Homework 4 due Thursday December 7 (Q1(c) corrected)

1. Cost functions and output prices in a price taking economy

$$q_A = (\tfrac{3}{2} z_1^A)^{2/3}(3z_2^A)^{1/3} = \frac{3}{2^{2/3}}(z_1^A)^{2/3}(z_2^A)^{1/3} \text{ ,}$$

$$q_B = (3z_1^B)^{1/3}(\tfrac{3}{2} z_2^B)^{2/3} = \frac{3}{2^{2/3}}(z_1^A)^{1/3}(z_2^A)^{2/3}$$

(a) For a budget C, output of A is maximized by spending two thirds of the budget on input 1. (Standard optimization problem.)

Then $r_1 z_1^A = \tfrac{2}{3}C$ and $r_2 z_2^A = \tfrac{1}{3}C$.

Maximized output is therefore

$$q_A = (\tfrac{3}{2} z_1^A)^{2/3}(3z_2^A)^{1/3} = (\frac{C}{r_1})^{2/3}(\frac{C}{r_2})^{1/3} = \frac{C}{r_1^{2/3} r_2^{1/3}} \text{ .}$$

Inverting,

$$C_A(q_A) = r_1^{2/3} r_2^{1/3} q_A \text{ .}$$

By a symmetrical argument,

$$C_B(q_B) = r_1^{1/3} r_2^{2/3} q_B$$

(b) If commodities A and B are both produced, the for both, price is equal to ,arginal cost.

$$p_A = r_1^{2/3} r_2^{1/3}$$

$$p_B = r_1^{1/3} r_2^{2/3}$$

Hence

$$\frac{p_A}{p_B} = (\frac{r_1}{r_2})^{1/3} \text{ .}$$

Hence if the output price ratio $\frac{p_A}{p_B}$ rises, then $\frac{r_1}{r_2}$ rises.

(c) Commodity A is more input 1 intensive if the MRTS is higher for commodity A at the aggregate endowment ω.

$$MRTS_A(\omega) = 2(\frac{\omega_2}{\omega_1})$$ and $$MRTS_B(\omega) = \tfrac{1}{2}(\frac{\omega_2}{\omega_1}) .$$

Therefore

$$\frac{MRTS_B(\omega)}{MRTS_A(\omega)} = \tfrac{1}{4} .$$

This is less than 1.

(d) From (b),

$$p_A = r_1^{2/3} r_2^{1/3}$$ and $$p_B = 1 = r_1^{1/3} r_2^{2/3}$$

From the second equation, $r_1 = \dfrac{1}{r_2^{\,2}}$. Substituting this into the first equation,

$$p_A = \frac{1}{r_1}$$

(e) From (b)

$p_A = (\frac{r_1}{r_2})^{1/3}$. Therefore $\frac{r_1}{r_2} = p_A^{\,3}$. Then $\varepsilon(\frac{r_1}{r_2}, p_A) = p_A \frac{\partial}{\partial p_A} \ln \frac{r_1}{r_2} = p_A \frac{\partial}{\partial p_A} \ln p_A^{\,3} = 3$

2. Two-input, two-output CRS economy

(a) $U = \ln q_A + \ln q_B = \tfrac{2}{3}\ln z_1^A + \tfrac{1}{3}\ln z_2^A + \tfrac{1}{3}\ln z_1^B + \tfrac{2}{3}\ln z_2^B$ + constant

Constraints: $z_1^A + z_1^B \le 1$, $z_2^A + z_2^B \le 1$.

We first consider input 1 and solve

$$Max\{\tfrac{2}{3}\ln z_1^A + \tfrac{1}{3}\ln z_1^B \mid 1 - z_1^A - z_1^B \ge 0\} .$$

Solution $\bar{z}_1^A = \tfrac{2}{3}$, $\bar{z}_1^B = \tfrac{1}{3}$.

We then consider input 2 and solve

$$Max\{\tfrac{1}{3}\ln z_2^A + \tfrac{2}{3}\ln z_2^B \mid 1 - z_2^A - z_2^B \ge 0\} .$$

Solution $\bar{z}_2^A = \tfrac{1}{3}$, $\bar{z}_2^B = \tfrac{2}{3}$.

Therefore $\bar{z}^A = (\tfrac{2}{3}, \tfrac{1}{3})$ and $\bar{z}^B = (\tfrac{1}{3}, \tfrac{2}{3})$

$\bar{q}_A = (\frac{3}{2}\bar{z}_1^A)^{2/3}(3\bar{z}_2^A)^{1/3} = 1$

$\bar{q}_B = (3\bar{z}_1^B)^{1/3}(\frac{3}{2}\bar{z}_2^B)^{2/3} = 1$

(b) $\dfrac{p_A}{p_B} = MRS(\bar{q}) = \dfrac{\partial U}{\partial q_A} / \dfrac{\partial U}{\partial q_B} = \dfrac{\bar{q}_B}{\bar{q}_A} = 1$.

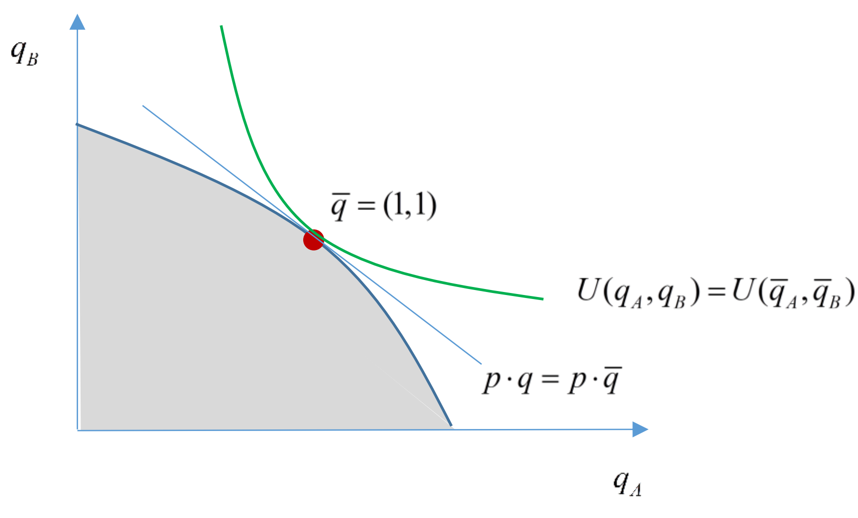
Appealing to exercise 1 the input price ratio is 1, ie $r_2 = r_1$

The MRTS is equal to the input price ratio.

Throughout we will normalize and assume that $p_B = 1$. What is the equilibrium price of commodity A. What is the equilibrium input price ratio? What is the equilibrium MRTS in bothe indistries?

(c) $p_B = 1 = r_1^{1/3} r_2^{2/3} = r_1^{1/3} r_1^{2/3} = r_1$

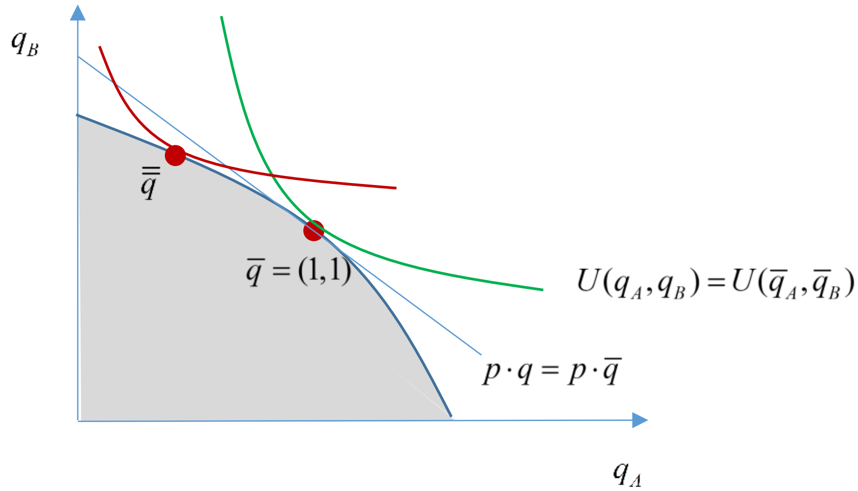
Thus the input prices are both equal to 1.



(d) There is no effect on output prices and so by our analysis no effect on input pruices either.

(e) Profit maiximization l=results in the same production is before. Consumers can trade anyway they like at therse prices. Thus international trade is almost certain.

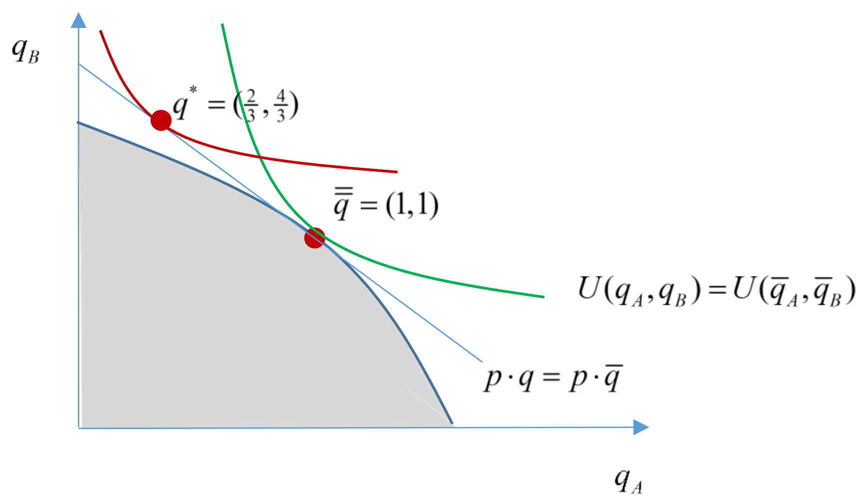
(f) The new level sets have a MRS that is half that in the other economy. Thus the new level sets are as depicted below.



The output vector $\bar{\bar{q}}$ in the second economy is to the North-West of the old economy where the steepness of the boundary is lower. Thus the price ratio in this economy is lower. Since $p_B = 1$ it follws that p_A is lower.

The results follow from exercise 1

(g) The second economy now produces the same output as the first economy $\bar{\bar{q}} = (1,1)$ and then trades at the world prices as depicted below.



Given the log utility, expecnditure on commodity 2 is twice as much as on commodity 1. Since $p_A = p_B = 1$ it follows that $q^* = (\frac{2}{3}, \frac{4}{3})$. Thus the exported quantity is $\frac{1}{3}$.

3. Price v. quantity competition

$$q_1 = 12 - p_1 + \tfrac{1}{2} p_2 \ , \ q_2 = 12 - p_2 + \tfrac{1}{2} p_1 \,,$$

$$C_1(q_1) = 6q_1 \ , \ C_2(q_2) = 6q_2$$

Suppose firms compete by setting prices rather then quantities.

(a) $\Pi_1 = (p_1 - 6)q_1 = (p_1 - 6)(12 - p_1 + \tfrac{1}{2} p_2)$

$$\frac{\partial \Pi_1}{\partial p_1} = (12 - p_1 + \tfrac{1}{2} p_2) - (p_1 - 6) = 18 - 2p_1 + \tfrac{1}{2} p_2 = 0$$

$$\frac{\partial \Pi_2}{\partial p_2} = 18 - 2p_2 + \tfrac{1}{2} p_1 = 0$$

$$(p_1, p_2) = (12, 12)$$

Is the unique solution.

(b)

Double the first equation

$$2q_1 = 24 - 2p_1 + p_2$$

$$q_2 = 12 - p_2 + \tfrac{1}{2} p_1$$

Add

$$2q_1 + q_2 = 36 - \tfrac{3}{2} p_1$$

$$p_1 = 24 - \tfrac{4}{3} q_1 - \tfrac{2}{3} q_2$$

Appealing to symmetry

$$p_2 = 24 - \tfrac{4}{3} q_2 - \tfrac{2}{3} q_1$$

$$\Pi_1 = (p_1 - 6)q_1 = (18 - \tfrac{4}{3} q_1 - \tfrac{2}{3} q_2) q_1$$

$$\frac{\partial \Pi_1}{\partial q_1} = 18 - \tfrac{8}{3} q_1 - \tfrac{2}{3} q_2 = 0 .$$

By the same argument

$$\frac{\partial \Pi_2}{\partial q_2} = 18 - \tfrac{8}{3} q_2 - \tfrac{2}{3} q_1 = 0$$

Solving $\overline{q} = (5.4, 5.4)$

4A. Auction with two identical items for sale

(a) Another buyer has a lower value than θ with probbaility $F(\theta)$ and a higher value with probability $1 - F(\theta)$.

(i) The joint probability that both opponents have a lower value is then $F(\theta) \times F(\theta) = F^2(\theta)$.

(ii) The joint probability that 2 has a higher and 3 has a lower probaility is then $F(\theta) \times (1 - F(\theta))$

(iii) Same as (ii).

Adding these, the win probability is

$$w(\theta) = F^2(\theta) + 2F(\theta)(1 - F(\theta) = 2F(\theta) - F^2(\theta)$$

(b) The joint probability that both have higher values (so you do not win an item is $(1 - F(\theta)) \times (1 - F(\theta)) = (1 - F(\theta))^2$

Therefore the win probability is

$$w(\theta) = 1 - (1 - F(\theta))^2 = 2F(\theta) - F^2(\theta)$$

The argument is exactly as in the slides

$$U(\theta) = w(\theta)(\theta - B(\theta) \qquad (*)$$

Then use the naïve buyer argument to show that $U'(\theta) = w(\theta)$

(c) $U'(\theta) = w(\theta) = 2F(\theta) - F^2(\theta) = 2\theta - \theta^2$

Also $U(0) = 0$. Therefore

$$U(v_i) = \int_0^{v_i} U'(\theta) d\theta = \int_0^{v_i} (2\theta - \theta^2) d\theta = v_i^2 - \tfrac{1}{3} v_i^3 \ .$$

Thisis true for all v_i . Therefore

$$U(\theta) = \theta^2 - \tfrac{1}{3}\theta^3$$

$$w(\theta) = 2\theta - \theta^2 .$$

Appealing to (*) , solve for $B(\theta)$

4B. Entry in a symmetric qunatity setting model

(b) Two firms

$p = 30 - q_1 - q_2$

$\Pi_1(q) = R_1 - C_1 = (30 - q_1 - q_2)q_1 - 60 - q_1^2$

$\frac{\partial \Pi_1}{\partial q_1}(q) = 30 - 2q_1 - q_2 - 2q_1 = 30 - 4q_1 - q_2 = 0$ for a best reponse.

Therefore firm 1's best reponse is

$\bar{q}_1 = \frac{1}{4}(30 - q_2)$

By the same argument, firm 2's best reponse is

$\bar{q}_2 = \frac{1}{4}(30 - q_1)$.

The mutuial best response vector $\bar{q}$ must satisfy both equations.

Solving, $\bar{q} = (6,6)$ then $\bar{p} = 18$ and

$\Pi_i(q) = pq_1 - 60 - q_1^2 = 12$

(c) 3 firms

$p = 30 - q_1 - q_2 - q_3$

$\Pi_1(q) = R_1 - C_1 = (30 - q_1 - q_2 - q_3)q_1 - 60 - q_1^2$

$\frac{\partial \Pi_1}{\partial q_1}(q) = 30 - 2q_1 - q_2 - q_3 - 2q_1 = 30 - 4q_1 - q_2 - q_3 = 0$ for a best reponse. (BR)

Therefore firm 1's best reponse is

$\bar{q}_1 = \frac{1}{4}(30 - q_2 - q_3)$

By the same argument, firm 2 and firm 3's best reponses are

$\bar{q}_2 = \frac{1}{4}(30 - q_1 - q_3)$ and $\bar{q}_3 = \frac{1}{4}(30 - q_1 - q_2)$

Solving for the mutual best reponse vector

$\bar{q} = (5,5,5)$.

Remark: Given the symmetry of the problem it is simpler to guess that the equilibrium is symmetric.

Setting $q_2 = q_3 = q_1$ in (BR)

$$\frac{\partial \Pi_1}{\partial q_1}(q) = 30 - 4q_1 - q_1 - q_1 = 30 - 6q_1 = 0.$$

And so $\bar{q}_1 = \bar{q}_2 = \bar{q}_3 = 5$

The price is $p = 30 - q_1 - q_2 - q_3 = 15$.

Thus the profit ot a firm is

$$\Pi_i = pq_i - 60 - q_i^2 = -10$$

Thus there is no NE with three firms in the market.

The NE is two firms in (see above) and the third firm out.

However we cannot really which ones will be in. What if Firm 3 does enter. Then one has to leave eventually but which one?

(d) $\Pi_3 = pq_3 - F_3 - q_i^2 = 50 - F_3$.

Therefore

Firm 3 will enter if it is profitable to do so. Then $F_3 < 50$ for profitable entry.

(e) By the same argument,

$$\frac{\partial \Pi_1}{\partial q_1}(q) = 30 - 4q_1 - q_2 - q_3 - q_4 = 0.$$

If there is an equilibrium it must be symmetric. Then $\bar{q}_i = \frac{30}{7}$.

But with this output the proft of each firm is negative.