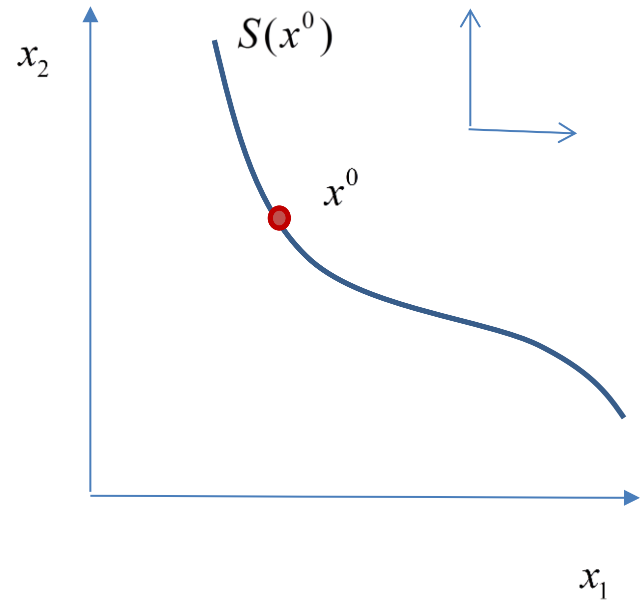
Mathematical Foundations II

48 pages

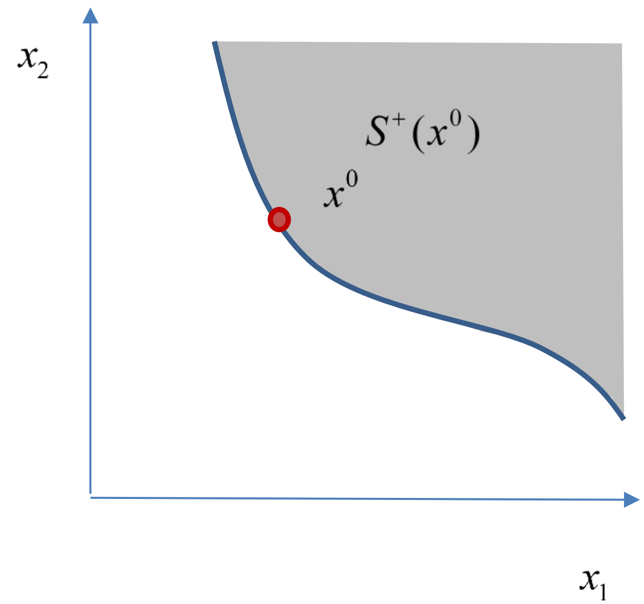
A. Level and superlevel sets of a function

$$S(x^0) \equiv \{x \mid h(x) = h(x^0)\}$$



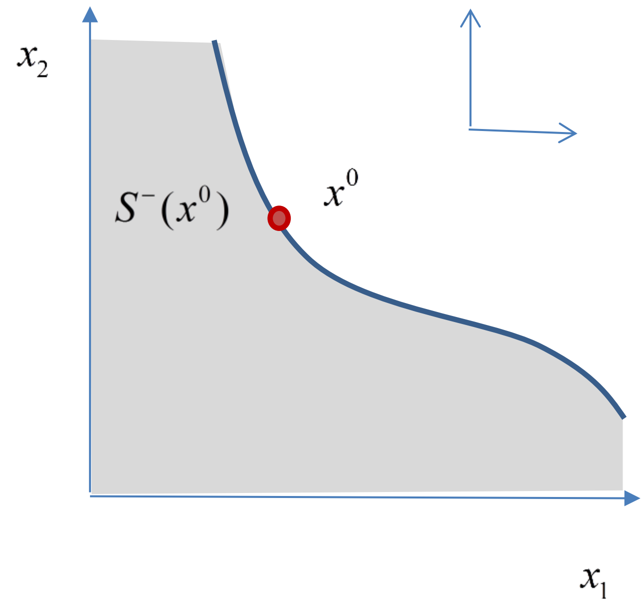
Superlevel set of a function

$$S^+(x^0) \equiv \{x \mid h(x) \geq h(x^0)\}$$



Sublevel set of a function

$$S^{-}(x^0) \equiv \{x \mid h(x) \leq h(x^0)\}$$



B. Convex sets and concave functions

Convex combination of two vectors

Consider any two vectors z^0 and z^1. The set of weighted average of these two vectors can be written as follows.

$$z^\lambda = (1-\lambda)z^0 + \lambda z^1 \text{ , } 0 < \lambda < 1$$

Such averages where the weighs are both strictly positive and add to 1 are called the convex combinations of z^0 and z^1.

*

B. Convex sets and concave functions

Convex combination of two vectors

Consider any two vectors z^0 and z^1. The set of weighted average of these two vectors can be written as follows.

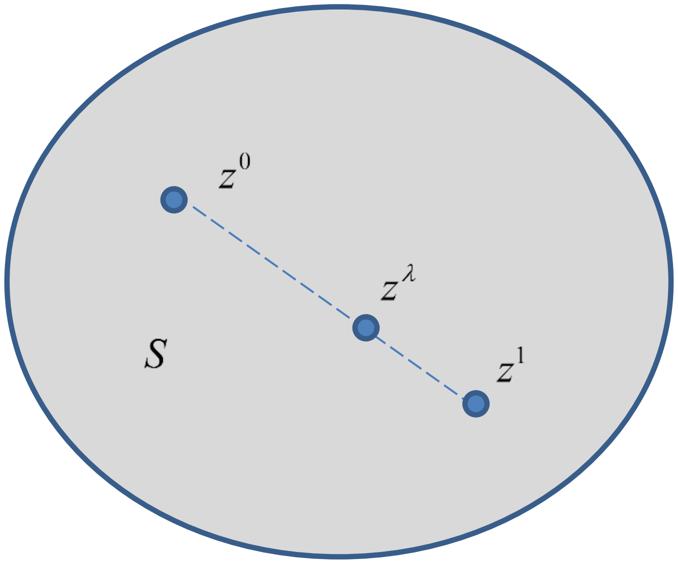
$$z^\lambda = (1-\lambda)z^0 + \lambda z^1 \ , \ 0 < \lambda < 1$$

Such averages where the weighs are both strictly positive and add to 1 are called the convex combinations of z^0 and z^1.

Convex set

The set $S \subset \mathbb{R}^n$ is convex if for any z^0 and z^1 in *S*,

every convex combination is also in S



A convex set

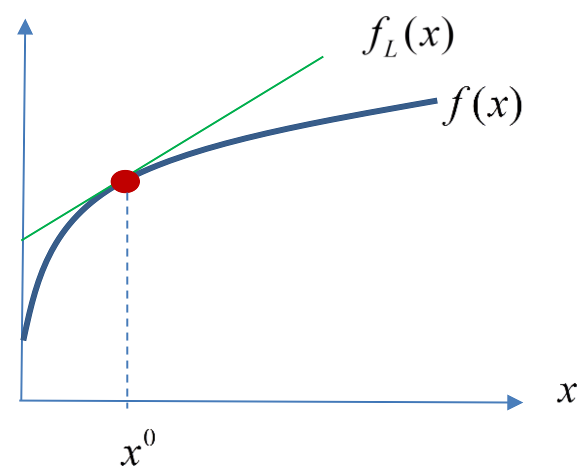
Concave functions of 1 variable

Definition: Linear approximation of the function f at x^0

$$f_L(x) \equiv f(x^0) + f'(x^0)(x - x^0) \ .$$

Note that the linear approximation has the same value at x^0 and the same first derivative (the slope.)

In the figure $f_L(x)$ is a line tangent to the graph of the function.



*

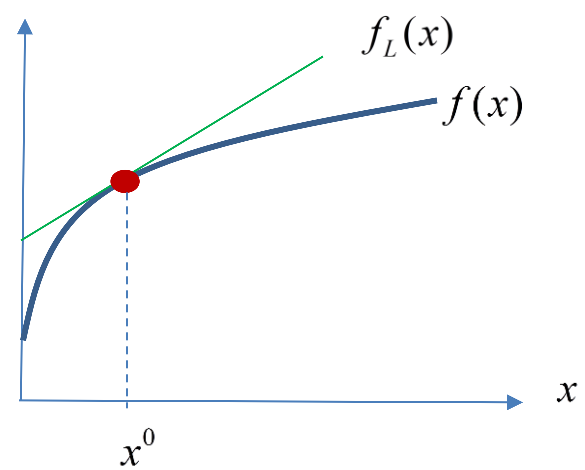
Concave functions of 1 variable

Definition: Linear approximation of the function f at x^0

$$f_L(x) \equiv f(x^0) + f'(x^0)(x - x^0) \ .$$

Note that the linear approximation has the same value at x^0 and the same first derivative (the slope.)

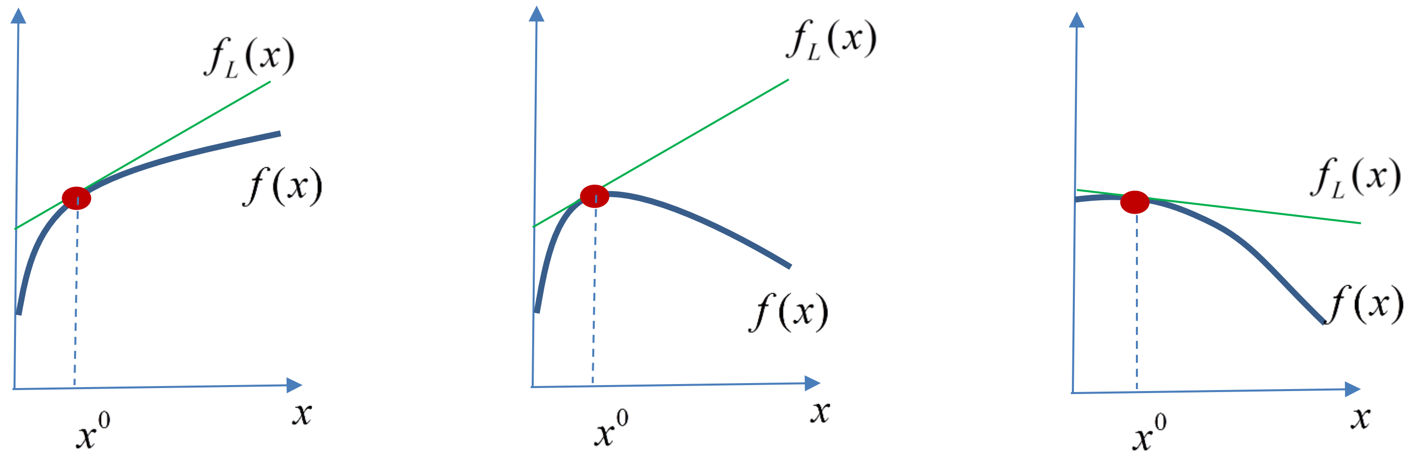
In the figure $f_L(x)$ is a line tangent to the graph of the function.



Definition 1: Concave Function

A differentiable function f defined on an interval X is concave if $f'(x)$, the derivative of $f(x)$ is decreasing.

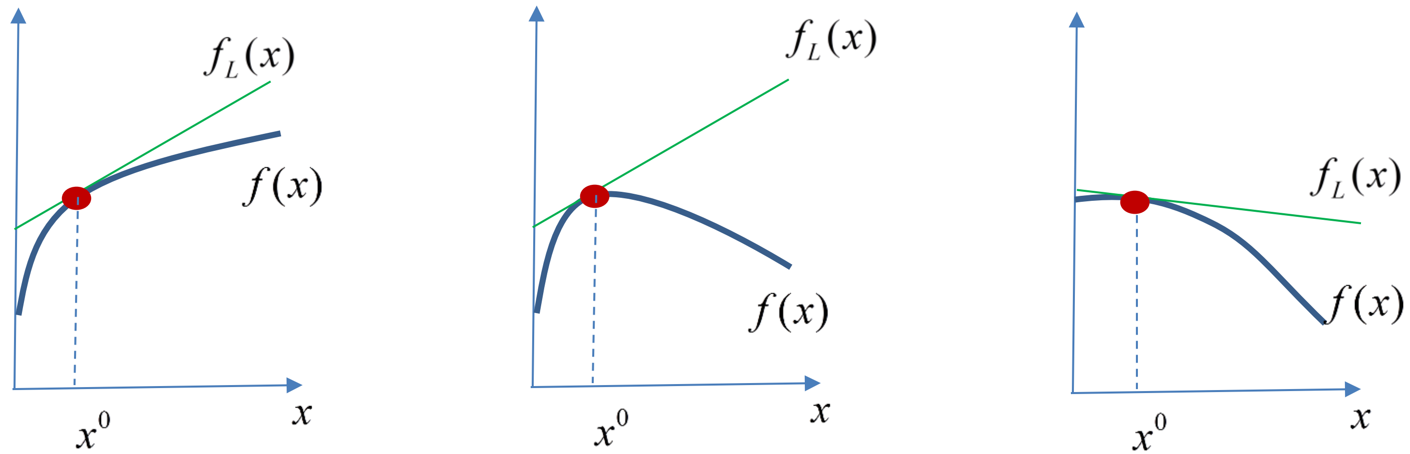
The three types of differentiable concave function are depicted below.



Note that in each case the linear approximations at any point x^0 lie above the graph of the function.

*

The three types of differentiable concave function are depicted below.



Note that in each case the linear approximations at any point x^0 lie above the graph of the function.

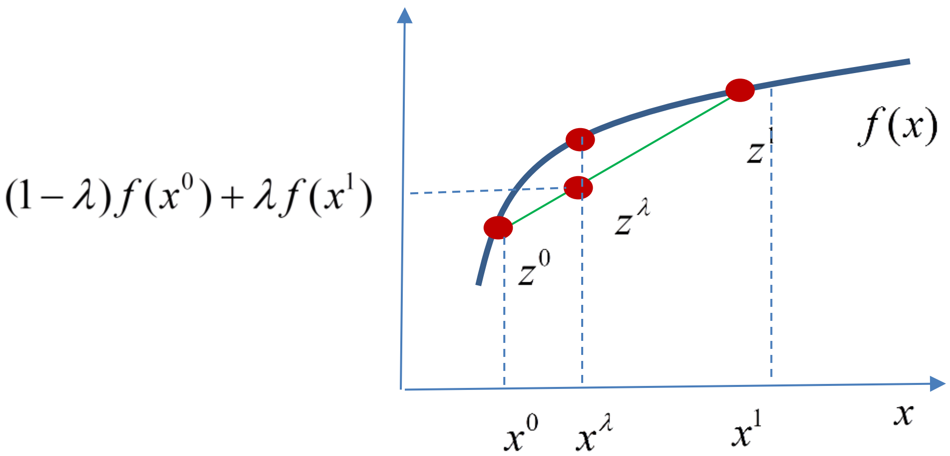
While we will not formally prove it, the following is an equivalent definition.

Definition 2: Concave Function

A differentiable function f defined on an interval X is concave if the tangent lines lie above the graph of the function, i.e. for any $x^0 \in X$

$$f(x) \leq f(x^0) + f'(x^0)(x - x^0)$$

Consider the two points $z^0 = (x^0, f(x^0))$ and $z^1 = (x^1, f(x^1))$ depicted below.



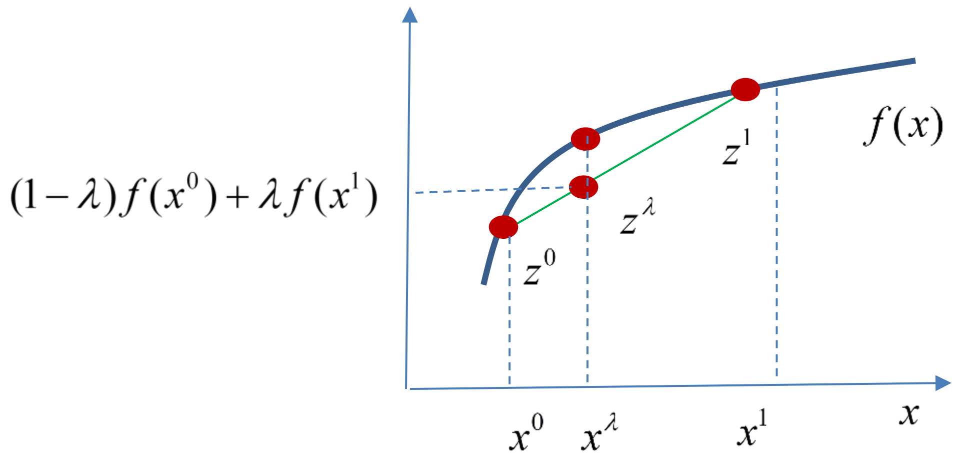
The convex combination is

$$(1-\lambda)(x^0, f(x^0)) + \lambda(x^1, f(x^1)) = (x^\lambda, (1-\lambda)f(x^0) + \lambda f(x^1)) \ .$$

In the figure, the convex combination is on the line segment joining the two points.

*

Consider the two points $z^0 = (x^0, f(x^0))$ and $z^1 = (x^1, f(x^1))$ depicted below.



The convex combination is

$$(1-\lambda)(x^0, f(x^0)) + \lambda(x^1, f(x^1)) = (x^\lambda, (1-\lambda) f(x^0) + \lambda f(x^1)) \ .$$

In the figure, the convex combination is on the line segment joining the two points.

For a concave function, it is clear from the figure that this line segment must lie below the graph of the function. i.e.

$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1) \ .$$

In fact this is another equivalent definition.

Definition 3: Concave Function

A function f defined on an interval X is concave if , for any $x^0, x^1 \in X$ and any convex combination x^λ ,

$$f(x^\lambda) \geq (1-\lambda) f(x^0) + \lambda f(x^1)$$

Concave functions of n variables

Propositions

1. The sum of concave functions is concave

2. If f is linear (i.e. $f(x)=a_0+b\cdot x$) and g is concave then $h(x)=g(f(x))$ is concave.

3. An increasing concave function of a concave function is concave.

4. If $f(x)$ is homogeneous of degree 1 (i.e. $f(\theta x)=\theta f(x)$ for all $\theta>0$)
   and for some increasing function $g(\cdot)$, $h(x)=g(f(x))$ is concave, then $f(x)$ is concave.

Convex functions

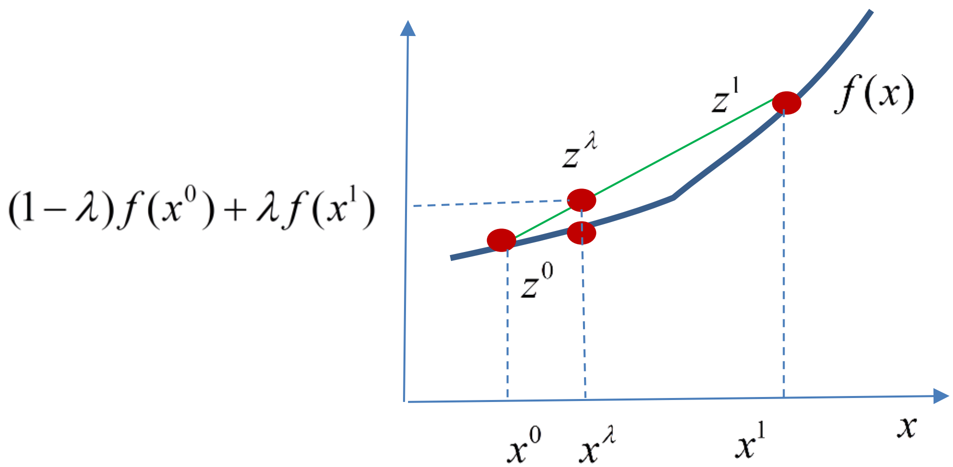
Convex combinations, convex sets and convex functions. You have got to be kidding. How confusing is that?

I think it is best to think of a convex function as being the "opposite" of a concave function

Definition: Convex Function

A function f defined on an interval X is convex if , for any $x^0, x^1 \in X$ and any convex combination x^λ ,

$$f(x^\lambda) \leq (1-\lambda) f(x^0) + \lambda f(x^1)$$



Remark: As you can show by rearranging the terms in the inequality, a function $f(x)$ is convex if and only if $-f(x)$ is concave.

Proposition

If $f(x)$ is concave, and $\overline{x}$ satisfies the necessary conditions for the maximization problem

$$\underset{x}{Max}\{f(x)\}$$

then $\overline{x}$ solves the maximization problem.

Proposition

If $f(x)$ and $h(x)$ are concave, and $\overline{x}$ satisfies the necessary conditions for the maximization problem

$$\underset{x}{Max}\{f(x) \mid h(x) \geq 0\}$$

then $\overline{x}$ solves the constrained maximization problem.

Remark: This result continues to hold if there are multiple constraints $h_i(x) \geq 0$ and each function $h_i(x)$ is concave.

Group Exercise: Output maximization with a fixed budget

A plant has the CES production function

$$F(z) = (z_1^{1/2} + z_2^{1/2})^2 .$$

The CEO gives the plant manager a budget B and instructs her to maximize output. The input price vector is $r = (r_1, r_2)$. Solve for the maximum output $q(r, B)$.

Class Exercise: What is the firm's cost function

C. Parameter Changes and the Envelope Theorem

Impact of a parameter change on the maximized value

Consider a profit-maximizing multi-product firm. The market for one of the outputs is competitive market, i.e. the firm has so little monopoly power that it takes the output price p as given. We call such a firm a price taker in this market.

Let $q(p)$ be the firm's supply function, i.e. the profit-maximizing output when the price is p . The current price is $\hat{p}$ so the profit maximizing output is $\hat{q} = q(\hat{p})$.

As the output price rises, how rapidly does maximized profit rise?

**

D. Parameter Changes and the Envelope Theorem

Impact of a parameter change on the maximized value

Consider a profit-maximizing multi-product firm. The market for one of the outputs is competitive market, i.e. the firm has so little monopoly power that it takes the output price p as given. We call such a firm a price taker in this market.

Let $q(p)$ be the firm's supply function, i.e. the profit-maximizing output when the price is p . The current price is $\hat{p}$ so the profit maximizing output is $\hat{q} = q(\hat{p})$.

As the output price rises, how rapidly does maximized profit rise?

Naïve answer: Don't change output then cost is constant so the extra profit is the extra revenue

$$\Delta\Pi = \Delta R = (p - \hat{p})\hat{q} = \Delta p\hat{q}$$

Therefore $\dfrac{\Delta\Pi}{\Delta p} = \hat{q}$

*

E. Parameter Changes and the Envelope Theorem

Impact of a parameter change on the maximized value

Consider a profit-maximizing multi-product firm. The market for one of the outputs is competitive market, i.e. the firm has so little monopoly power that it takes the output price p as given. We call such a firm a price taker in this market.

Let $q(p)$ be the firm's supply function, i.e. the profit-maximizing output when the price is p. The current price is $\hat{p}$ so the profit maximizing output is $\hat{q} = q(\hat{p})$.

As the output price rises, how rapidly does maximized profit rise?

Naïve answer: Don't change output then cost is constant so the extra profit is the extra revenue

$$\Delta\Pi = \Delta R = (p - \hat{p})\hat{q} = \Delta p\hat{q}$$

Therefore $\dfrac{\Delta\Pi}{\Delta p} = \hat{q}$

Sophisticated answer:

For any finite price change Δp, the firm can do better by increasing the output until marginal cost is equal to the new price. Therefore $\Delta\Pi > \Delta p\hat{q}$ and so

$$\frac{\Delta\Pi}{\Delta p} > \hat{q}$$

Super sophisticated argument

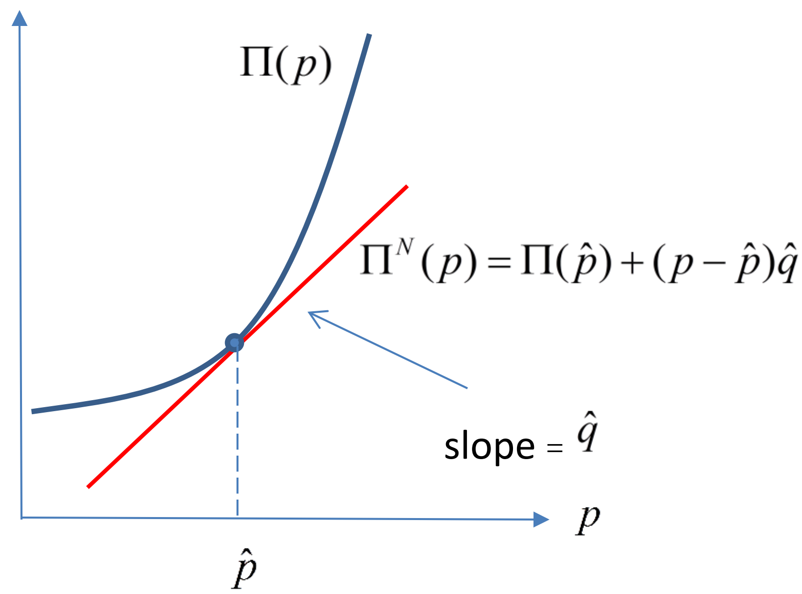
The naïve profit (with output constant) is

$$\Pi^N(p) = \Pi(\hat{p}) + \hat{q}(p - \hat{p}) \ .$$

This the line of slope $\hat{q}$ depicted opposite.

Maximized profit, $\Pi(p)$ is the same at $\hat{p}$ and

Greater for all $p \neq \hat{p}$.



*

Super sophisticated argument

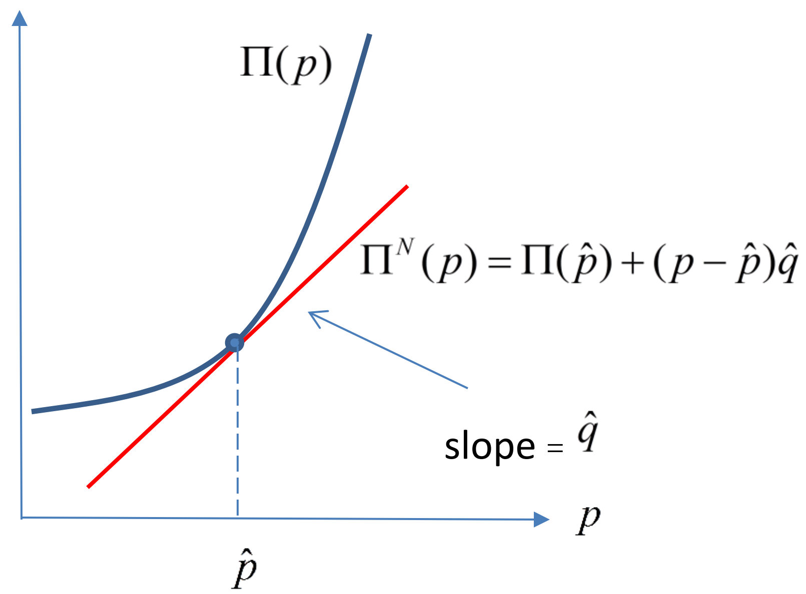
The naïve profit (with output constant) is

$$\Pi^N(p) = \Pi(\hat{p}) + \hat{q}(p - \hat{p}) \ .$$

This the line of slope $\hat{q}$ depicted opposite.

Maximized profit, $\Pi(p)$ is the same at $\hat{p}$ and

Greater for all $p \neq \hat{p}$.



The only way this can be true is if the line and the curve have the same slope at $\hat{p}$.

We have therefore proved that

$$\frac{d\Pi}{dp}(\hat{p}) = \frac{d\Pi^N}{dp}(\hat{p}) = \hat{q} = q(\hat{p}) \ .$$

This argument holds for any price $\hat{p}$. Therefore

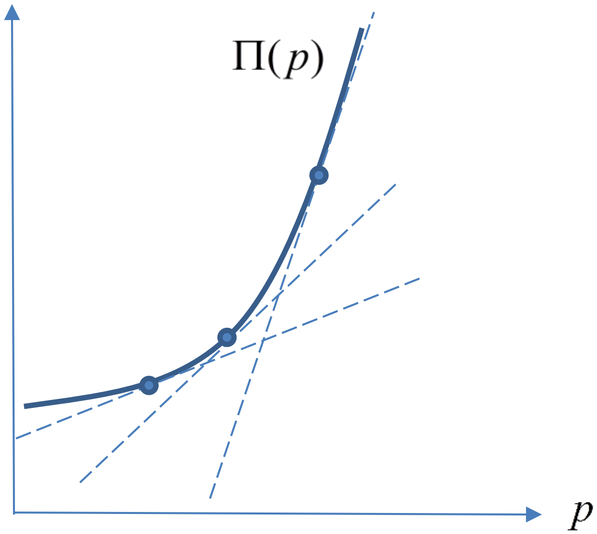
$$\frac{d\Pi}{dp}(p) = q(p)$$

Thus the rate at which maximized profit rises is the rate predicted by the naïve analysis.

The naïve profit lines are depicted for three different prices.

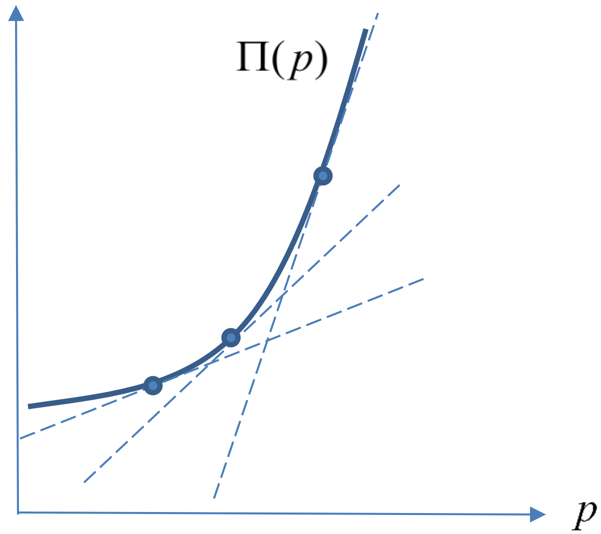
Maximized profit touches all three lines.

*

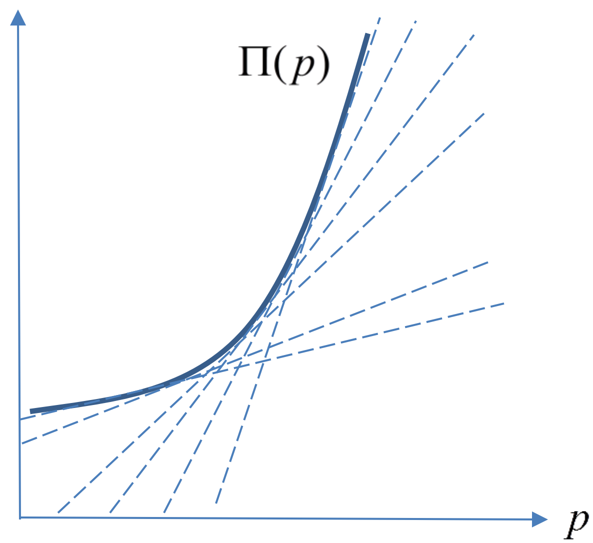


The naïve profit lines are depicted for three different prices.

Maximized profit touches all three lines.



In the second figure the naïve profit lines are drawn for many prices. In the limit, maximized profit is an upper envelope of all these lines.



The result that we have derived is thus called an Envelope Theorem.

Summary:

Suppose that $q(\theta)$ solves $\underset{q}{Max}\{\pi(q,p)\,|\,q\in Q\}$.

The maximized value of the maximand

$$V(p)=\underset{q}{Max}\{\pi(p,q)\,|\,q\in Q\}$$

is called the value function.

Envelope Theorem: If $q(p)$ is continuous then the slope of the value function is $V'(p)=\dfrac{\partial\pi}{\partial p}(p,q(p))$

Another look at the profit maximizing firm

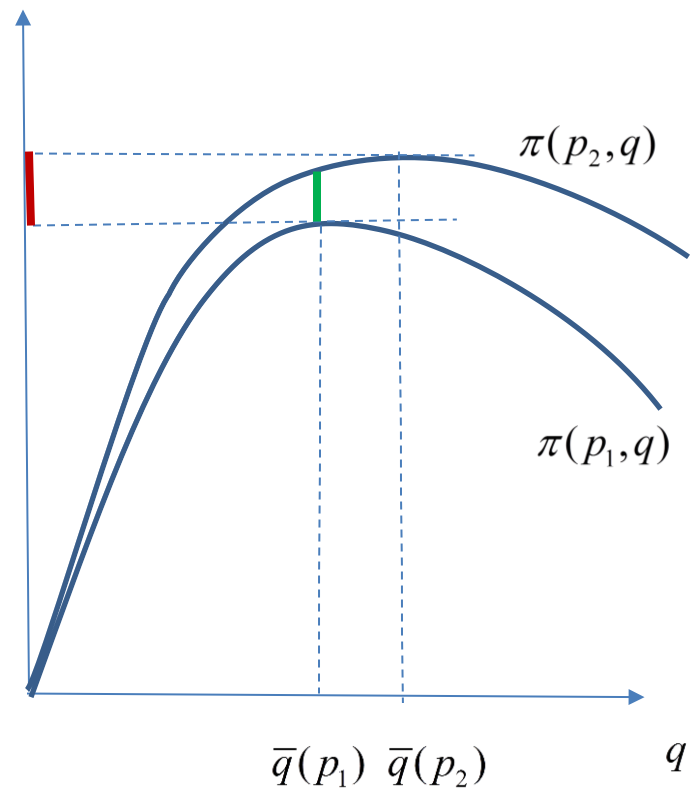
For any output price p , the firm chooses an optimal response $\bar{q}(p)$ to maximize

$$\pi(q,p) = pq - C(q) \ .$$

We consider the case in which $\bar{q}(p_1) > 0$. The price rises to p_2. The two curves $\pi(p_1,q)$ and $\pi(p_2,q)$ are depicted. The profit-maximizing outputs are the outputs where the slope of each profit function is zero.

The heavy green line segment is the naïve increase in profit (output remains at $q(p_1)$).

Note that it is an underestimate of the increase in maximized profit (shown in red.)



However note that the slope of each profit curve is close to zero near the maximum. So the error in using the naïve answer is small. The Envelope Theorem reveals that in the limit, there is no error.

Calculus proof

For a calculus proof, note that maximized profit is

$$\Pi(p) = \pi(p, \bar{q}(p))$$

With $\bar{q}(p_1) > 0$ the FOC for a maximum is

$$\frac{\partial \pi}{\partial q}(p, \bar{q}(p)) = 0$$

We totally differentiate $\Pi(p) = \pi(p, \bar{q}(p)) = pq - C(q)$.

There is the direct effect of the price change on the profit function and the effect on the maximizing output as the price rises.

$$\frac{d\Pi}{dp}(p) = \frac{\partial \pi}{\partial p}(p, \bar{q}(p)) + \frac{\partial \pi}{\partial q}(p, \bar{q}(p))\frac{d\bar{q}}{dp}$$

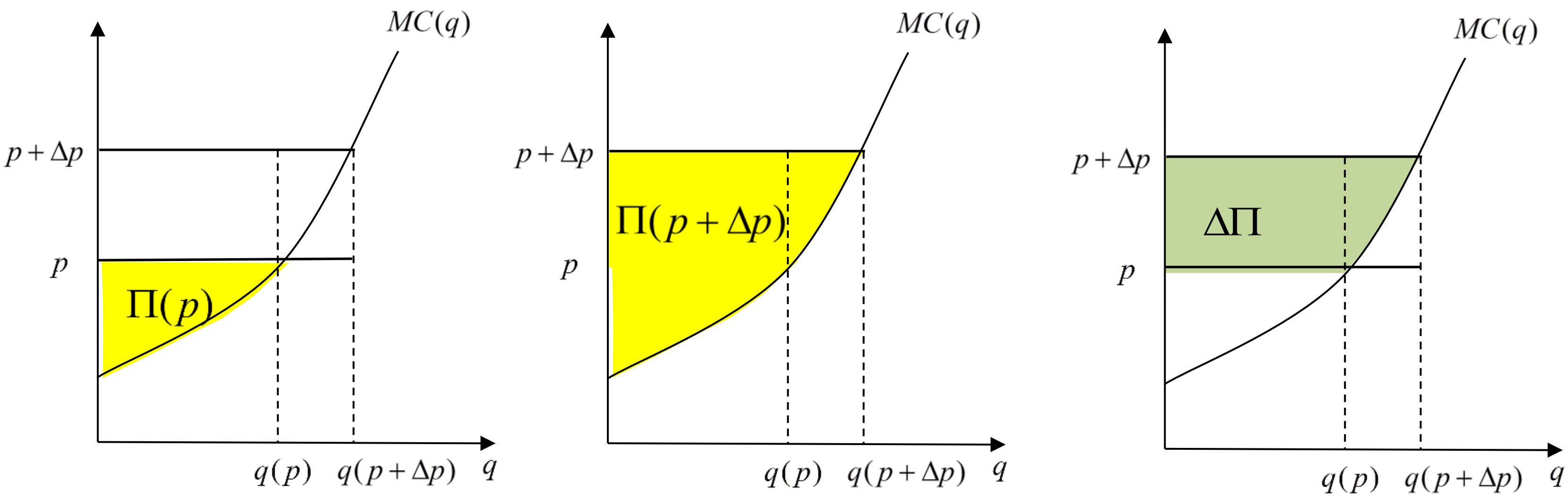
Note that $\frac{\partial \pi}{\partial p}(p, \bar{q}(p)) = \bar{q}(p)$ and $\frac{\partial \pi}{\partial q}(p, \bar{q}(p)) = 0$.

Therefore

$$\frac{d\Pi}{dp}(p) = \bar{q}(p)$$

A third look at the profit maximizing firm

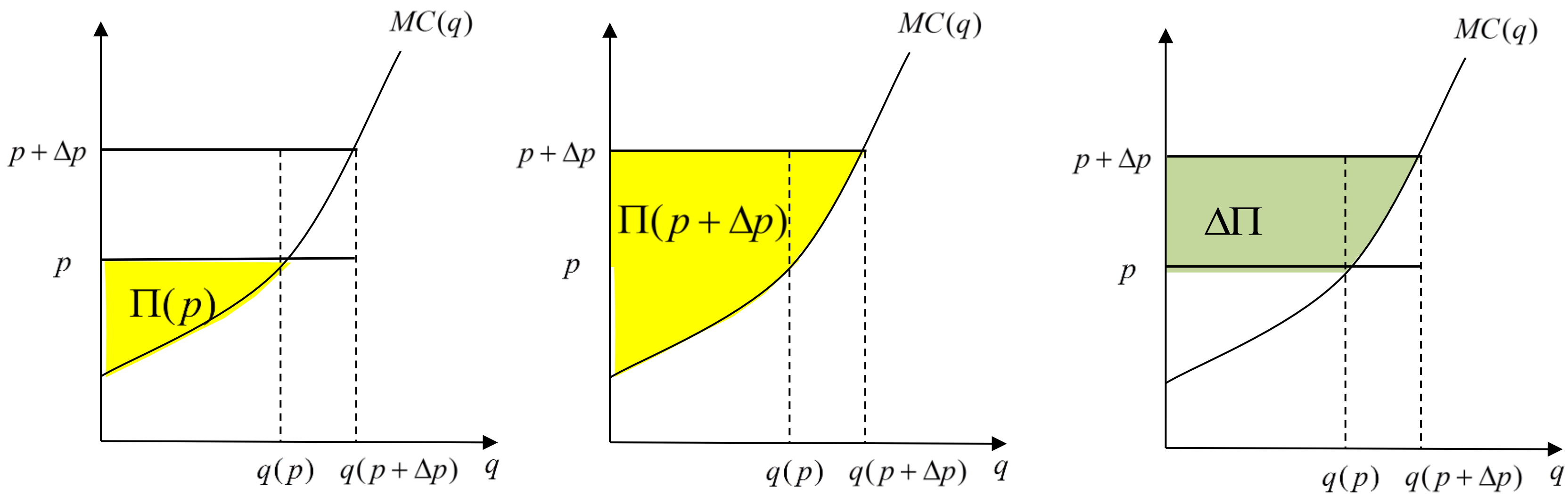
$$\pi = pq - c(q) \qquad \frac{\partial \pi}{\partial q} = p - MC(q)$$



++

Profit maximizing firm

$$\pi = pq - c(q) \qquad \frac{\partial \pi}{\partial q} = p - MC(q)$$



Let $\Delta\Pi$ be the increase in profit if the price rises by Δp

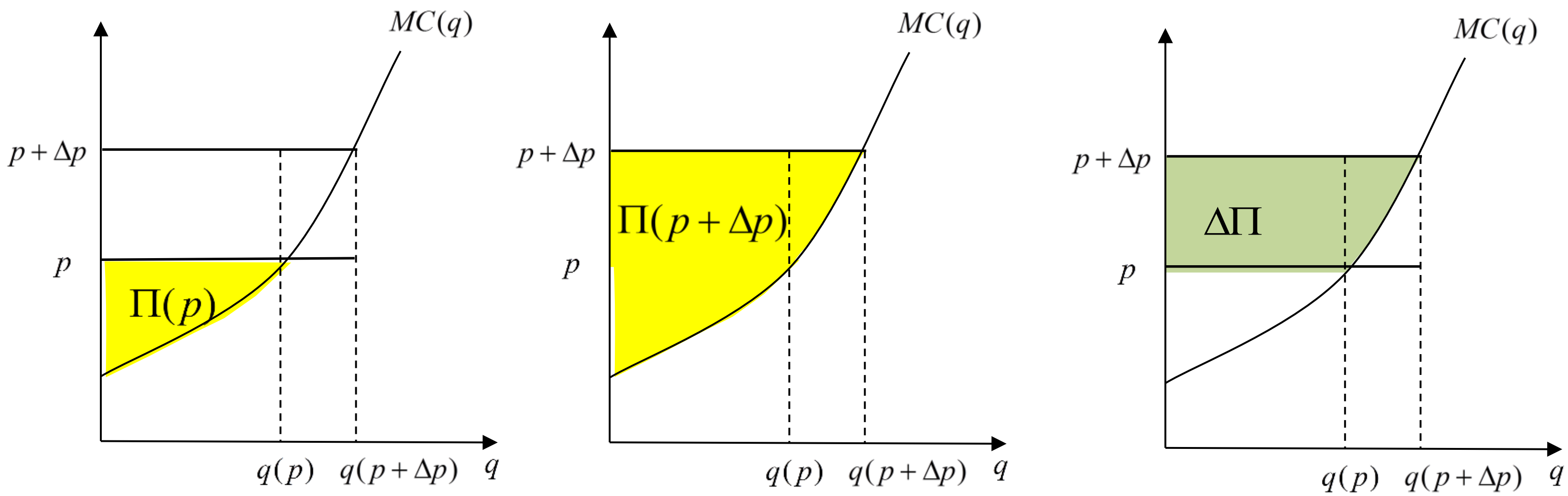
Note that

$$q^*(p)\Delta p \le \Delta\Pi \le q^*(p+\Delta p)\Delta p$$

+

Profit maximizing firm

$$\pi = pq - c(q) \qquad \frac{\partial \pi}{\partial q} = p - MC(q)$$



Let $\Delta\Pi$ be the increase in profit if the price rises by Δp

Note that

$$q^*(p)\Delta p \le \Delta\Pi \le q^*(p+\Delta p)\Delta p$$

Therefore $q^*(p) \le \dfrac{\Delta\Pi}{\Delta p} \le q^*(p+\Delta p)$. So if $q^*(p)$ is continuous, $\dfrac{d\Pi}{dp} = q(p)$

Envelope Theorem I

Suppose that $x^*(\alpha)$ uniquely solves $\underset{x\geq 0}{Max}\{f(x,\alpha)\,|\,x\in X\}$ for each $\alpha\in A\subset\mathbb{R}$.

Define $F(\alpha)\equiv f(x^*(\alpha),\alpha)$

If $x^*(\alpha)$ is continuous then $\dfrac{dF}{d\alpha}=\dfrac{\partial f}{\partial\alpha}(x^*(\alpha),\alpha)$.

++++

Envelope Theorem I

Suppose that $x^*(\alpha)$ uniquely solves $\underset{x\geq 0}{Max}\{f(x,\alpha)\,|\,x\in X\}$ for each $\alpha\in A\subset\mathbb{R}$.

Define $F(\alpha)\equiv f(x^*(\alpha),\alpha)$

If $x^*(\alpha)$ is continuous then $\dfrac{dF}{d\alpha}=\dfrac{\partial f}{\partial\alpha}(x^*(\alpha),\alpha)$.

<u>Proof</u>: Consider any α^1 and α^2 and maximizers $x^*(\alpha^1)$ and $x^*(\alpha^2)$.

Upper bound for $F(\alpha^2)-F(\alpha^1)$

$x^*(\alpha^1)$ is optimal with parameter α^1 so $f(x^*(\alpha^1),\alpha^1)\geq f(x^*(\alpha^2),\alpha^1)$.

+++

Envelope Theorem I

Suppose that $x^*(\alpha)$ uniquely solves $\underset{x\geq 0}{Max}\{f(x,\alpha)\,|\,x\in X\}$ for each $\alpha\in A\subset\mathbb{R}$.

Define $F(\alpha)\equiv f(x^*(\alpha),\alpha)$

If $x^*(\alpha)$ is continuous then $\dfrac{dF}{d\alpha}=\dfrac{\partial f}{\partial\alpha}(x^*(\alpha),\alpha)$.

Proof: Consider any α^1 and α^2 and maximizers $x^*(\alpha^1)$ and $x^*(\alpha^2)$.

Upper bound for $F(\alpha^2)-F(\alpha^1)$

$x^*(\alpha^1)$ is optimal with parameter α^1 so $f(x^*(\alpha^1),\alpha^1)\geq f(x^*(\alpha^2),\alpha^1)$.

Then $F(\alpha^2)-F(\alpha^1)\equiv f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^1),\alpha^1)\leq f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)$

++

Envelope Theorem I (feasible set is independent of the parameter)

Suppose that $x^*(\alpha)$ uniquely solves $\underset{x\geq 0}{Max}\{f(x,\alpha)\,|\,x\in X\}$ for each $\alpha\in A\subset\mathbb{R}$.

Define $F(\alpha)\equiv f(x^*(\alpha),\alpha)$

If $x^*(\alpha)$ is continuous then $\dfrac{dF}{d\alpha}=\dfrac{\partial f}{\partial\alpha}(x^*(\alpha),\alpha)$.

<u>Proof</u>: Consider any α^1 and α^2 and maximizers $x^*(\alpha^1)$ and $x^*(\alpha^2)$.

Upper bound for $F(\alpha^2)-F(\alpha^1)$

$x^*(\alpha^1)$ is optimal with parameter α^1 so $f(x^*(\alpha^1),\alpha^1)\geq f(x^*(\alpha^2),\alpha^1)$.

Then $F(\alpha^2)-F(\alpha^1)=f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^1),\alpha^1)\leq f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)$

Lower bound for $F(\alpha^2)-F(\alpha^1)$

$x^*(\alpha^2)$ is optimal with parameter α^2 so $f(x^*(\alpha^1),\alpha^2)\leq f(x^*(\alpha^2),\alpha^2)$.

+

Envelope Theorem I

Suppose that $x^*(\alpha)$ uniquely solves $\underset{x\geq 0}{Max}\{f(x,\alpha)\,|\,x\in X\}$ for each $\alpha\in A\subset\mathbb{R}$.

Define $F(\alpha)\equiv f(x^*(\alpha),\alpha)$

If $x^*(\alpha)$ is continuous then $\dfrac{dF}{d\alpha}=\dfrac{\partial f}{\partial\alpha}(x^*(\alpha),\alpha)$.

<u>Proof</u>: Consider any α^1 and α^2 and maximizers $x^*(\alpha^1)$ and $x^*(\alpha^2)$.

Upper bound for $F(\alpha^2)-F(\alpha^1)$

$x^*(\alpha^1)$ is optimal with parameter α^1 so $f(x^*(\alpha^1),\alpha^1)\geq f(x^*(\alpha^2),\alpha^1)$.

Then $F(\alpha^2)-F(\alpha^1)=f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^1),\alpha^1)\leq f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)$

Lower bound for $F(\alpha^2)-F(\alpha^1)$

$x^*(\alpha^2)$ is optimal with parameter α^2 so $f(x^*(\alpha^1),\alpha^2)\leq f(x^*(\alpha^2),\alpha^2)$.

Then $F(\alpha^2)-F(\alpha^1)\equiv f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^1),\alpha^1)\geq f(x^*(\alpha^1),\alpha^2)-f(x^*(\alpha^1),\alpha^1)$

Combining these results

$$f(x^*(\alpha^1),\alpha^2)-f(x^*(\alpha^1),\alpha^1)\le F(\alpha_2)-F(\alpha_1)\le f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)$$

+++

Combining these results

$$f(x^*(\alpha^1),\alpha^2)-f(x^*(\alpha^1),\alpha^1)\le F(\alpha_2)-F(\alpha_1)\le f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)$$

Therefore,

$$\frac{f(x^*(\alpha^1),\alpha^2)-f(x^*(\alpha^1),\alpha^1)}{\alpha^2-\alpha^1}\le\frac{F(\alpha_2)-F(\alpha_1)}{\alpha^2-\alpha^1}\le\frac{f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)}{\alpha^2-\alpha^1}$$

++

Combining these results

$$f(x^*(\alpha^1),\alpha^2) - f(x^*(\alpha^1),\alpha^1) \le F(\alpha_2) - F(\alpha_1) \le f(x^*(\alpha^2),\alpha^2) - f(x^*(\alpha^2),\alpha^1)$$

Therefore,

$$\frac{f(x^*(\alpha^1),\alpha^2) - f(x^*(\alpha^1),\alpha^1)}{\alpha^2 - \alpha^1} \le \frac{F(\alpha_2) - F(\alpha_1)}{\alpha^2 - \alpha^1} \le \frac{f(x^*(\alpha^2),\alpha^2) - f(x^*(\alpha^2),\alpha^1)}{\alpha^2 - \alpha^1}$$

By assumption $x^*(\alpha)$ is continuous at α^1 . Taking the limit,

$$\frac{\partial f}{\partial \alpha}(x^*(\alpha^1),\alpha^1) \le \frac{dF}{d\alpha}(\alpha^1) \le \frac{\partial f}{\partial \alpha}(x^*(\alpha^1),\alpha^1).$$ Q.E.D.

+

Combining these results

$$f(x^*(\alpha^1),\alpha^2)-f(x^*(\alpha^1),\alpha^1)\le F(\alpha_2)-F(\alpha_1)\le f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)$$

Therefore,

$$\frac{f(x^*(\alpha^1),\alpha^2)-f(x^*(\alpha^1),\alpha^1)}{\alpha^2-\alpha^1}\le\frac{F(\alpha_2)-F(\alpha_1)}{\alpha^2-\alpha^1}\le\frac{f(x^*(\alpha^2),\alpha^2)-f(x^*(\alpha^2),\alpha^1)}{\alpha^2-\alpha^1}$$

By assumption $x^*(\alpha)$ is continuous at α^1 . Taking the limit,

$$\frac{\partial f}{\partial\alpha}(x^*(\alpha^1),\alpha^1)\le\frac{dF}{d\alpha}(\alpha^1)\le\frac{\partial f}{\partial\alpha}(x^*(\alpha^1),\alpha^1).$$ Q.E.D.

Envelope Theorem for a minimization problem

Suppose that $x^*(\alpha)$ uniquely solves $\underset{x\geq 0}{Min}\{f(x,\alpha)\,|\,x\in X\}$ for each $\alpha \in A \subset \mathbb{R}$.

Define $F(\alpha) \equiv f(x^*(\alpha),\alpha)$

If $x^*(\alpha)$ is continuous then $\dfrac{dF}{d\alpha} = \dfrac{\partial f}{\partial \alpha}(x^*(\alpha),\alpha)$.

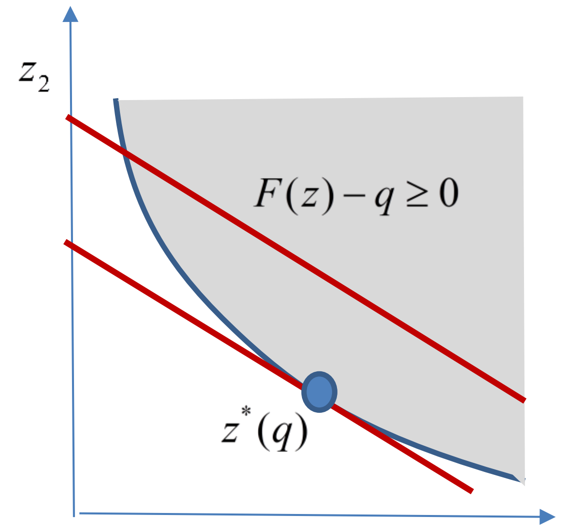
To show this is true one can use the method of proof above. All the inequalities are reversed.

Or note that if $f(x,\alpha)$ is being minimized, then $-f(x,\alpha)$ is being maximized.

Application: The impact of an input price rise on marginal cost and hence output

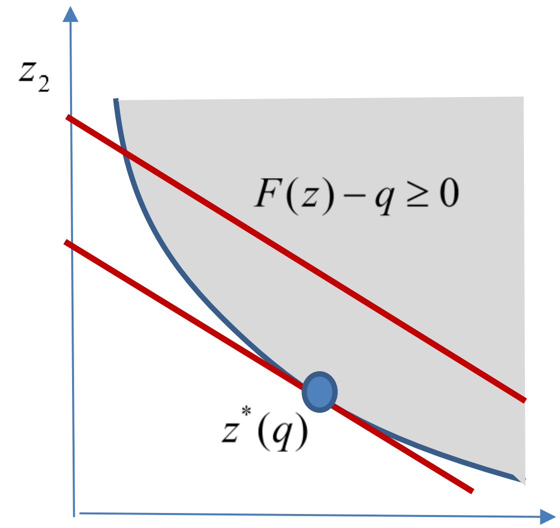
$$C(r,q) = \underset{z}{Min}\{r \cdot z \mid F(z) - q \geq 0\}$$

***



Application: The impact of an input price rise on marginal cost and hence output

$$C(r,q) = \underset{z}{Min}\{r \cdot z \mid F(z) - q \geq 0\}$$



Since the feasible set

$$X = \{z \mid F(z) - q \geq 0\}$$

is independent of the price vector we can appeal to

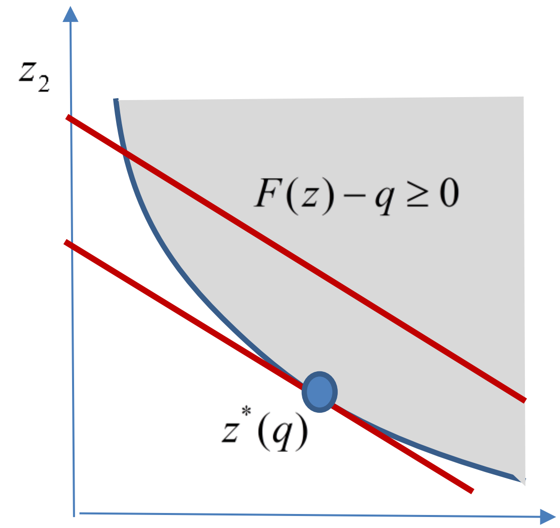
Envelope Theorem I.

$$\frac{\partial C}{\partial r_j}(r,q) = z_j^{\,*}(r,q)$$

*

Application: The impact of an input price rise on marginal cost and hence output

$$C(r,q) = \underset{z}{Min}\{r \cdot z \mid F(z) - q \geq 0\}$$



Since the feasible set is independent of

the price vector we can appeal to

Envelope Theorem I.

$$\frac{\partial C}{\partial r_j}(r,q) = z_j^*(r,q)$$

Differentiate both sides by q

$$\frac{\partial}{\partial r_j}MC = \frac{\partial}{\partial r_j}(\frac{\partial C}{\partial q}) = \frac{\partial}{\partial q}(\frac{\partial C}{\partial r_j}) = \frac{\partial z_j^*}{\partial q} .$$

Thus marginal cost may fall as an input price rises

It depends upon whether the j-th input is a normal or inferior input

D: Envelope Theorem when the feasible set and maximand vary with the parameter

$$F(\alpha) \equiv \underset{x \geq 0}{Max}\{ f(x,\alpha) \mid x \in X(\alpha)\}$$

$$X(\alpha) = \{x \mid h_i(x,\alpha) \geq 0,\ \ i = 1,...,m\}$$

Envelope Theorem II

Define $F(\alpha) = \underset{x}{Max}\{ f(x,\alpha) \mid x \geq 0,\ \ h_i(x,\alpha) \geq 0,\ \ i = 1,...,m\}$ where $f, h \in \mathbb{C}^2$

If $x^*(\alpha)$ and $\lambda^*(\alpha)$ are continuously differentiable functions, then

$$\frac{dF}{d\alpha} = \frac{\partial L}{\partial \alpha}(x^*(\alpha), \lambda^*(\alpha), \alpha).$$

Proof: We consider the one constraint case. An almost identical proof can be constructed with multiple constraints.

The Lagrangian is

$$L(x^*(\alpha),\alpha) = f(x^*(\alpha),\alpha) + \lambda(\alpha)h(x^*(\alpha),\alpha)$$

Since $x^*(\alpha)$ solves the maximization problem, the FOC must hold. Appealing to complementary slackness, the second term is zero. Therefore

$$F(\alpha) = f(x^*(\alpha),\alpha) = L(x^*(\alpha),\alpha)$$

We differentiate this equation as follows:

$$\frac{dF}{d\alpha}(\alpha) = \frac{d}{d\alpha}L(x^*(\alpha),\alpha) = \sum_{j=1}^{n} \frac{\partial L}{\partial x_j}\frac{dx_j^*}{d\alpha} + \frac{\partial L}{\partial \lambda}\frac{d\lambda}{d\alpha} + \frac{\partial L}{\partial \lambda} \ .$$

Suppose that the constraint is binding and $x^*(\alpha) >> 0$

Then, from the FOC each term on the right hand side except the last is zero.

Therefore

$$\frac{dF}{d\alpha}(\alpha) = \frac{\partial L}{\partial \lambda}$$

Complete proof:

This is provided for any student who may be interested.

We consider the one constraint case. An almost identical proof can be constructed with multiple constraints.

As argued above,

$$F(\alpha) = f(x^*(\alpha),\alpha) = L(x^*(\alpha),\alpha)$$

Therefore

$$\frac{dF}{d\alpha}(\alpha) = \frac{d}{d\alpha}L(x^*(\alpha),\alpha) = \sum_{j=1}^{n} \frac{\partial L}{\partial x_j}\frac{dx_j^*}{d\alpha} + \frac{\partial L}{\partial \lambda}\frac{d\lambda}{d\alpha} + \frac{\partial L}{\partial \lambda} \ .$$

The FOC are

$$\frac{\partial L}{\partial x_j}(x^*(\alpha),\alpha) \le 0 \text{ with equality if } x_j^*(\alpha) > 0 \qquad (1)$$

$$\frac{\partial L}{\partial \lambda}(x^*(\alpha),\alpha) = h(x^*(\alpha),\alpha) \ge 0 \text{ with equality if } \lambda > 0 \qquad (2)$$

***

The FOC are

$$\frac{\partial L}{\partial x_j}(x^*(\alpha),\alpha) \le 0 \text{ with equality if } x_j^*(\alpha) > 0 \qquad (1)$$

$$\frac{\partial L}{\partial \lambda}(x^*(\alpha),\alpha) = h(x^*(\alpha),\alpha) \ge 0 \text{ with equality if } \lambda > 0 \qquad (2)$$

Suppose that k of the conditions in (1) hold with strict inequality. Re-label the variables so that the strict inequality holds for the first k variables. Since $\frac{\partial L}{\partial x_j}(x^*(\alpha),\alpha) < 0$ for $j = 1,...,k$, it follows that $x_j^*(\alpha) = 0, \;\; j = 1,...,k$.

**

The FOC are

$$\frac{\partial L}{\partial x_j}(x^*(\alpha),\alpha) \le 0 \text{ with equality if } x_j^*(\alpha) > 0 \qquad (1)$$

$$\frac{\partial L}{\partial \lambda}(x^*(\alpha),\alpha) = h(x^*(\alpha),\alpha) \ge 0 \text{ with equality if } \lambda > 0 \qquad (2)$$

Suppose that k of the conditions in (1) hold with strict inequality. Re-label the variables so that the strict inequality holds for the first k variables. Since $\frac{\partial L}{\partial x_j}(x^*(\alpha),\alpha) < 0$ for $j = 1,...,k$, it follows that

$x_j^*(\alpha) = 0, \;\; j = 1,...,k$.

Also, given the continuity assumption, for sufficiently small $\Delta\alpha$

$$\frac{\partial L}{\partial x_j}(x^*(\alpha + \Delta\alpha), \alpha + \Delta\alpha) < 0.$$

Therefore $x_j^*(\alpha + \Delta\alpha) = 0$

Thus for the first k variables there is no change in x_j and so $\frac{dx_j^*}{d\alpha}(\alpha) = 0$.

The FOC are

$$\frac{\partial L}{\partial x_j}(x^*(\alpha),\alpha)\le 0 \text{ with equality if } x_j^*(\alpha)>0 \qquad (1)$$

$$\frac{\partial L}{\partial \lambda}(x^*(\alpha),\alpha)=h(x^*(\alpha),\alpha)\ge 0 \text{ with equality if } \lambda>0 \qquad (2)$$

Suppose that k of the conditions in (1) hold with strict inequality. Re-label the variables so that the strict inequality holds for the first k variables. Since $\frac{\partial L}{\partial x_j}(x^*(\alpha),\alpha)<0$ for $j=1,...,k$, it follows that

$x_j^*(\alpha)=0,\ \ j=1,...,k$.

Also, given the continuity assumption, for sufficiently small $\Delta\alpha$

$$\frac{\partial L}{\partial x_j}(x^*(\alpha+\Delta\alpha),\alpha+\Delta\alpha)<0.$$

Therefore $x_j^*(\alpha+\Delta\alpha)=0$

Thus for the first k variables there is no change in x_j and so $\frac{dx_j^*}{d\alpha}(\alpha)=0$.

For the remaining variables $\frac{\partial L}{\partial x_j}=0$. Thus for all j, $\frac{\partial L}{\partial x_j}\frac{dx_j^*}{d\alpha}(\alpha)=0$

Case (i) $\dfrac{\partial L}{\partial \lambda} = h(x^*(\alpha),\alpha) = 0$

Case (ii) $\dfrac{\partial L}{\partial \lambda} = h(x^*(\alpha),\alpha) > 0$ and so $\lambda(\alpha) = 0$

In this case, for sufficiently small $\Delta\alpha$ the constraint is still not binding at $x^*(\alpha + \Delta\alpha)$ and so $\lambda(\alpha + \Delta\alpha) = 0$

Therefore $\lambda(\alpha) = \lambda(\alpha + \Delta\alpha)$ and so $\dfrac{d\lambda}{d\alpha}(\alpha) = 0$.

Combining the two cases, it follows that

$$\frac{\partial L}{\partial \lambda}\frac{d\lambda}{d\alpha} = 0 \ .$$

$$\frac{dF}{d\alpha}(\alpha) = \frac{d}{d\alpha}L(x^*(\alpha),\alpha) = \sum_{j=1}^{k} \frac{\partial L}{\partial x_j}\frac{dx_j^*}{d\alpha} + \sum_{j=k+1}^{n} \frac{\partial L}{\partial x_j}\frac{dx_j^*}{d\alpha} + \frac{\partial L}{\partial \lambda}\frac{d\lambda}{d\alpha} + \frac{\partial L}{\partial \lambda} \ .$$

From the above arguments, each of the terms on the right hand side except the last one is zero.

Therefore

$$\frac{dF}{d\alpha}(\alpha) = \frac{\partial L}{\partial \lambda}$$