41 pages

Introduction

Four questions…

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Who are you going to learn most from at UCLA?

What do economists do?

Discuss in 3 person groups

Maximization

1. Profit-maximizing firm

Example 1:

Cost function

$$C(q) = 5 + 12q + 3q^2$$

Demand price function

$$p(q) = 20 - q$$

Group exercise: Solve for the profit maximizing output and price.

Example 2: Two products

MODEL 1

Cost function

$$C(q) = 10q_1 + 15q_2 + 2q_1^{\,2} + 3q_1q_2 + 2q_2^{\,2}$$

Demand price functions

$p_1 = 85 - \frac{1}{4}q_1$ **and** $p_2 = 90 - \frac{1}{4}q_2$

Group 1 exercise: How might you solve for the profit maximizing outputs?

MODEL 2

Cost function

$$C(q) = 10q_1 + 15q_2 + q_1^{\,2} + 3q_1q_2 + q_2^{\,2}$$

Demand price functions

$p_1 = 65 - \frac{1}{4}q_1$ **and** $p_2 = 70 - \frac{1}{4}q_2$

Group 2 exercise: How might you solve for the profit maximizing outputs?

MODEL 1:

Revenue

$$R_1 = p_1 q_1 = (85 - \tfrac{1}{4} q_1) q_1 = 85 q_1 - \tfrac{1}{4} q_1^{\,2} \quad , \quad R_2 = p_2 q_2 = (90 - \tfrac{1}{4} q_2) q_2 = 90 q_2 - \tfrac{1}{4} q_2^{\,2}$$

Profit

$$\pi = R_1 + R_2 - C$$

$$= 85 q_1 - \tfrac{1}{4} q_1^{\,2} + 90 q_2 - \tfrac{1}{4} q_2^{\,2} - (10 q_1 + 15 q_2 + 2 q_1^{\,2} + 3 q_1 q_2 + 2 q_2^{\,2})$$

$$= 75 q_1 + 75 q_2 - \tfrac{9}{4} q_1^{\,2} - \tfrac{9}{4} q_2^{\,2} - 3 q_1 q_2$$

Think on the margin

Marginal profit of increasing q_1

$$\frac{\partial \pi}{\partial q_1} = 75 - \tfrac{9}{2}q_1 - 3q_2 \ .$$

Therefore the profit-maximizing choice is

$$q_1 = m_1(q_2) = \tfrac{2}{9}(75 - 3q_2) = \tfrac{2}{3}(25 - q_2) \ .$$

$$\pi = 75q_1 + 75q_2 - \tfrac{9}{4}q_1^{\,2} - \tfrac{9}{4}q_2^{\,2} - 3q_1q_2$$

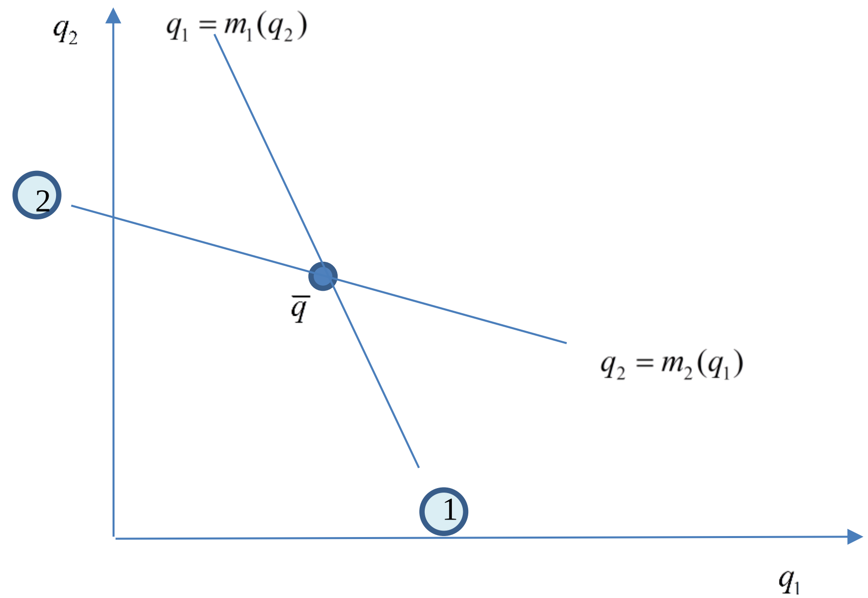
Marginal profit of increasing q_2

$$\frac{\partial \pi}{\partial q_2} = 75 - 3q_1 - \tfrac{9}{2}q_2 \ .$$

Therefore the profit-maximizing choice is

$$q_2 = m_2(q_1) = \tfrac{2}{9}(75 - 3q_1) = \tfrac{2}{3}(25 - q_1) \ .$$

The two profit-maximizing lines are depicted.



Model 1: Profit-maximizing lines

$q_1 = m_1(q_2) = \frac{2}{3}(25 - q_2)$, $q_2 = m_2(q_1) = \frac{2}{3}(25 - q_1)$

If you solve for $\bar{q}$ satisfying both equations you will find that the unique solution is

$\bar{q} = (\bar{q}_1, \bar{q}_2) = (10,10)$.

MODEL 2

Cost function

$$C(q) = 10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2$$

Demand price functions

$p_1 = 65 - \frac{1}{4}q_1$ **and** $p_2 = 70 - \frac{1}{4}q_2$

Revenue

$R_1 = p_1q_1 = (65 - \frac{1}{4}q_1)q_1 = 65q_1 - \frac{1}{4}q_1^2$ **,** $R_2 = p_2q_2 = (70 - \frac{1}{4}q_2)q_2 = 70q_2 - \frac{1}{4}q_2^2$

Profit

$$\pi = R_1 + R_2 - C$$

$$= 65q_1 - \tfrac{1}{4}q_1^2 + 70q_2 - \tfrac{1}{4}q_2^2 - (10q_1 + 15q_2 + q_1^2 + 3q_1q_2 + q_2^2)$$

$$= 55q_1 + 55q_2 - \tfrac{5}{4}q_1^2 - \tfrac{5}{4}q_2^2 - 3q_1q_2$$

Think on the margin

Marginal profit of increasing q_1

$$\frac{\partial \pi}{\partial q_1} = 55 - \tfrac{5}{2}q_1 - 3q_2 \ .$$

Therefore, for any q_2 the profit-maximizing q_1 is

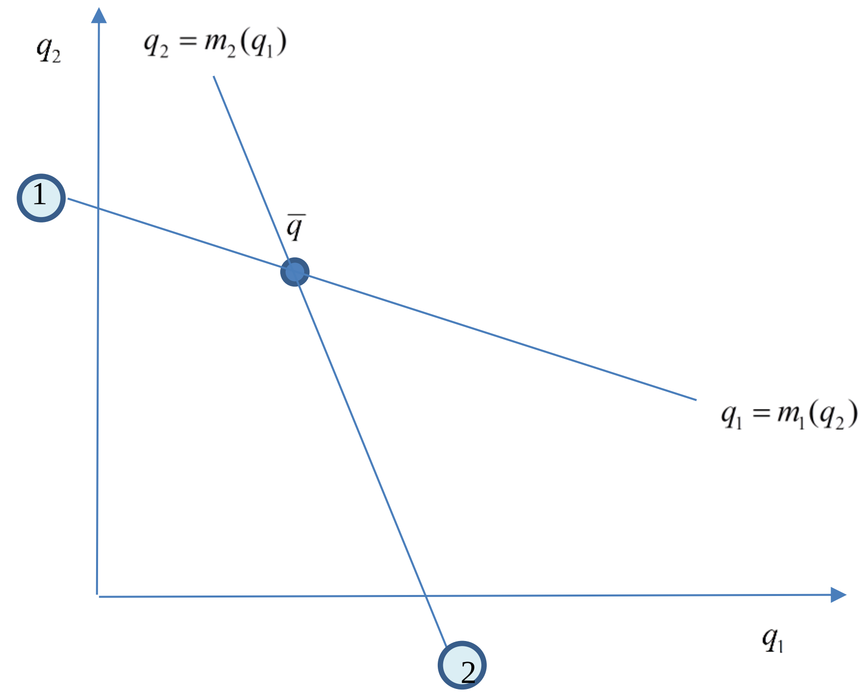
$$q_1 = m_1(q_2) = \tfrac{2}{5}(55 - 3q_2) \ .$$

Marginal profit of increasing q_2

$$\frac{\partial \pi}{\partial q_2} = 55 - 3q_1 - \tfrac{5}{2}q_2 \ .$$

Therefore, for any q_1 the profit-maximizing q_2 is

$$q_2 = m_2(q_1) = \tfrac{2}{5}(55 - 3q_1)$$



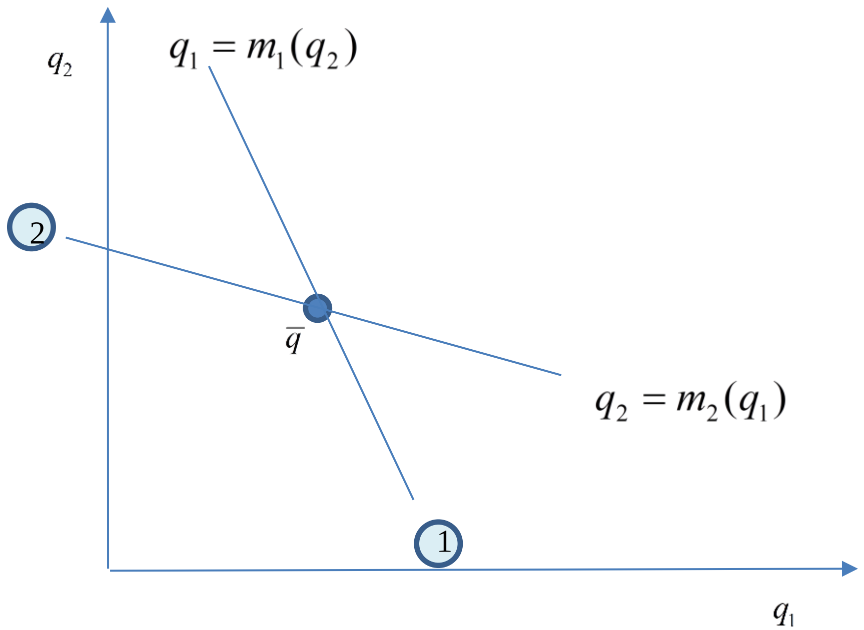
Model 2: Profit-maximizing lines.

The two profit-maximizing lines are depicted.

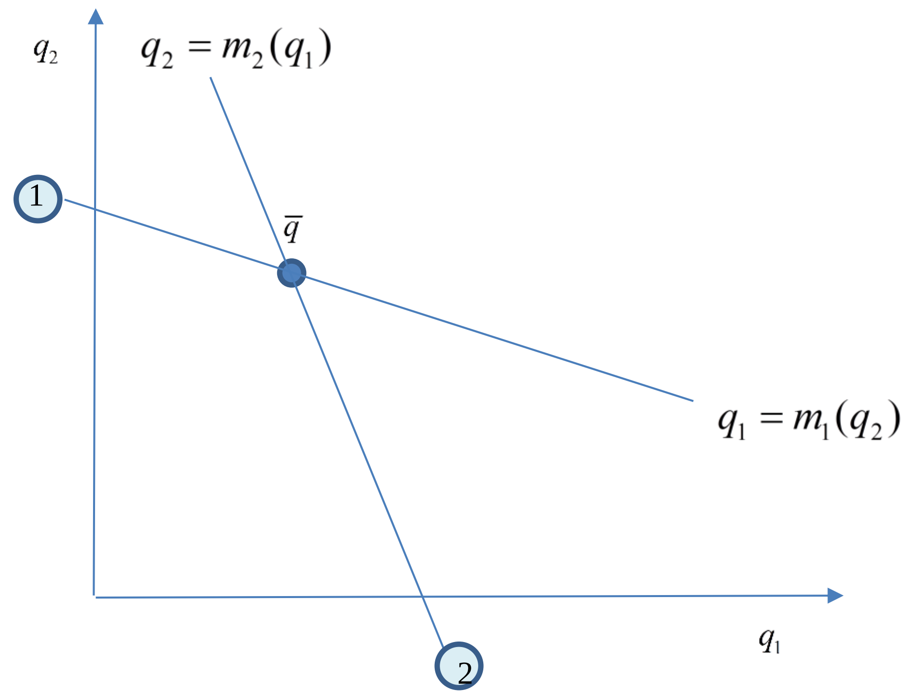
If you solve for $\bar{q}$ satisfying both equations you will find that the unique solution is

$\bar{q} = (\bar{q}_1, \bar{q}_2) = (10,10)$.

These look very similar to the profit-maximizing lines in Model 1. However now the profit-maximizing line for q_2 is steeper (i.e. has a more negative slope).



Model 1: Profit-maximizing lines

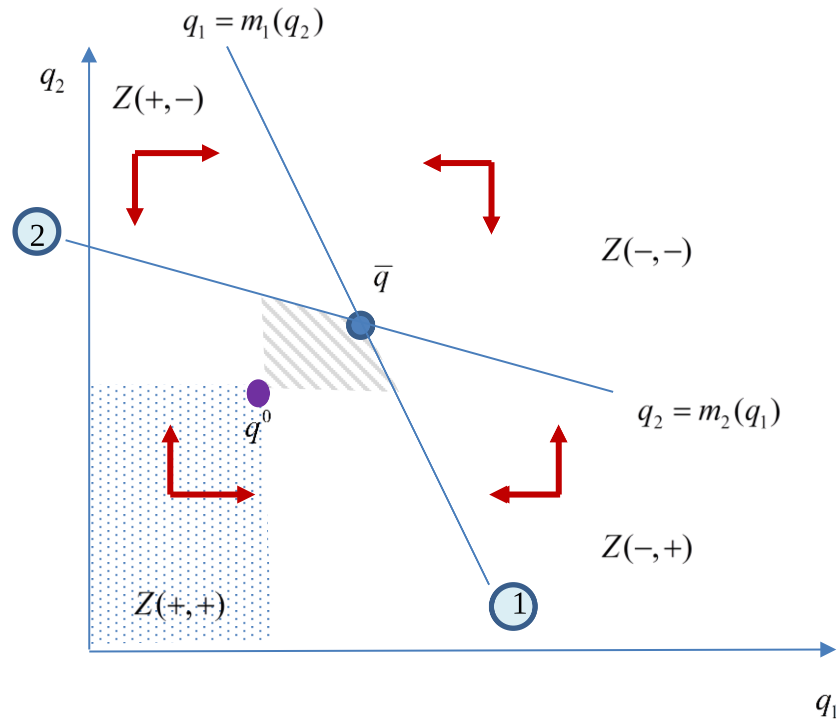


Model 2: Profit-maximizing lines.

As we shall see, this makes a critical difference.

Model 1:

Is $\bar{q} = (\bar{q}_1, \bar{q}_2)$ the profit-maximizing output vector?



Model 1: Profit-maximizing lines

The profit-maximizing lines divide the

positive quadrant into four zones.

The arrows indicate the directions of

in which $\pi(q_1, q_2)$ increases.

Consider the point q^0 .

Output is higher in the diagonally shaded region

and lower in the dotted region.

Thus the level set through q^0 must have a negative slope.

A similar argument can be used in the other three quadrants.

The level set $\pi(q) = \pi(q^0)$ in the $Z(+,+)$ region

Profit is higher in the shaded region

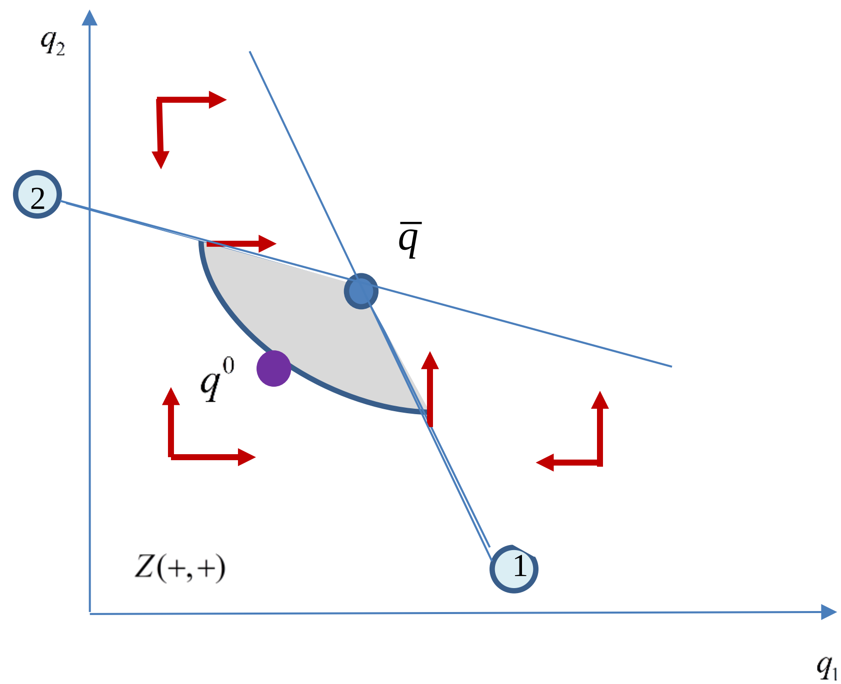
Note that the level set is parallel to the q_2 axis

at the point of intersection with the maximizing

line for q_2 and is parallel to the horizontal

axis at the point of intersection with the

maximizing line for q_1

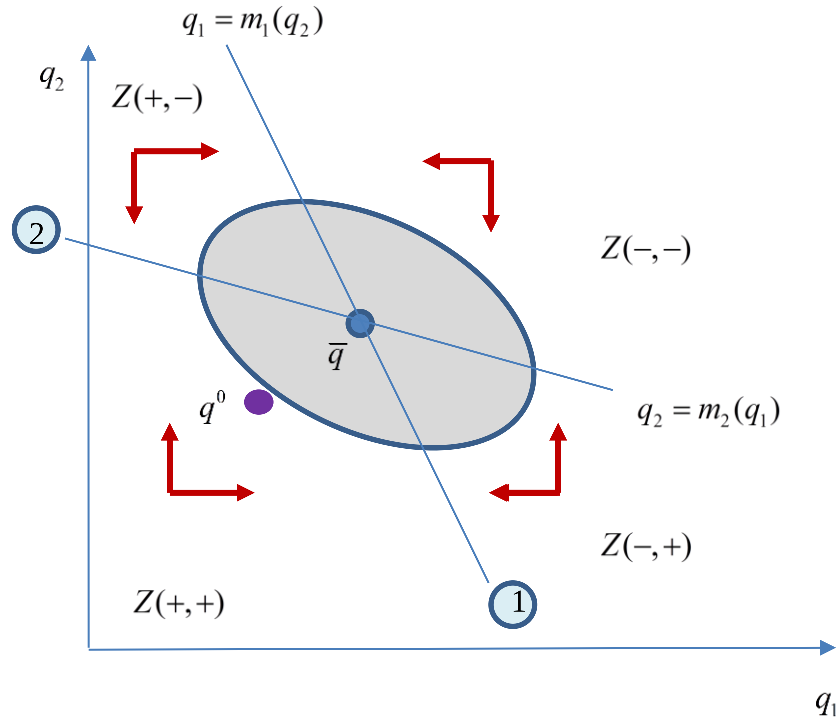


Model 1: Level set for profit

The level set $\pi(q) = \pi(q^0)$

and superlevel set $\pi(q) \geq \pi(q^0)$

are depicted opposite.



Model 1: Level set for profit

Model 1

Suppose we alternate, first maximizing

with respect to q_1 , then q_2 and so on.

There are four zones.

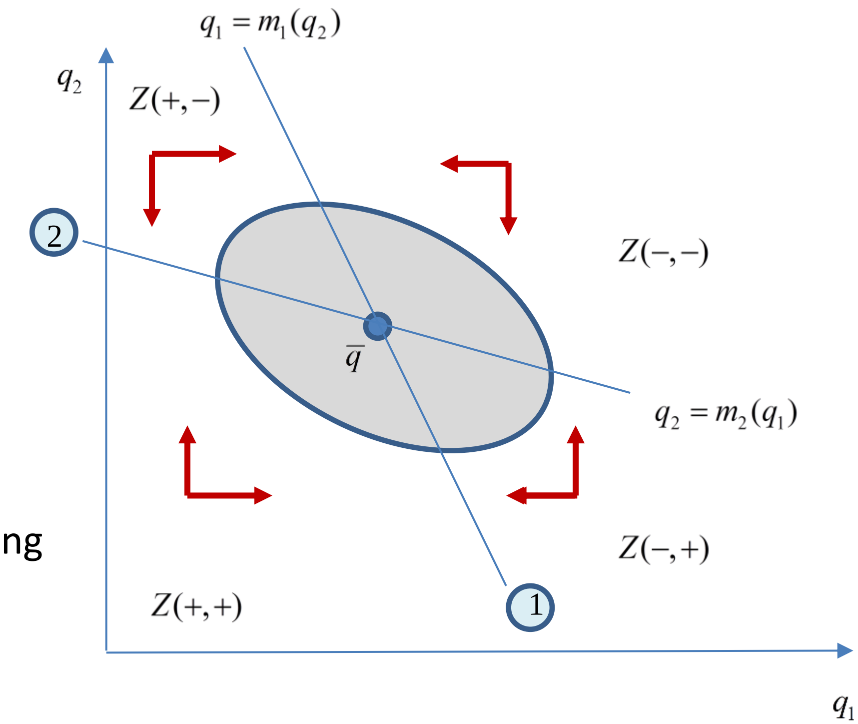
$Z(+,+)$:

The zone in which q_1 is increasing and q_2 is increasing

$Z(+,-)$:

The zone in which q_1 is increasing and q_2 is decreasing

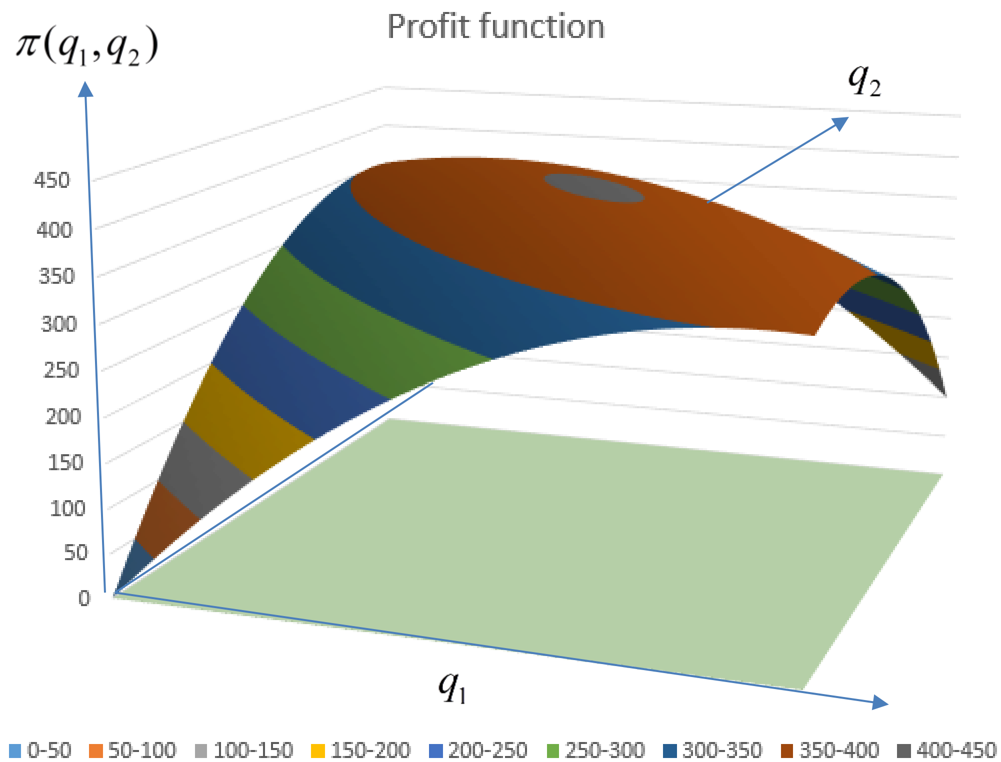
and so on…



Model 1: Profit-maximizing lines

If you pick any starting point you will find this process leads to the intersection point $\bar{q} = (10,10)$.

The profit is depicted below (using a spread-sheet)



Group Exercise: For model 2 solve for maximized profit if only one commodity is produced.

Compare this with the profit if $\bar{q} = (10,10)$ is produced.

MODEL 2

Suppose we alternate,

first maximizing with respect to q_1 , then q_2 and so on.

There are four zones.

$Z(+,+)$:

The zone in which q_1 is increasing and q_2 is increasing

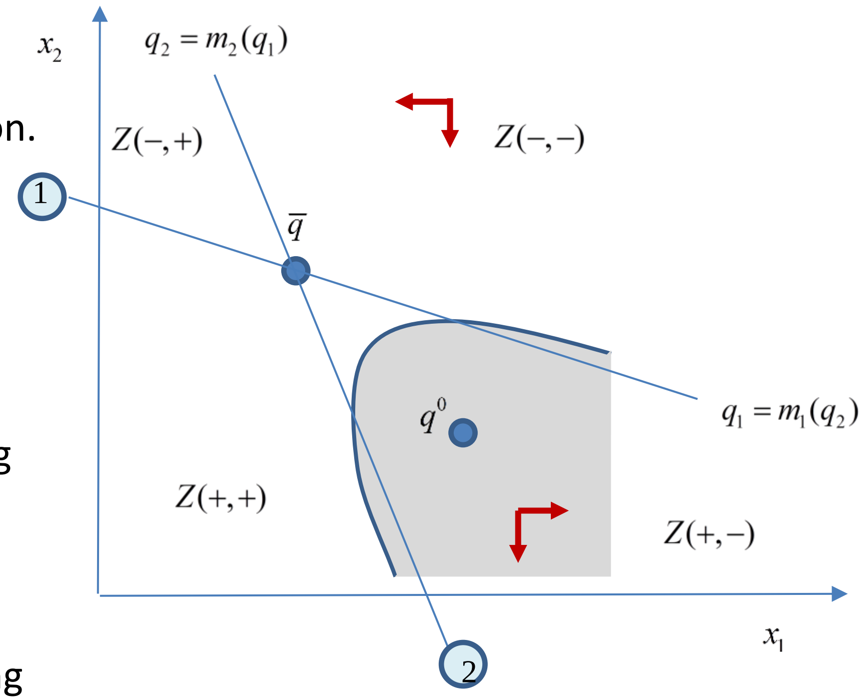
$Z(+,-)$:

The zone in which q_1 is increasing and q_2 is decreasing

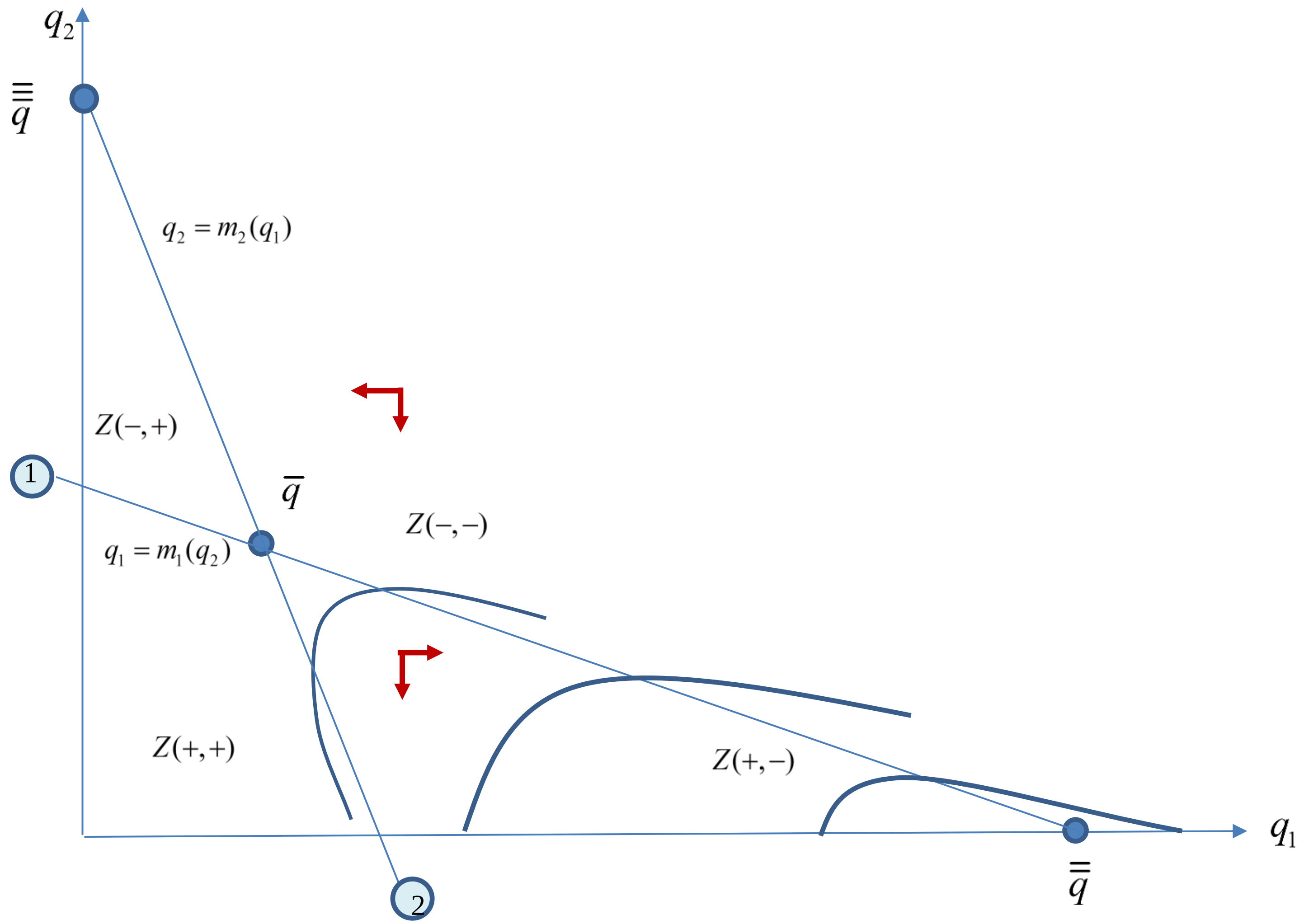
and so on…

If you pick any starting point you will find this process

never leads to the intersection point $\bar{q} = (10,10)$.

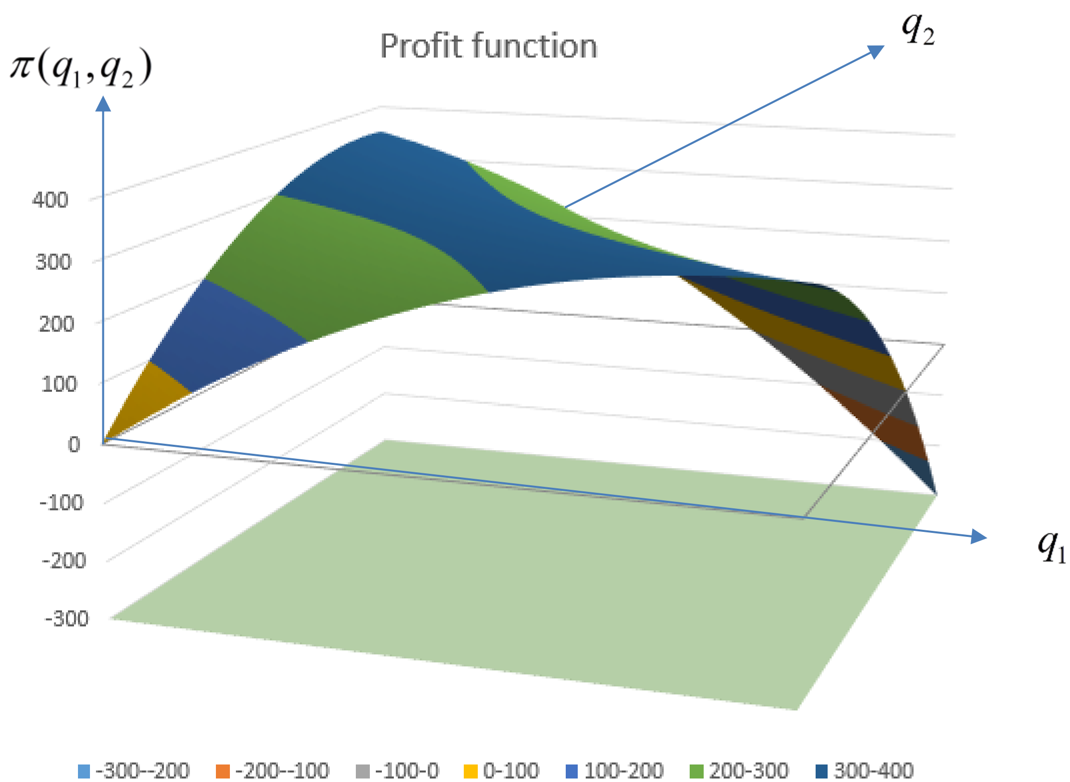


Local maximum $\bar{\bar{q}} = (\bar{\bar{q}}_1, 0)$ **on the** q_1 **axis**



By an essentially identical argument, there is a second local maximum $\bar{\bar{q}}$ on the q_2 axis.

The profit function has the shape of a saddle. The output vector $\bar{q}$ where the slope in the direction of each axis is zero is called a saddle-point.



2. General results

Consider the two variable problem

$$\underset{q}{Max}\{f(q_1,q_2)\}$$

Necessary conditions

Consider any $\bar{q} >> 0$. If the slope in the cross section parallel to the q_1-axis is not zero, then by standard one variable analysis, the function is not maximized. The same holds for the cross section parallel to the q_2-axis. Thus for $\bar{q}$ to be a maximizer, the slope of both cross sections must be zero.

First order necessary conditions for a maximum

For $\bar{q} >> 0$ to be a maximizer the following two conditions must hold

$$\frac{\partial f}{\partial q_1}(\bar{q}) = 0 \text{ and } \frac{\partial f}{\partial q_2}(\bar{q}) = 0 \qquad (3\text{-}1)$$

Suppose that the first order necessary conditions hold at $\bar{q}$. Also, if the slope of the cross section parallel to the q_1-axis is strictly increasing in q_1 at $\bar{q}$, then $\bar{q}_1$ is not a maximizer. Thus a necessary condition for a maximum is that the slope must be decreasing. Exactly the same argument holds for $\bar{q}_2$.

We therefore have a second set of necessary conditions for a maximum. Since they are restrictions on second derivatives they are called the second order conditions.

Second order necessary conditions for a maximum

If $\bar{q} >> 0$ is a maximizer of $f(q)$, then

$$\frac{\partial}{\partial q_1}\frac{\partial f}{\partial q_1}(\bar{q}_1,\bar{q}_2) \le 0 \text{ and } \frac{\partial}{\partial q_2}\frac{\partial f}{\partial q_2}(\bar{q}_1,\bar{q}_2) \le 0 \qquad (3\text{-}2)$$

As we have seen, these conditions are necessary for a maximum but they do not, by themselves guarantee that $\bar{q}$ satisfying these conditions is the maximum.

However, if the step by step approach does lead to $\bar{q}$ then this point is a least a local maximizer.

Proposition: Sufficient conditions for a local maximum

If the first and second order necessary conditions hold at $\bar{q}$ and the level sets are closed loops around $\bar{q}$, then the function $f(q)$ has a local maximum at $\bar{q}$

Proposition: Sufficient conditions for a global maximum

If the first and second order necessary conditions hold at $\bar{q}$ and the level sets are closed loops around $\bar{q}$ and the FOC hold only at $\bar{q}$, then this is the global maximizer.

3. Non-negativity constraints

Many economic variables cannot be negative. Suppose this is true for all variables

Let $\bar{x} = (\bar{x}_1,...,\bar{x}_n)$ solve $\underset{x\geq 0}{Max}\{f(x)\}$.

We will consider the first variable.

It is helpful to write the optimal value

of all the other variables as $\bar{x}_{-1}$. Then

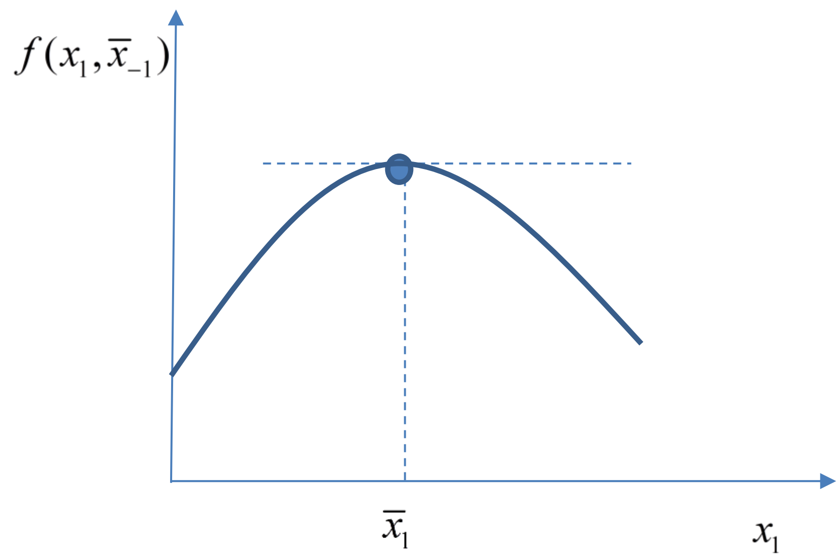
$$\bar{x}_1 \text{ solves } \underset{x_1\geq 0}{Max}\{f(x_1,\bar{x}_{-1})\} .$$

Case (i) $\bar{x}_1 > 0$

This is depicted opposite.

For $\bar{x}_1$ to be the maximizer,

the graph of $f(x_1,\bar{x}_{-1})$ must be zero at $\bar{x}_1$.



Case (i): Necessary condition for a maximum

Case (ii) $\bar{x}_1 = 0$

This is depicted opposite.

For $\bar{x}_1$ to be the maximizer,

the graph of $f(x_1, \bar{x}_{-1})$ cannot be strictly

positive at $\bar{x}_1$.

Taking the two cases together,

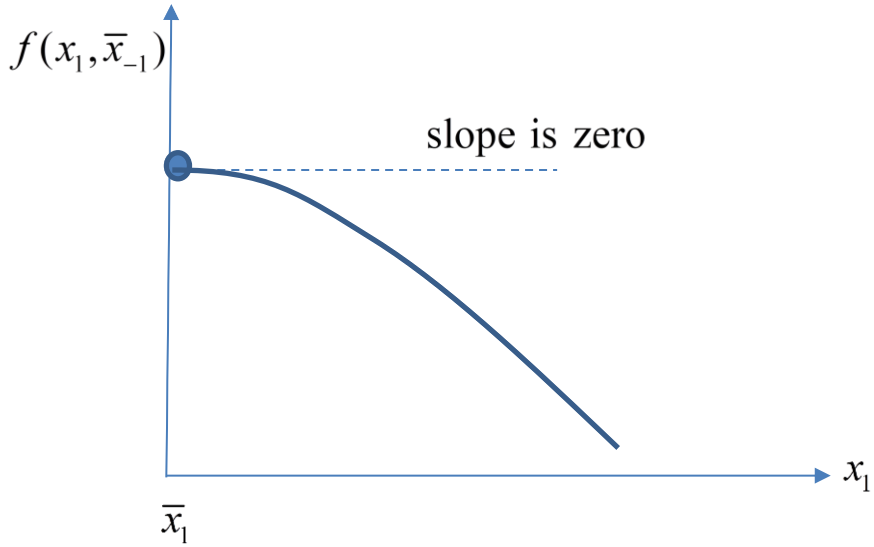
$\frac{\partial f}{\partial x_1}(\bar{x}) \le 0$, with equality if $\bar{x}_1 > 0$

An identical argument holds for all of the

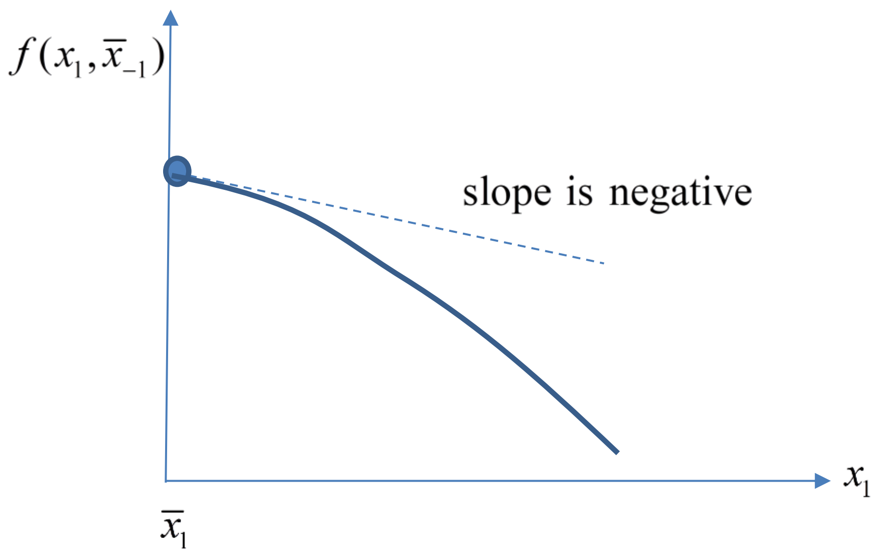
variables.

Necessary conditions

$\frac{\partial f}{\partial x_1}(\bar{x}) \le 0$, with equality if $\bar{x}_1 > 0$



Case (ii): Necessary condition for a maximum



Case (ii): Necessary condition for a maximum

4. Laws of supply and input demand

The first law of firm supply

As an output price p rises, the maximizing output $q(p)$ increases (at least weakly).

Case (i) $p > MC(0)$ **Case (ii)** $p < MC(0)$

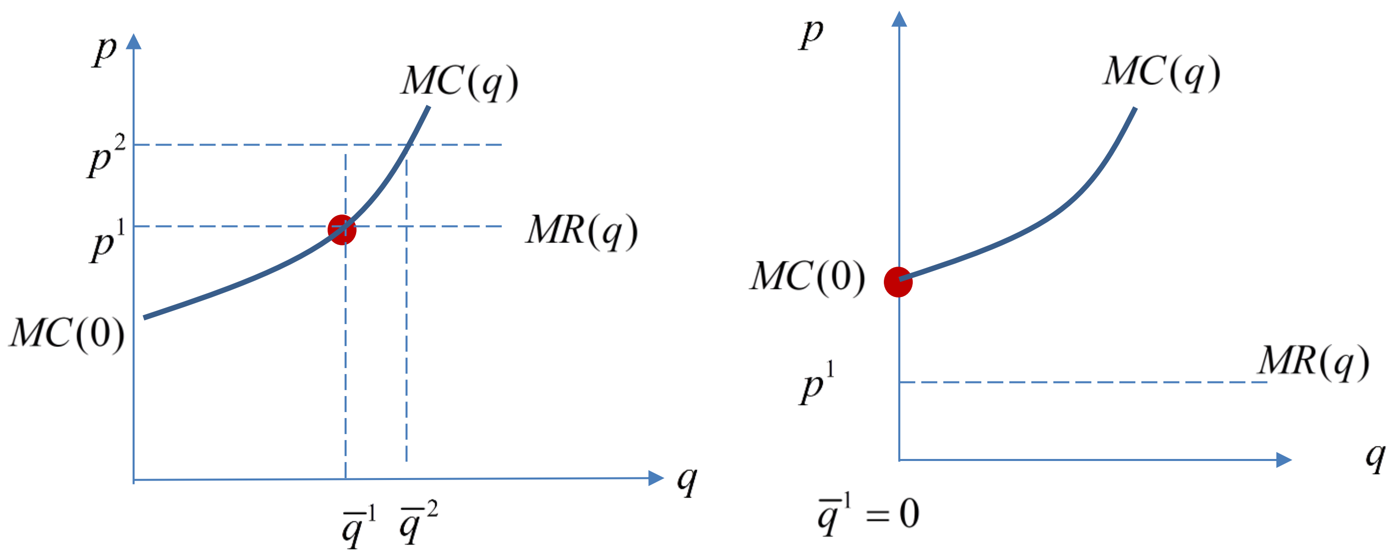


Fig. 1: Profit-maximizing output

As the output price rises, the profit-maximizing output increases (at least weakly).

The firm's supply curve

For prices below $MC(0)$, supply is zero. For higher prices the graph of marginal cost $MC(q)$ is the supply curve.

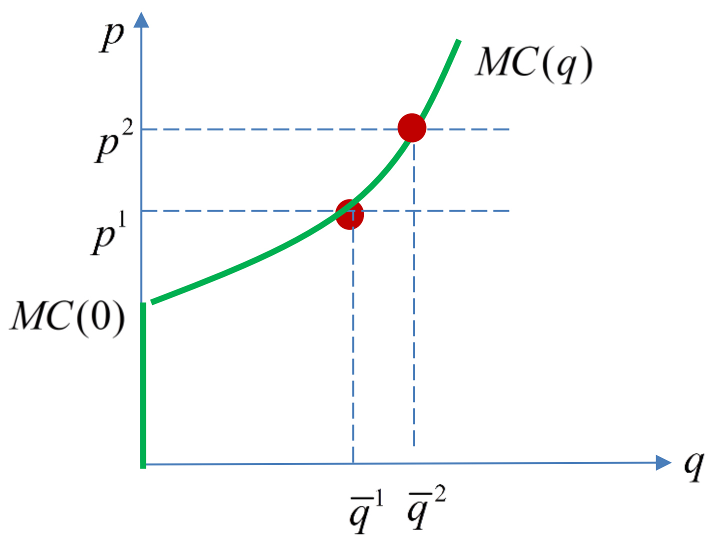


Fig. 2: Firm's supply curve

The first law of input demand

As an input price r rises, the maximizing input $z(r)$ decreases (at least weakly).

The rate at which revenue rises as the input (and hence output) rises is called the Marginal Revenue Product (MRP).

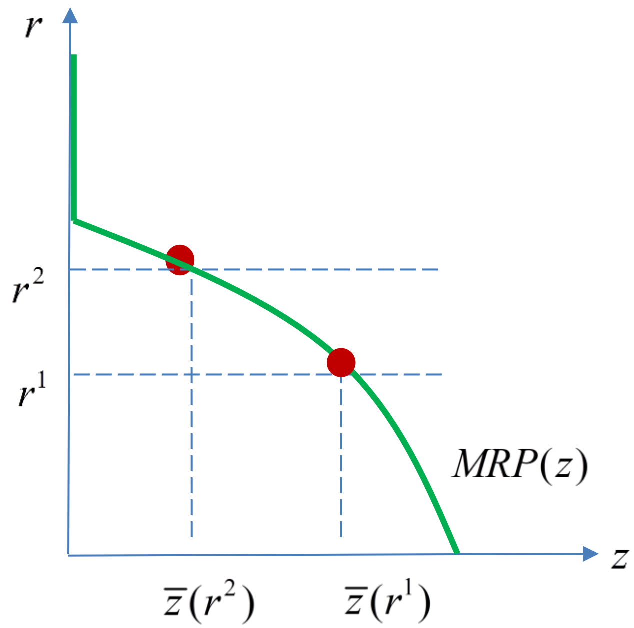


Fig. 3: Firm's input demand curve

5. Resource constrained maximization - - an economic approach

Problem: $\underset{x\geq 0}{Max}\{f(x) \mid b-g(x)\geq 0\}$

NOTE: Always write a resource constraint as $h(x)\geq 0$

Let $\bar{x}$ be the solution to this problem.

Interpretation, if the firm chooses x it requires $g(x)$ units of a resource that is fixed in supply (.e. Floor space of plant, highly skilled workers)

We interpret $q=f(x)$ as the output of the firm. The price of the output is 1 so this is also the revenue of the firm. There is a single input $z=g(x)$. There are b units of this input available.

To solve this problem, we consider the "relaxed problem" in which the firm can purchase additional units at the price λ . Since this is a hypothetical opportunity, economists refer to the price as the "shadow price" of the resource rather than a market price.

Suppose that the firm purchases $g(x)-b$ additional units. Its profit is then

$$\mathfrak{L} = f(x) - \lambda(g(x)-b) = f(x) + \lambda(b - g(x))$$

The relaxed problem is then

$$\underset{x\geq 0}{Max}\{\mathfrak{L} = f(x) + \lambda(b - g(x))\}$$

First Order Necessary Conditions:

Necessary conditions for $\bar{x}(\lambda)$ **to solve** $\underset{x\geq 0}{Max}\{\mathfrak{L}(x,\lambda)\}$

$$\frac{\partial \mathfrak{L}}{\partial x_j}(\bar{x},\lambda) = \frac{\partial f}{\partial x_j}(\bar{x}) - \lambda \frac{\partial g}{\partial x_j}(\bar{x}) \leq 0, \text{ with equality if } \bar{x}_j > 0 , \; j = 1,2$$

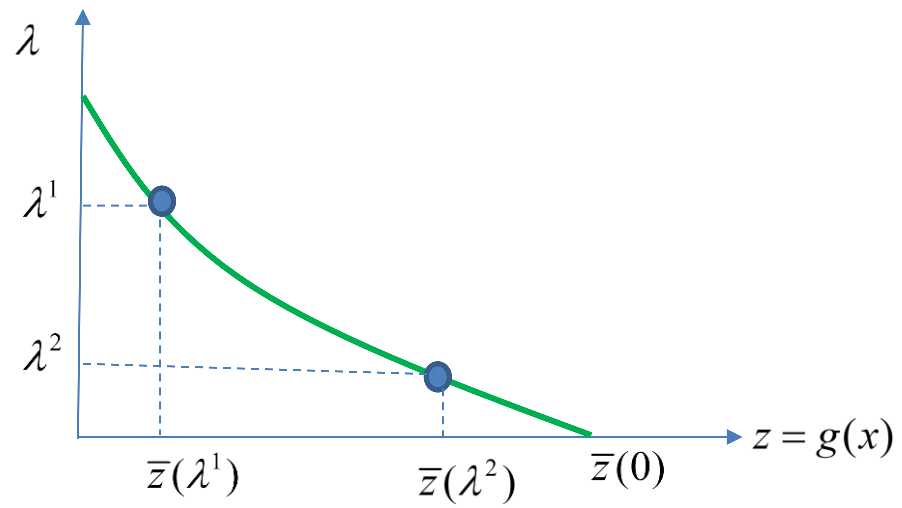
Let $\bar{z} = g(\bar{x})$ be demand for the resource.

In Section 4 it was argued that

Demand, $\bar{z}(r)$ declines as the input price rises.

If the resource price is sufficiently high it is

more profitable to sell all of the resource.



Demand for the resource in the relaxed problem

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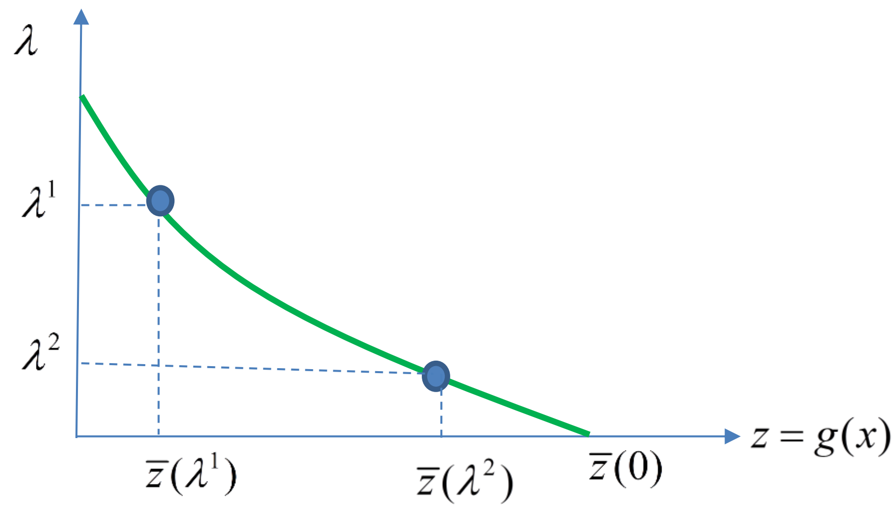
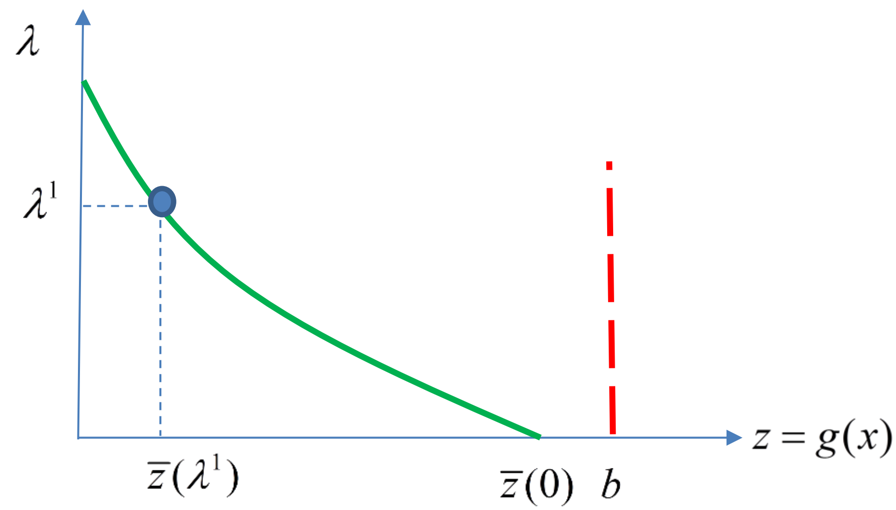


Demand for the resource in the relaxed problem

Case (i) $\overline{z}(0) < b$

Supply exceeds demand at every price

So the market clearing price $\overline{\lambda} = 0$.



Demand for the resource in the relaxed problem

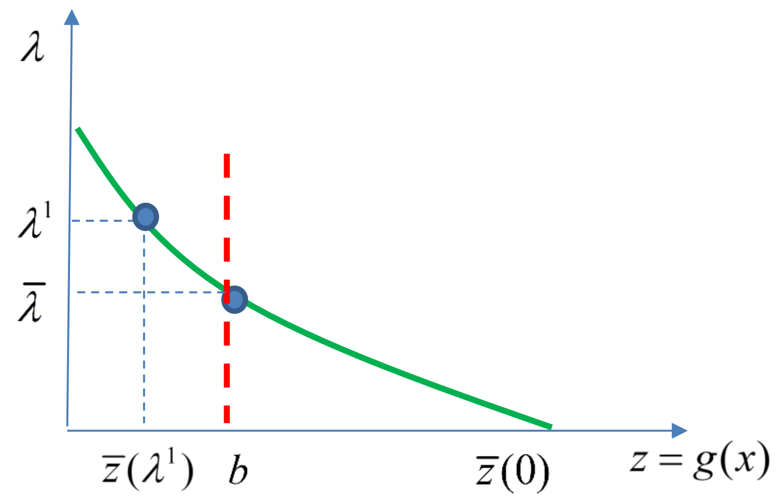
Case (ii): $\bar{z}(0) > b$

At the price $\bar{\lambda}$, demand for the resource is equal to b.

Suppose we find such a price $\bar{\lambda}$.

SInce $\bar{x}$ is profit-maximizing,

$$\bar{\mathfrak{L}} = f(\bar{x}) - \bar{\lambda}(g(\bar{x}) - b) \geq f(x) + \bar{\lambda}(b - g(x))$$



Demand for the resource equals supply at price $\bar{\lambda}$

**

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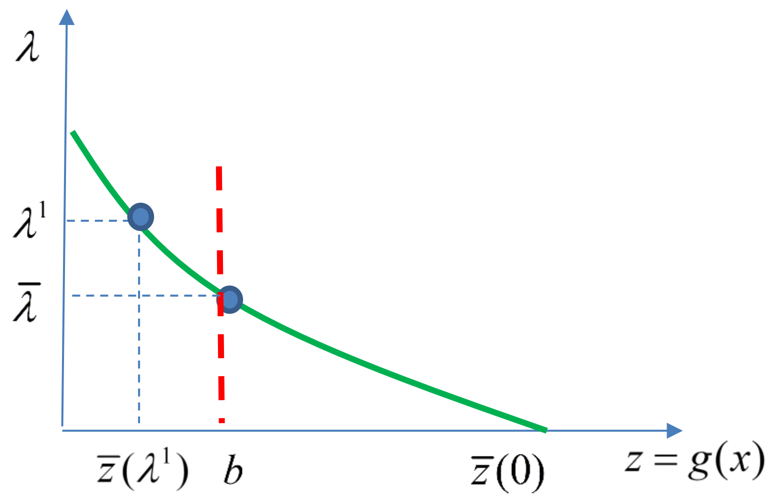
At the price $\bar{\lambda}$, demand for the resource equals supply

It follows that

$$\bar{\mathfrak{L}} = f(\bar{x}) \geq f(x) + \bar{\lambda}(b - g(x)) \qquad (*)$$

*



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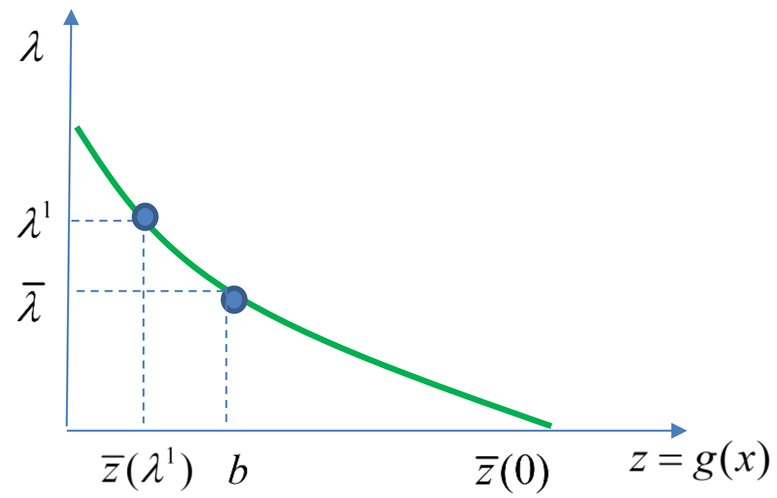
At the price $\bar{\lambda}$, demand for the resource equals supply



Demand for the resource equals supply at price $\bar{\lambda}$

It follows that

$$\bar{\mathfrak{L}} = f(\bar{x}) \geq f(x) + \bar{\lambda}(b - g(x)) \qquad (*)$$

Now consider the original problem, $\underset{x \geq 0}{Max}\{f(x) \mid b - g(x) \geq 0\}$.

For any feasible x it follows that $b - g(x) \geq 0$. Appealing to (*),

$$\bar{\mathfrak{L}} = f(\bar{x}) \geq f(x) + \bar{\lambda}(b - g(x)) \geq f(x)$$

Thus $\bar{x}$ solves the original problem.

Summary: Necessary conditions for a maximum with a resource constraint

$$\underset{x\geq 0}{Max}\{f(x)\,|\,b-g(x)\geq 0\}$$

NOTE: Always write a resource constraint as $h(x)\geq 0$

Consider the relaxed problem in which there is a market for the resource and the firm owns b units of the resource. If the price of the resource is λ , then profit in the relaxed problem is

$$\mathfrak{L}=f(x)-\lambda(g(x)-b)=f(x)+\lambda(b-f(x))\ .$$

Since this market is a theoretical rather than an actual market we call the price a shadow price.

*

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$$\underset{x\geq 0}{Max}\{f(x)\,|\,b-g(x)\geq 0\}$$

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$$\mathfrak{L}=f(x)-\lambda(g(x)-b)=f(x)+\lambda(b-f(x)) \ .$$

Since this market is a theoretical rather than an actual market we call the price a shadow price.

Suppose we find a shadow price $\bar{\lambda}\geq 0$ and $\bar{x}$ such that the Necessary First Order Conditions for the relaxed problem are satisfied and in addition,

(i) $b-g(\bar{x})>0\Rightarrow\bar{\lambda}=0$ (ii) $\bar{\lambda}>0\Rightarrow b-g(\bar{x})=0$.

Then these conditions are the necessary conditions for the resource constrained problem.

Solving for the maximum

Example 1: Output maximization with a budget constraint

$$\underset{x\in X}{Max}\{q = f(x) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}\} \text{ where } X = \{x \geq 0 \mid p \cdot x \leq \bar{b}\} \text{ and } p >> 0$$

Preliminary analysis

If $q = f(x)$ takes on its maximum at $\bar{x}$, then, for any strictly increasing function $g(q)$,

$h(x) = g(f(x))$ also takes on its maximum at $\bar{x}$.

In this case the function $g(q) = \ln q$ simplifies the problem since

$$h(x) = \ln f(x) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2 + \alpha_3 \ln x_3, \text{ where } \sum_{j=1}^{3} \alpha_j = 1$$

The derivatives of $\ln q$ are very simple since each term has only one variable. The new problem is

$$\underset{x\geq 0}{Max}\{h(x) = \sum_{j=1}^{3} \alpha_j \ln x_j \mid \bar{b} - p \cdot x \geq 0\}.$$

$$\underset{x\geq 0}{Max}\{h(x)=\sum_{j=1}^{3}\alpha_j \ln x_j \mid \bar{b}-p\cdot x\geq 0\}$$

We write down the profit in the relaxed problem in which there is a market price λ for the resource.

Mathematicians call this the Lagrangian.

If the firm sells $\bar{b}-p\cdot x$ units of the resource, then the profit of the firm is

$$\mathfrak{L}=\sum_{j=1}^{3}\alpha_j \ln x_j+\lambda(\bar{b}-\sum_{j=1}^{3}p_j x_j)$$

Necessary conditions for profit maximization

$$\frac{\partial\mathfrak{L}}{\partial x_j}=\frac{\alpha_j}{x_j}-\lambda p_j\leq 0\text{, with equality if } \bar{x}_j>0,\quad j=1,2,3\ .$$

Note that as $x_j\to 0$ the first term on the right hand side increases without bound. Therefore the right hand side cannot be negative. Then

$$\frac{\partial\mathfrak{L}}{\partial x_j}=\frac{\alpha_j}{x_j}-\lambda p_j=0,\ j=1,2,3\ .\qquad \text{Therefore } p_j x_j=\frac{\alpha_j}{\lambda}\ ,\ j=1,2,3$$

We have shown that

$$p_j x_j = \frac{\alpha_j}{\lambda}\ , \ j = 1,2,3 \qquad \text{(2-1)}$$

Summing over the commodities,

$$\bar{b} = \sum_{j=1}^{3} p_j x_j = \sum_{j=1}^{3} \frac{\alpha_j}{\lambda} = \frac{1}{\lambda}, \text{ since } \sum_{j=1}^{3} \alpha_j = 1$$

Appealing to (2-1) it follows that

$$\bar{x}_j = \frac{\alpha_j \bar{b}}{p_j},\ j = 1,2,3$$

Example 2: Utility maximization

A consumer's preferences are represented by a strictly increasing utility function $U(x)$, where $U(x) > 0$ if and only if $x >> 0$. The consumer's budget constraint is $p \cdot x = p_1 x_1 + ... + p_n x_n \leq I$ where the price vector $p >> 0$.

The consumer chooses $\bar{x}$ to solve $\underset{x \geq 0}{Max}\{U(x) \mid p \cdot x \leq I\}$.

Group Exercise:

(1) Explain why $\bar{x} >> 0$ and $p \cdot \bar{x} = I$

(ii) Show that the FOC can be written as follows:

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = ... = \frac{\frac{\partial U}{\partial x_n}}{p_n} .$$

(iii) Provide the intuition behind these conditions.

A graphical approach

Suppose $\bar{x}$ solves $\underset{x\geq 0}{Max}\{U(x) \mid p\cdot x \leq I\}$. Define $\bar{z}=(\bar{x}_3,...,\bar{x}_n)$. Then

$$(\bar{x}_1,\bar{x}_2) \text{ solves } \underset{x\geq 0}{Max}\{U(x_1,x_2,\bar{z}) \mid p_1x_1+p_2x_2+p\cdot\bar{z} \leq I\}.$$

Hence

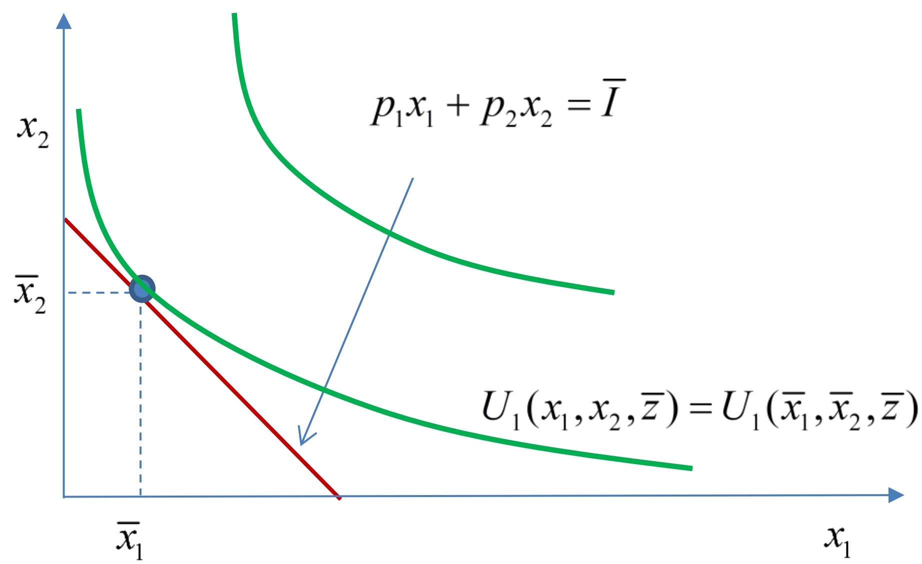
$$(\bar{x}_1,\bar{x}_2) \text{ solves } \underset{x\geq 0}{Max}\{U(x_1,x_2,\bar{z}) \mid p_1x_1+p_2x_2 \leq \bar{I} = I - p\cdot\bar{z}\}.$$

We can illustrate this two variable problem

in a figure showing the 2 commodity budget

constraint and level sets of the function

$U(x_1,x_2,\bar{z})$.



Choosing commodities 1 and 2

The slope of the budget line is $-\frac{p_1}{p_2}$

But what is the slope of the level set?

Note that the level set implicitly defines

a function $x_2 = \phi(x_1)$. That is

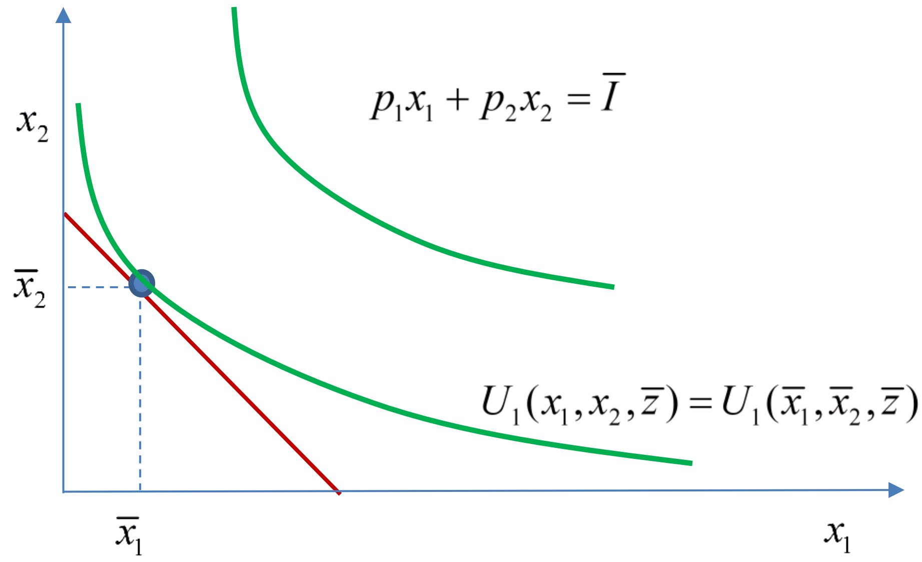
$$U(x_1, \phi(x_1), \bar{z}) = U(\bar{x}_1, \bar{x}_2, , \bar{z})$$

Differentiate with respect to x_1

$$\frac{d}{dx_1} U(x_1, \phi(x_1), \bar{z}) = \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} \phi'(x_1) = 0$$

Therefore the slope of the level set is

$$\phi'(x_1) = -\frac{\partial U}{\partial x_1} / \frac{\partial U}{\partial x_2}$$



Choosing commodities 1 and 2

At the maximum the slopes are equal.

Therefore

$$\frac{p_1}{p_2} = \frac{\partial U}{\partial x_1}(\bar{x}_1, x_2, \bar{z}) / \frac{\partial U}{\partial x_2}(\bar{x}_1, x_2, \bar{z})$$

i.e.

$$\frac{p_1}{p_2} = \frac{\partial U}{\partial x_1}(\bar{x}) / \frac{\partial U}{\partial x_2}(\bar{x})$$

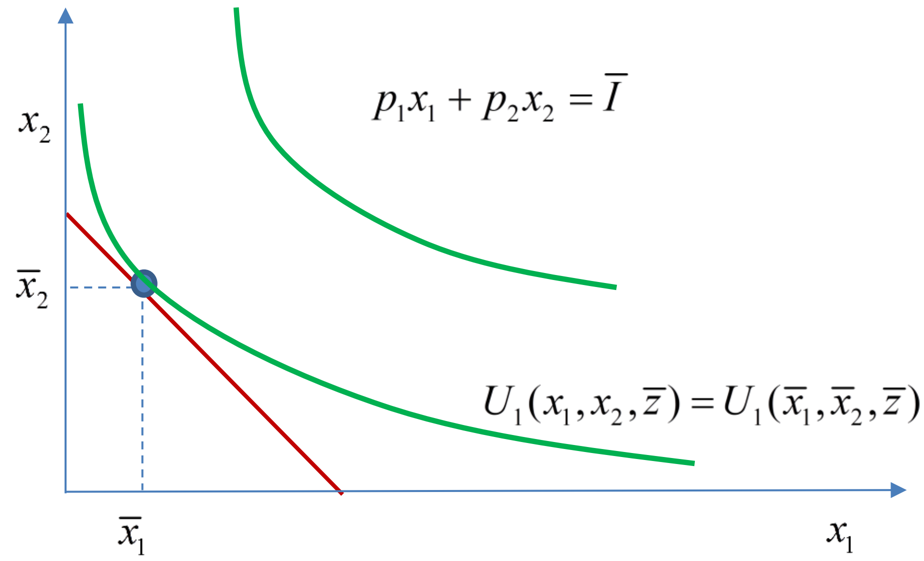
Exactly the same argument holds for every

pair of commodities.

Therefore

$$\frac{p_i}{p_j} = \frac{\partial U}{\partial x_i}(\bar{x}) / \frac{\partial U}{\partial x_j}(\bar{x}) \text{ for all } i, j$$

Rearranging this equation, $\dfrac{\dfrac{\partial U}{\partial x_i}(\bar{x})}{p_i} = \dfrac{\dfrac{\partial U}{\partial x_j}(\bar{x})}{p_j}$.



Choosing commodities 1 and 2