Answers to Midterm practice questions

Here are my proposed answers. Please let me know ASAP if you think I have made a mistake.

1. Walrasian Equilibrium

(a) Maximize $u(x) = \ln U(x)$. First $u(x)$ is easier to differentiate. Second it is concave so the FOC are both necessary and sufficient.

FOC

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} \quad \text{so} \quad \frac{x_1^{-1}}{p_1} = \frac{x_2^{-1}}{p_2} \quad \text{i.e.} \quad \frac{1}{p_1 x_1} = \frac{1}{p_2 x_2} \ .$$

Appealing to the Ratio rule

$$\frac{1}{p_1 x_1} = \frac{1}{p_2 x_2} = \frac{2}{p \cdot x} = \frac{2}{I^h}$$

Therefore

$$\overline{x}_i^h(p, I^h) = \frac{1}{2}\frac{I^h}{p_i} \ .$$

(b) Summing over consumers,

$$\sum_{h=1}^{H} \overline{x}_i^h = \sum_{h=1}^{H} \frac{1}{2}\frac{I^h}{p_i} = \frac{I}{2p_i} \quad \text{where } I = \sum_{j=1}^{H} I^h$$

and

$$\overline{x}_i \ (p, I) = \frac{1}{2}\frac{I}{p_i}$$

Therefore

$$\overline{x}_i \ (p, I) = \sum \overline{x}_i^h(p, I^h)$$

(c) $x_2 = q_2 = 4z^{1/3}$, $x_1 = \omega_1 - z = 32 - z$.

$$u = \ln U = \ln x_1 + \ln x_2 = \ln(32 - z) + \ln 4z^{1/3}$$

$$= \ln(32 - z) + \ln 4 + \tfrac{1}{3}\ln z$$

FOC

$$\frac{du}{dz} = \frac{1}{32-z} - \frac{\frac{1}{3}}{z} = 0 \text{ for a Maximum.}$$

Note that the second derivative $u''(z)$ is negative so $u(z)$ is concave. Thus the FOC is both necessary and sufficient.

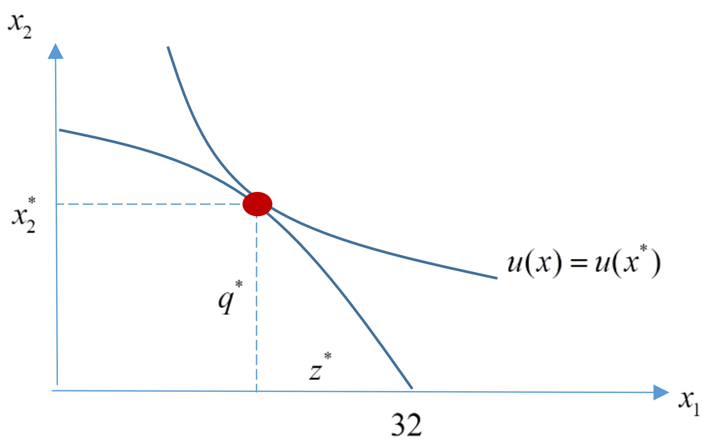
Appeal to the Ratio Rule

$$\frac{1}{32-z} = \frac{\frac{1}{3}}{z} = \frac{\frac{4}{3}}{32}.$$

Therefore $z^* = 8$ and so $x_1^* = 32 - z^* = 24$ and $x_2^* = q^* = 4z^{1/3} = 8$

The solution is depicted below. (A diagram is worth 1000 words!)

The utility of the representative consumer is maximized at $z^* = 8$ where the indifference curve is tangential to the production function.



(d) The profit of the firm is

$$\pi = p_2 q - p_1 z = 4 p_2 z^{1/3} - p_1 z$$

For profit maximization,

$$\frac{d\pi}{dz} = 4 p_2 \tfrac{1}{3} z^{-2/3} - p_1 = 0$$

To support $z^* = 8$

$$\frac{d\pi}{dz} = 4 p_2 \tfrac{1}{3} (\frac{1}{z^*})^{2/3} - p_1 = 0$$

Therefore

$$\frac{d\pi}{dz} = \frac{4}{12} p_2 - p_1 = 0 \text{ and so } p_1 = \tfrac{1}{3} p_2 \ .$$

(e) The zero profit line is the green line through the blue marker (the zero production vector $(z,q) = (0,0)$). The parallel profit maximizing line must be tangential to the boundary of the production set (and the indifference curve.) As shown in (d) the slope of the line must be equal to -1/6.

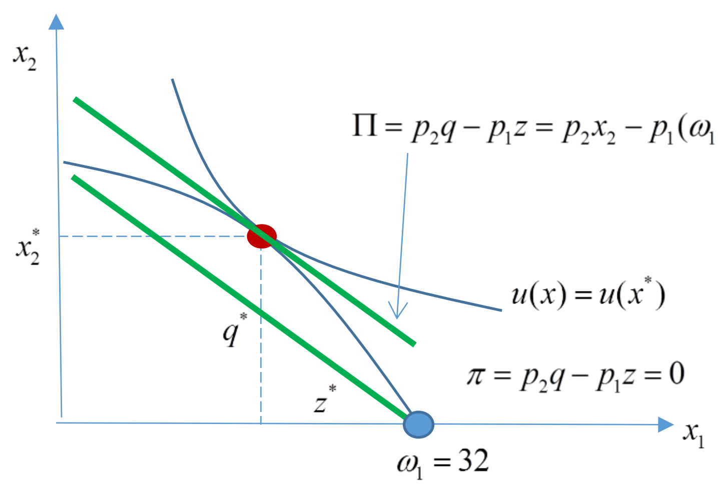
I use Π to denote maximized profit. Then

$$\Pi \geq p_2 q - p_1 z = p_2 x_2 - p_1(\omega_1 - x_1) = p \cdot x - p_1 \omega_1 .$$

Rearranging this inequality,

$$p \cdot x \leq p_1 \omega_1 + \Pi$$

Thus the profit maximizing line is also the budget line of the representative consumer. We have already shown that the level set for $u(x)$ is tangential to the boundary of the production set at the red marker so it must also be tangential to the budget line.



2. Utility maximization and elasticity of substitution

(a) $1/U(x)$ has the same level sets as $U(x)$ but is a decreasing function. Therefore

$$u(x) = -1/U(x) = -x_1^{-1} - x_2^{-2}$$

Is an increasing function with the same level sets.

(b) FOC

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} \quad \text{so} \quad \frac{x_1^{-2}}{p_1} = \frac{x_2^{-2}}{p_2} \ .$$

Therefore

$$\frac{1}{p_1 x_1^{\;2}} = \frac{1}{p_2 x_2^{\;2}} \ .$$

Take the square root.

$$\frac{1}{p_1^{\;1/2} x_1} = \frac{1}{p_2^{\;1/2} x_2}$$

Multiply top and bottom by the square root of the price

$$\frac{p_1^{\;1/2}}{p_1 x_1} = \frac{p_2^{\;1/2}}{p_2 x_2}$$

(c) Appealing to the Ratio Rule,

$$\frac{1}{p_1^{\;1/2} x_1} = \frac{1}{p_2^{\;1/2} x_2} = \frac{p_1^{\;1/2}}{p_1 x_1} = \frac{p_2^{\;1/2}}{p_2 x_2} = \frac{p_1^{\;1/2} + p_2^{\;1/2}}{p_1 x_1 + p_2 x_2} = \frac{p_1^{\;1/2} + p_2^{\;1/2}}{I} \ .$$

Therefore

$$\frac{1}{p_1^{\;1/2} x_1} = \frac{1}{p_2^{\;1/2} x_2} = \frac{p_1^{\;1/2} + p_2^{\;1/2}}{I} .$$

And we can solve for $\bar{x}(p, I)$

(d) Define

$$\bar{U} = \underset{x}{Max}\{U(x) \mid p \cdot x \leq I \ .$$

The FOC for the solution $\bar{x}(p, I)$ are

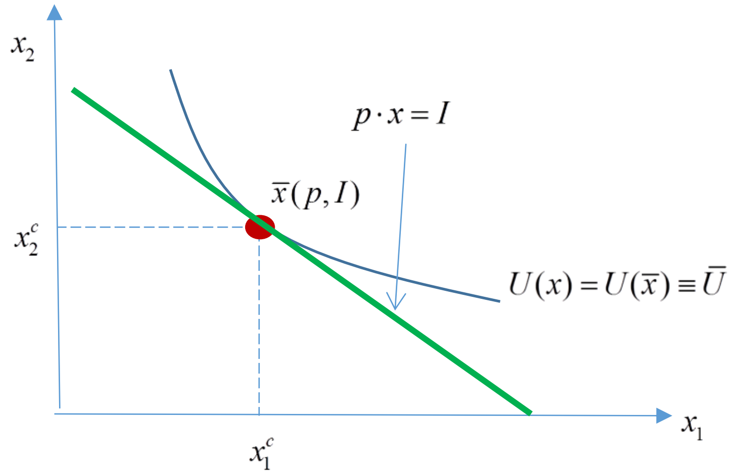
$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2}$$

Now consider the following minimization problem.

$$M(p, \bar{U}) = \underset{x}{Min}\{p \cdot x \,|\, U(x) \geq \bar{U}\} .$$

From the figure it is clear that $\bar{x}$ that solves the first problem is the same as x^c that solves the second problem. Thus the FOC must be the same.

(Alternatively write out the FOC using the Lagrange method.)



From (a)

$$\frac{1}{p_1^{1/2} x_1^{\ c}} = \frac{1}{p_2^{1/2} x_2^{\ c}} .$$

Therefore

$$\frac{x_2}{x_1} = (\frac{p_1}{p_2})^{1/2}$$

(d) $\mathcal{E}(y,x) = x\frac{\partial}{\partial x}\ln y$

Therefore

$$\mathcal{E}(\frac{x_2^{\ c}}{x_1^{\ c}}, p_1) = p_1 \frac{\partial}{\partial p_1} \ln \frac{x_2}{x_1} = p_1 \frac{\partial}{\partial p_1} \ln (\frac{p_1}{p_2})^{1/2} = \tfrac{1}{2} p_1 \frac{\partial}{\partial x}[\ln p_1 - \ln p_2] = \tfrac{1}{2} .$$

Appealing to symmetry, $\sigma_2 = \frac{1}{2}$.

Exercise 3: Production and cost

Appealing to 2(b)

$$\frac{1}{r_1^{1/2} z_1} = \frac{1}{r_2^{1/2} z_2} = \frac{r_1^{1/2} + r_2^{1/2}}{B} .$$

Therefore

$$z_1(B) = \frac{B}{r_1^{1/2}(r_1^{1/2} + r_2^{1/2})} \text{ and } z_2(B) = \frac{B}{r_2^{1/2}(r_1^{1/2} + r_2^{1/2})}$$

Write these as follows:

$$z_1(B)^{-1} = \frac{r_1^{1/2}(r_1^{1/2}+r_2^{1/2})}{B} \text{ and } z_2(B)^{-1} = \frac{r_2^{1/2}(r_1^{1/2}+r_2^{1/2})}{B}.$$

Then

$$z_1(B)^{-1} + z_2(B)^{-1} = \frac{(r_1^{1/2}+r_2^{1/2})^2}{B}$$

Therefore

$$q^* = (z_1^{-1}+z_2^{-1})^{-1} = \frac{B}{(r_1^{1/2}+r_2^{1/2})^2} .$$

With budget B , maximum output is q^* . With a smaller budget the maximum output is lower. Therefore the lowest cost of producing q^* is B.

Inverting the above expression, for any q^*

$$C(r,q^*) = (r_1^{1/2}+r_2^{1/2})^2 q^* .$$

Then

$$AC(q) = MC(q) = (r_1^{1/2}+r_2^{1/2})^2 .$$

Remark:

If a price taking firm produces a strictly positive output, it must be the case that for a WE a firm must be profit maximizing so $p = MC$. Therefore the WE price is $p = (r_1^{1/2}+r_2^{1/2})^2$.

Since $p = AC$ it follows that maximized profit is

$$\Pi = pq - C(q) = q(p - AC(q)) = 0 .$$

(d) $q^* = (z_1^{-1}+z_2^{-1})^{-1/2} = [\frac{B}{(r_1^{1/2}+r_2^{1/2})^2}]^{1/2} = \frac{B^{1/2}}{r_1^{1/2}+r_2^{1/2}}$

Inverting,

$$B = (r_1^{1/2}+r_2^{1/2})^2 q^2$$

Therefore the cost function is

$$C(q,r)=(r_1^{1/2}+r_2^{1/2})^2q^2$$

(e) $AC(q,r)=(r_1^{1/2}+r_2^{1/2})^2q$ and $MC(q,r)=2(r_1^{1/2}+r_2^{1/2})^2q$

Remark:

Now $MC(q)>AC(q)$ so for any price $p>AC(0)$ the profit maximizing firm makes a strictly positive profit.

4. Choice over time

(a) Period 1: $S_1=K_1+y_1-c_1$

Period 2: $K_2=(1+r)S_1$, $c_2=K_2+y_2$

Hence

$$\frac{c_2}{1+r}=S_1+\frac{y_2}{1+r}=K_1+y_1-c_1+\frac{y_2}{1+r}$$

This can be rewritten as the following life-time PV constraint.

$$c_1+\frac{c_2}{1+r}=K_1+y_1+\frac{y_2}{1+r}$$

(b) In the standard model the budget constraint is

$$p_1x_1+p_2x_2=I$$

Thus in the lifetime budget constraint, $p_1=1$ and $p_2=\dfrac{1}{1+r}$. An increase in the interest rate is a decrease in the price of period 2 goods. (You have to save less to consume the same in period 2. Thus if you are a saver $\overline{S}_1>0$ the higher interest rate makes you better off.

Substitution effect: As the steepness of the budget line p_1/p_2 rises, that is, the interest rate rises, the compensated c_1 falls and c_2 rises.

Income effect: A saver is strictly better off as the interest rate rises. Then to be "compensated", the consumer must be taxed. Now give the tax back. Assuming that goods are normal, both c_1 and c_2 rise. (In part (c) we will show that this is the case.) So the two effects are reinforcing on period 2 consumption and offsetting for period 1 consumption (and hence $S_1=K_1+y_1-c_1$)

Remark: You were asked only to discuss saving. For a borrower the substation effect is the same. But a higher interest rate make a borrower worse off. Thus he has to be given money to be compensated. For the income effect the money is taken away again and this results in a negative income effect. The period 1 effects are then reinforcing (less consumption and so less borrowing). The period 2 effects are off setting.

$$U(c_1,c_2)=\ln c_1+\tfrac{4}{5}\ln c_2$$

FOC

$$\frac{MU_1}{p_1}=\frac{MU_2}{p_2} \quad \text{so} \quad \frac{1}{c_1}=\frac{\tfrac{4}{5}}{\dfrac{c_2}{1+r}}=\frac{\tfrac{9}{5}}{K_1+y_1+\dfrac{y_2}{1+r}} \quad ..$$

Then

$$c_1=\tfrac{5}{9}(K_1+y_1+\frac{y_2}{1+r}) \quad \text{and} \quad c_2=\tfrac{4}{9}(K_1+y_1+\frac{y_2}{1+r})$$

With endowments the solution is

$$c_1=\tfrac{5}{9}(\omega_1+\frac{\omega_2}{1+r}) \quad \text{and} \quad c_2=\tfrac{4}{9}(\omega_1+\frac{\omega_2}{1+r})$$

The consumer is a first period saver if

$$c_1=\tfrac{5}{9}(\omega_1+\frac{\omega_2}{1+r})<\omega_1$$

i.e.

$$\omega_1-\tfrac{5}{9}\omega_1>\tfrac{4}{9}\frac{\omega_2}{1+r}$$

i.e.

$$1+r>\tfrac{4}{5}\frac{\omega_2}{\omega_1}\ .$$

Given the data

$$1+r>\tfrac{4}{5}\frac{125}{100}=1$$

Thus the consumer is a saver for all positive interest rates.