Homework 3 Due Tuesday, Nov 28

Answers

Answer to question 1

(a) Double both sides of the second equation and subtract the second equation

$160q_a + 120q_b = 20$.

$60q_a + 120q_b = 10$

$100q_a = 10$. Then $q_a = \frac{1}{10}$. Substituting, $q_b = \frac{1}{30}$.

(b) The value of the fund is

$$P^a q_a + P^b q_b = \frac{1}{10}200 + \frac{1}{30}300 = 30$$

(c)

$$\begin{bmatrix} 60 \\ 80 \end{bmatrix} q_a + \begin{bmatrix} 120 \\ 60 \end{bmatrix} q_b = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

Solve to obtain $\hat{q} = (-\frac{3}{30}, \frac{4}{30})$.

(d) The value of the fund is

$$P^a q_a + P^b q_b = -\frac{3}{30}200 + \frac{4}{30}300 = 20 .$$

(e)

$$\begin{bmatrix} 60 \\ 80 \end{bmatrix} q_a + \begin{bmatrix} 120 \\ 60 \end{bmatrix} q_b = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

Solve to obtain $\hat{\hat{q}} = (\frac{2}{10}, -\frac{1}{10})$.

(f) The value of this fund is 10

Owning the first fund costs 20 and yields 10 in state 1. Thus a claim to a unit in state 1 has a market value of $p_1 = 2$. Owning the second fund has a value of 10 and yields 10 in state 2. Thus a claim to state 2 costs 1.

From (e) and (f) the value of the mutual fund is 10. Thus a claim to a unit in state 1 has a market value of $p_1 = 2$.

We could get this another way. Asset 1 with return $(60,80)$ is worth 200. Therefore $p_1 60 + p_2 80 = 200$ The value of its state 1 claim is $60 p_1 = 120$. Therefore the market value of its state 2 claims is 80 and so $p_2 80 = 80$.

2. Betting on "The Game".

(a) $x_1 = \hat{w} + q,\ \ x_2 = \hat{w} - (\frac{p_1}{p_2})q$

Therefore

$p_1 x_1 = p_1 \hat{w} + p_1 q,\ \ p_2 x_2 = p_2 \hat{w} - p_1 q$.

Adding these equations,

$p_1 x_1 + p_2 x_2 = (p_1 + p_2)\hat{w}$

(b) Tommy's utility function is $U^T = \pi_1^T \ln x_1^T + \pi_2^T \ln x_2^T$. Tommy therefore solves the following problem

$$\underset{x}{Max}\{U^T(x) \,|\, (p_1 + p_2)\hat{w} - p_1 x_1 + p_2 x_2 \geq 0\}$$

FOC

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1} = \frac{1}{p_2}\frac{\partial U}{\partial x_2} \qquad \frac{\pi_1}{p_1}\frac{1}{x_1} = \frac{\pi_2}{p_2}\frac{1}{x_2}$$

Applying the Ratio Rule

$$\frac{\pi_1}{p_1}\frac{1}{x_1} = \frac{\pi_2}{p_2}\frac{1}{x_2} = \frac{1}{p_1 x_1 + p_2 x_2} = \frac{1}{(p_1 + p_2)\hat{w}} = \frac{1}{\hat{w}}$$

Therefore

$$\overline{x}_1^T = \frac{\pi_1^T \hat{w}}{p_1} \ .$$

(c) For Bev the FOC is

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1} = \frac{1}{p_2}\frac{\partial U}{\partial x_2} \qquad \frac{\pi_1}{p_1}\frac{1}{b + x_1} = \frac{\pi_2}{p_2}\frac{1}{b + x_2}$$

Appealing to the Ratio Rule

$$\frac{\pi_1}{p_1}\frac{1}{b+x_1}=\frac{\pi_2}{p_2}\frac{1}{b+x_2}=\frac{1}{p_1(b+x_1)+p_2(b+x_2)}=\frac{1}{b+\hat{w}}$$

Therefore

$$b+\overline{x}_1^B=\frac{\pi_1^B(b+\hat{w})}{p_1}$$

(d) If $b=0$

$$\overline{x}_1^B=\frac{\pi_1^B\hat{w}}{p_1}.$$

Therefore total demand for state 1 wealth is

$$\overline{x}_1^T+\overline{x}_1^B=\frac{\pi_1^T\hat{w}}{p_1}+\frac{\pi_1^B\hat{w}}{p_1}$$

Total supply is the same of the wealth of the two representative bettors, i.e. $2\hat{w}$.

For equilibrium supply = demand so

$$\overline{x}_1^T+\overline{x}_1^B=\frac{\pi_1^T\hat{w}}{p_1}+\frac{\pi_1^B\hat{w}}{p_1}=2\hat{w}.$$

Rearranging this expression

$$p_1=\tfrac{1}{2}\pi_1^T+\tfrac{1}{2}\pi_1^B=0.6 \ .$$

Therefore

$$\frac{p_1}{p_2}=\frac{0.6}{0.4}=\frac{3}{2} \ .$$

(e)

$$\overline{x}_1^B=\frac{\pi_1^B(b+\hat{w})}{p_1}-b\,,\quad \overline{x}_1^T=\frac{\pi_1^T\hat{w}}{p_1}.$$

Therefore

$$\overline{x}_1^T+\overline{x}_1^B=\frac{\pi_1^T\hat{w}}{p_1}+\frac{\pi_1^B(b+\hat{w})}{p_1}-b=2\hat{w}$$

Therefore

$$\frac{\pi_1^T \hat{w} + \pi_1^B (b + \hat{w})}{p_1} = 2\hat{w} + b \,.$$

Therefore

$$p_1 = \frac{\pi_1^T \hat{w} + \pi_1^B (b + \hat{w})}{2\hat{w} + b}$$

The derivative with respect to b is negative. Thus p_1 declines. Since we have normalized, $p_2 = 1 - p_1$ must increases. Therefore the market odds decrease.

(f) If you compute Bev's relative aversion to risk ($RRA^B(w)$) you will find that it is a decreases as b Increases. As Bev becomes less risk averse she is willing to risk more and therefore bet more. Thus the supply of bets on the Bruins increases. This lowers the relative price *the market odds of a Trojan victory.

3. Two asset economy

(a) $MRS^h(x^h) = \dfrac{\pi_1}{\pi_2}\dfrac{v'(x_1^h)}{v'(x_2^h)} = \dfrac{\frac{2}{3}}{\frac{1}{3}}\dfrac{x_2^h}{x_1^h} = 2\dfrac{x_2^h}{x_1^h}$.

For an allocation to be PE

$$MRS^A = MRS^A = 2\frac{x_2^A}{x_1^A} = 2\frac{x_2^B}{x_1^B} \,.$$

Therefore

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{x_2^A + x_2^B}{x_1^A + x_1^B} = \frac{\omega_2}{\omega_1} = \frac{1}{4}$$

Thus the consumption ratios are the same as the aggregate endowment ratio.

(b) It follows from (a) that $MRS^A = MRS^B = \dfrac{1}{2}$ for any PE allocation.

By the first welfare theorem a WE is a PE. In a WE, it follows from the necessary conditions for utility maximization that

$$MRS^A = MRS^B = \frac{p_1}{p_2} \,.$$

Then the WE price ratio is 2. Then choose the price vector $p = (1, 2)$

(d) The value of the endowments are as follows:

$P^A = p \cdot z^A = (1,2)\cdot(200,100) = 400$

$P^B = p \cdot z^B = (1,2)\cdot(200,0) = 200$.

(d) Alex's endowment has a value equal to $\frac{2}{3}$ of the total endowment thus in equilibrium he consumes $\frac{2}{3}$ of the total endowment i.e. $\overline{x}^A = (\frac{800}{3},\frac{200}{3})$. Bev's consumption is equal to $\frac{1}{3}$ of the total endowment thus in equilibrium she consumes $\frac{1}{3}$ of the total endowment i.e. $\overline{x}^B = (\frac{400}{3},\frac{100}{3})$.

The asset price ratio is $P^A / P^P = 2$. Thus Alex sells $\frac{1}{3}$ of his plantation (and so retains $\frac{2}{3}$) to purchase $\frac{2}{3}$ of Bev's plantation. Bev is on the other side of this trade.

The resulting allocation is a PE allocation. The MRS are equal so neither consumer will wish to make any additional trade.

(f) Arguing as above, in a PE allocation each consumer has a share of the total endowment if state claims markets are open. Thus Bev and Charles taken together do exactly what Bev did above. So there is no change in any of the conclusions.

4. WE and PE in a three commodity model

(a) $U(\theta x^h) = \theta^{1/2}U(x^h)$ so utility is homothetic. Then we can consider the repr4esentative agent. His endowment I $\omega^R = (100,400,900)$.

(b) His utility maximizing demand must equate the marginal utility per dollar.

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1} = \frac{1}{p_2}\frac{\partial U}{\partial x_2} = \frac{1}{p_3}\frac{\partial U}{\partial x_3}$$

$$\frac{\frac{1}{2}}{p_1}\frac{\pi_1}{\overline{x}_1^{1/2}} = \frac{\frac{1}{2}}{p_2}\frac{\pi_2}{\overline{x}_2^{1/2}} = \frac{\frac{1}{2}}{p_3}\frac{\pi_3}{\overline{x}_3^{1/2}} .$$

To clear all markets, $\overline{x} = \omega$.

$$\frac{1}{p_1}\frac{\frac{1}{6}}{\omega_1^{1/2}} = \frac{2}{p_2}\frac{\frac{2}{6}}{\omega_2^{1/2}} = \frac{1}{p_3}\frac{\frac{3}{6}}{\omega_3^{1/2}}$$

$$\frac{1}{p_1}\frac{1}{10} = \frac{1}{p_2}\frac{2}{20} = \frac{1}{p_3}\frac{3}{30} .$$

(c) Suppose that $\{\hat{x}^A, \hat{x}^B\}$ is a PE allocation.

Holding constant the allocation of x_3 we can consider the other two commodities. If the MRS are not equal therew are opportunities for mutual gain. Then

$MRS(x_1^A, x_2^A) = MRS(x_1^B, x_2^B)$ is a necessary condition for an allocation to be PE.

$MRS(x_1^A, x_2^A) = \dfrac{\pi_1 \frac{1}{2}(x_1^A)^{-1/2}}{\pi_2 \frac{1}{2}(x_2^A)^{-1/2}} = \dfrac{\pi_1}{\pi_2}(\dfrac{x_2^A}{x_1^A})^{1/2}$ and $MRS(x_1^B, x_2^B) = \dfrac{\pi_1}{\pi_2}(\dfrac{x_2^B}{x_1^B})^{1/2}$.

Therefore

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} .$$

Appealing to the ratio rule

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{x_2^A + x_2^B}{x_1^A + x_1^B} = \frac{\omega_2}{\omega_1} .$$

Same argument for any pair of commodities.

It follows that each consumes a fraction of the aggregate endowment.

The value of Alex's endowment is 350 and the value of Bev's is 1050.

Thus Alex has one quarter of the total wealth. Then his consumption is one quarter of the aggregate endowment.