Homework 3 Due Tuesday, Nov 28

1. Financial engineering in a coconut economy

There are two risky assets. Plantation a has a gross state contingent return of $z^a = (60,80)$. The market value of this plantation (financial asset) is $P_a = 200$. Plantation b has a state contingent return of $z^b = (120,60)$ and a market value of $P_b = 300$. The S&P2 therefore has a value of 500.

An example of a simple "engineered" asset is the "S&P2" mutual fund. It holds an equal share of the two assets. If its market value is 50, then its holding of the two assets must $q^{S\&P} = (\frac{50}{500}, \frac{50}{500}) = (\frac{1}{10}, \frac{1}{10})$

Financial engineers wish to design a riskless mutual fund that has the same return $(10,10)$ in the two states. They therefore purchase shares (q_a, q_b) such that

$$\begin{bmatrix} 60 \\ 80 \end{bmatrix} q_a + \begin{bmatrix} 120 \\ 60 \end{bmatrix} q_a = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

Then

$$60q_a + 120q_b = 10 \text{ and } 80q_a + 60q_b = 10.$$

(a) Solve these two equations for the shares q_a and q_b . (One is 3 times the other).

(b) What is the market value of this riskless mutual fund?

Next consider the design of a mutual fund that pays off 10 in state 1 and zero in state 2. With such an asset available it is easy for an investor increase his weath in state 1.

Remark: One of the quantities is negative. So the fund is selling that asset short.

(c) Solve for the shares $(\hat{q}_1, \hat{q}_2)$ in this new mutual fund.

(d) What is its market value?

Next consider the design of a mutual fund that pays off zero in state 1 and 10 in state 2. With such an asset available it is easy for an investor increase his weath in state 2.

(e) Solve for the shares $(\hat{\hat{q}}_1, \hat{\hat{q}}_2)$ in this new mutual fund.

(f) What is its market value?

Use these results to solve for p_s, $s = 1,2$, where p_s is the market value of a financial instrument payong only 1 unit only in state s.

Remark: The point of this excercise is to show that it there are at least an many financial assets as states, it may be possible to replicate all state contingent trading (insurance markets) by trading in asset markets.

2. Betting on "The Game" (2018).

In state 1 the Trojans win. In state 2 the Bruins win. Bev (a Bruin) thinks that the probability of a Trojan victory is $\pi_1^B = 0.4$ and Tommy (a Trojan) thinks that the probaility of of a Trojan victory is $\pi_1^T = 0.8$.

The Tommy's utility function is $v_T(x) = \ln(x)$ Bev's utility fucntion is $v_B(x) = \ln(b + x)$

Both Bev and Tommy have a wealth of $\hat{w}$ million. Think of them as "representative" Bruins and Trojans.

If Tommy makes a bet that pays off q in the event of a Trojan victory he pays the market odds of $\frac{p_1}{p_2}$ and so must pay the bookie $(\frac{p_1}{p_2})q$ if the Tojans lose. Then his state contingent wealth is

$$(x_1, x_2) = (\hat{w} + q, \hat{w} - (\frac{p_1}{p_2})q).$$

(a) Show that this is equvalent to the following state contingent wealth constraint.

$$p_1 x_1 + p_2 x_2 = p_1 \hat{w} + p_2 \hat{w} \ .$$

We normalize so that the sum of the prices is 1.

(b) Hence show that if Tommy makes his optimal bet, then his wealth if the outcome is state 1 (Trojans win) is

$$\overline{x}_1^T = \frac{1}{p_1} \pi_1^T \hat{w}.$$

(c) Show also that Bev's wealth if the outcome is state 1, $\overline{x}_1^B$ satisfies

$$b + \overline{x}_1^B = \frac{1}{p_1} \pi_1^B (b + \hat{w})$$

(d) Suppose that $b = 0$. Show that the equilibrium price of state 1 claims is $p_1 = \frac{1}{2}\pi_1^T + \frac{1}{2}\pi_1^B$. Hence show that the market odds are

$$\frac{p_1}{p_2} = \frac{3}{2}$$

Henceforth consider the case $b > 0$.

(e) Solve for the new equibrium price a of a state 1 claim and hence the new market odds.

How do the maket odds vary with b ?

(f) What is the intuition for the effect of a higher parameter b ?

HINT What is the degree of relative risk aversion of each bettor?

3. Two Asset economy (the coconut plantation economy)

Asset A has a return $z^a = (200,100)$. Asset B has a return $z^b = (200,0)$. Thus the total state contingent wealth is $\omega = (400,100)$. Each consumer has the same utility function $u(x) = \ln x$. The probability of state 1 is $\frac{2}{3}$. Initially suppose that Alex owns asset a and Bev owns asset b .

(a) Solve for the PE allocations in this economy.
(b) If there are competitive markets for state claims (insurance), explain why the equilibrium state claims price ratio is $\dfrac{p_1}{p_2} = \frac{1}{2}$ and hence explain why the equilibrium consumption ratios must equal the endowment ratio $4:1$.
(c) Compare the value of the two endowments. (Choose either $p = (1,2)$ or $p = (\frac{1}{3}, \frac{2}{3})$.) Use this (or some other method) to solve for the equilibrium allocation.
(d) Suppose that there are no state claims prices but the market value of the assets P_a and P_b are equal to the value in the state claims market equilibrium. Explain how it is possible for the two consumers to trade shares and reach the same outcome as in (c). What fraction of each asset would be traded?
(e) Suppose that this trade is completed. Then state contingent markets unexpectedly open. Would there be any further trade in these markets?
(f) Would any of the above results change if there were three investors. Alex owns Asset A. Bev and Charles have 50% ownership of asset B.

4. Walrasian Equilibrium and Pareto efficiency in a three commodity model

Suppose that Alex and Bev have the same utility function

$$U(x^h) = \tfrac{1}{6}(x_1^h)^{1/2} + \tfrac{1}{3}(x_2^h)^{1/2} + \tfrac{1}{2}(x_3^h)^{1/2}$$

Alex has endowment $\omega^A = (50,180,120)$ and Bev has endowment $\omega^B = (50,220,780)$ so that the aggregate endowment is $\omega = \omega^A + \omega^B = (100,400,900)$.

(a) Show that the utility function is homothetic and hence appeal to the represntative agent approach to show that $p = (1,1,1)$ is a WE price vector.

(b) Show that for an allocation to a PE allocation

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{\omega_2}{\omega_1} = 4 \text{ and } \frac{x_3^A}{x_1^A} = \frac{x_3^B}{x_1^B} = \frac{\omega_3}{\omega_1} = 9 .$$

(c) Compare the market values of each consumer's endowment. Hence or otherwise solve for the equilibrium consumption vectors $\overline{x}^A$ and $\overline{x}^B$.

Other exercises (Do not hand in answers)

5. State claims equilibria and asset market equilibria (do not hand in)

(a) Reinterpret the model of question 4 as one in which there are three states. The vector of probabilities is $(\pi_1, \pi_2, \pi_3) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$. Each "endowment" is the state contingent number of coconut trees surving three different types of storm.
(b) Use the equilbrium state claims price vector to solve for the value of the two coconut plantations.
(c) What are the equilibrium outcomes $\overline{x}^A$ and $\overline{x}^B$?
(d) Suppose there are no state claims markets but there is a stock market where the plantations acan be traded. Explan why it is possible to achive the same outcome as in (c) by exchanging shares.
(e) Given these prices, is there any other trade which would make one of the consumers better off?

6. Almost homothetic economy

Each of H consumers has the same utility function $U(x_1^h, x_2^h) = v(a_1^h + x_1^h) + v(a_2^h + x_2^h)$ where $a^h = (a_1^h, a_2^h)$ is an individual specific parameter. The endowment of individual h is ω^h .

Define $\hat{x}^h = x^h + a^h$ and $\hat{\omega}^h = \omega^h + a^h$.

(a) Explain why the following is an equivalent economy. Each of H consumers has the same utility function $u(\hat{x}_1^h, \hat{x}_2^h) = v(\hat{x}_1^h) + v(\hat{x}_2^h)$. The endowment of individual h is $\hat{\omega}^h$.

(b) Give two examples of functions $v(\cdot)$ (excluding $v(x) = \ln x$) for which the transformed economy is homothetic.

Henceforth assume that $v(\cdot) = \ln x$.

(c) Solve for a WE price ratio in this second economy as a function of $\hat{\omega}^h = (\hat{\omega}_1, \hat{\omega}_2)$

(d) As long as the parameters satisfy a^h are sufficiently small, explain why these are also equilibrium prices in the original economy.

7. Two period economy with production

Alex and Bev have the same utility function $U(x_1^h, x_2^h) = v(2 + x_1^h) + v(x_2^h)$ The endowments are $\omega^A = (20,0)$ and $\omega^B = (24,0)$

The first period commodity can be transformed into period 2 output according to

$$q_2 = 2z_1^{1/2}.$$

(a) Use the answer to question 6 to transform the problem into an equivalent economy with homothetic preferences.

Henceforth assume that $v(\hat{x}^h) = \ln \hat{x}^h$.

(b) Show that in the transformed economy the optimal first period input is $z_1^* = \frac{1}{3}\hat{\omega}_1$. What are the WE prices in this economy?

(c) Explain why these are also WE prices in the original economy.