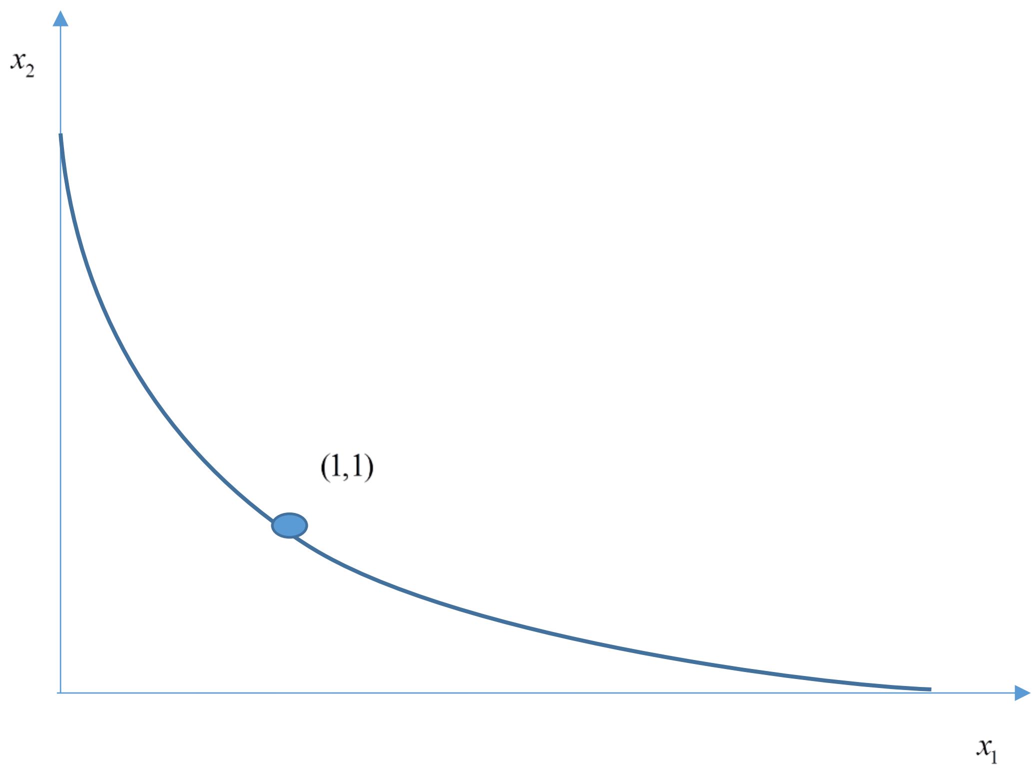
$$u(x)=-a_1x_1^{-1}=-a_1-a_2$$

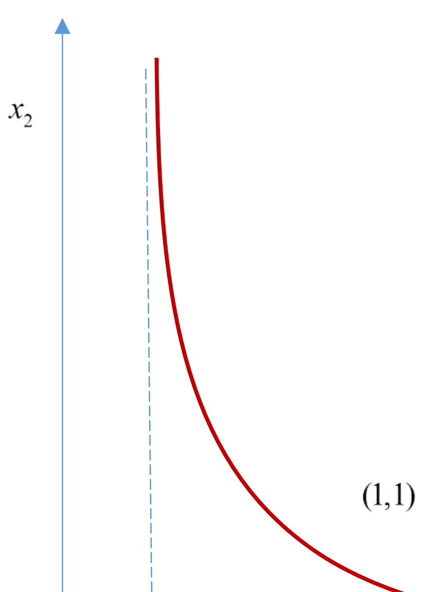
Then

$$x_1^{-1}=1+\frac{a_2}{a_1} \text{ and so } x_1=\frac{1}{1+\frac{a_2}{a_1}}$$

(f) The three level sets are depicted below.




x_2
(1,1)
x_1


x_2
(1,1)

Technical Remark (not examinable and only for those who like calculus).

For any z^0 define $z(\theta) = \theta z^0 = (\theta z_1^0, \theta z_2^0)$

The derivative of $F(z(\theta))$ with respect to θ is

$$\frac{\partial F}{\partial z_1}(\theta z)z_1^0 + \frac{\partial F}{\partial z_2}(\theta z)z_2^0 \ .$$

Under CRS, $F(z(\theta)) = \theta F(z^0)$. Therefore the derivative of $F(z(\theta))$ is $F(z^0)$.

Therefore

$$F(z^0) = \frac{\partial F}{\partial z_1}(\theta z)z_1^0 + \frac{\partial F}{\partial z_2}(\theta z)z_2^0 \text{ for all } \theta$$

In particular, setting $\theta = 1$,

$$F(z^0) = \frac{\partial F}{\partial z_1}(z^0)z_1^0 + \frac{\partial F}{\partial z_2}(z^0)z_2^0 \text{ for all } z^0 \ .$$

Therefore

$$p_3 F(z) = p_3 \frac{\partial F}{\partial z_1}(z)z_1 + p_3 \frac{\partial F}{\partial z_2}(z)z_2 \qquad (1)$$

From 2(d) the FOC are

$$\frac{\partial \pi}{\partial z_1}(z) = p_3 \frac{\partial F}{\partial z_1}(z) - p_1 = 0 \qquad (2)$$

and

$$\frac{\partial \pi}{\partial z_2}(z) = p_3 \frac{\partial F}{\partial z_2}(z) - p_2 = 0 \ . \qquad (3)$$

Substituting (2) and (3) into (1), $p_3 q^* = p_1 z_1^* + p_2 z_2^*$.

So revenue is equal to cost and so profit is zero.