Choice over time

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First draft
T period model

Initial financial capital $K_1$

Consumption sequence $\{c_t\}_{t=1}^T$

Income sequence $\{y_t\}_{t=1}^T$

Financial capital sequence $\{K_t\}_{t=1}^{T+1}$

Life time utility:

$$U(c) = u(c_1) + \delta u(c_2) + \ldots + \delta^{T-1} u(c_T) \quad \text{where } u(c_t) \text{ is strictly increasing and strictly concave.}$$

Interest rate $r$
B. Analysis of the first two periods

Assumption 1: \( \delta < 1 \) The future is weighed less heavily (i.e. “discounted”)

Let \( K_3^* \) be the optimal period 3 financial capital.

Fix this and consider the first two periods of this model.

Period budget constraints

\[
K_2 \leq (1 + r)(K_1 + y_1 - c_1) \quad (1)
\]

\[
K_3^* \leq (1 + r)(K_3 + y_2 - c_2) \quad (2)
\]

In present values,

\[
\frac{K_2}{1 + r} \leq K_1 + y_1 - c_1
\]

\[
\frac{K_3^*}{(1 + r)^2} \leq \frac{K_2}{1 + r} + \frac{y_2}{1 + r} - \frac{c_2}{1 + r}
\]

Add and rearrange

\[
c_1 + \frac{c_2}{1 + r} \leq y_1 + \frac{y_2}{1 + r} + K_1 - \frac{K_3^*}{(1 + r)^2}
\]
Economists refer to the present value of the future wage income stream as the consumer’s human capital. The first period wealth of the consumer is the sum of the initial financial capital and the human capital.

\[ W_1 = K_1 + PV(y_1, y_2) \]

Then the two period budget constraint is

\[ PV(c_1, c_2) \leq W_1 - \frac{K_3}{(1 + r)^2} \]
A necessary condition for lifetime utility maximization is that \((c_1, c_2)\) must solve the following maximization problem

\[
\max_c \{u(c_1) + \delta u(c_2) \mid W_1 - \frac{K^*_3}{(1+r)^2} - c_1 - \frac{c_2}{1+r}\}
\]

The Lagrangian:

\[
L = U(c) + \lambda(W_1 - \frac{K^*_3}{(1+r)^2} - PV(c)) = u(c_1) + \delta u(c_2) + \lambda(W_1 - \frac{K^*_3}{(1+r)^2} - c_1 - \frac{c_2}{1+r})
\]

We will choose a CES function so that \(c^*_1 \gg 0\). Then the FOC are as follows:

\[
\frac{\partial L}{\partial c_1} = u'(c_1) - \lambda = 0 \; , \; t = 1,\ldots,T
\]

\[
\frac{\partial L}{\partial c_2} = \delta u'(c_2) - \frac{\lambda}{1+r} = 0 \; , \; t = 1,\ldots,T
\]

Therefore \(\frac{u'(c_1^*)}{u'(c_2^*)} = \theta\) where \(\theta = (1+r)\delta\)
C. Analysis of period \( t \) and period \( t+1 \)

Next we note that we can also fix \( \{c_s^t\}_{t=1}^T = \{c_s^t\}_{t=1}^T \) and hence \( W_t^* \) and also set \( K_{t+2} = K_{t+2}^* \). A necessary condition for life-time utility maximization is that \( (c_t, c_{t+1}) \) must solve the following maximization problem

\[
Max \{u(c_t) + \delta u(c_{t+1}) | W_t^* - \frac{K_{t+2}^*}{(1+r)^2} - c_t - \frac{c_{t+1}}{1+r} \}
\]

The FOC are the same as for the first two periods. Hence the following is the growth equation for all periods.

**Consumption growth equation**

\[
\frac{u'(c_t^*)}{u'(c_{t+1}^*)} = \theta
\]

This equation implicitly defines a consumption growth equation

\[
c_{t+1} = f(c_t)
\]

Case (i) \( \theta > 1 \) then from the growth equation, \( u'(c_t^*) > u'(c_{t+1}^*) \).

Given the concavity assumption it follows that
\( \{c_t^*\}_{t=1}^T \) is a strictly increasing sequence.

Case (ii) \( \theta < 1 \) then \( \{c_t^*\}_{t=1}^T \) is a strictly decreasing sequence.
Complete solution for the CES family

**Consumption growth equation**

\[
\frac{u'(c_t^*)}{u'(c_{t+1})} = \theta
\]

CES utility \( u'(c_t) = c_t^{-1/\sigma} \)

\[
U(c) = \frac{1}{(1-1/\sigma)} \sum_{t=1}^{T} \delta^{t-1} c_t^{1-1/\sigma}, \quad \sigma \neq 1
\]

\[
U(c) = \sum_{t=1}^{T} \delta^{t-1} \ln(c_t), \quad \sigma = 1
\]

Then \( u'(c_t) = c_t^{-1/\sigma} \) and so \( \frac{u'(c_t)}{u'(c_{t+1})} = \left(\frac{c_{t+1}}{c_t}\right)^{1/\sigma} \).

**Growth equation**

**FOC**

\[
\frac{u'(c_t)}{u'(c_{t+1})} = \left(\frac{c_{t+1}}{c_t}\right)^{1/\sigma} = (1 + r)\delta
\]
Therefore

\[ \frac{c_{t+1}}{c_t} = ((1 + r)\delta)^\sigma = \theta. \]

Assumption 2: \( \theta < 1 + r \)

Financial capital growth equation

\[ K_{t+1} = (1 + r)(K_t + y_t - c_t) \]

Hence

\[ \frac{K_2}{1 + r} = K_1 + y_1 - c_1 \]
D. Wealth growth equation

\[ W_i = K_1 + y_1 + \frac{y_2}{1 + r} + \frac{y_3}{(1 + r)^2} + \frac{y_4}{(1 + r)^3} + \ldots. \]

Since \( \frac{K_2}{1 + r} = K_1 + y_1 - c_i \)

\[ W_i - c_i = (K_1 + y_1 - c_1) + \frac{y_2}{1 + r} + \frac{y_3}{(1 + r)^2} + \frac{y_4}{(1 + r)^3} + \ldots. \]

\[ = \frac{K_2}{1 + r} + \frac{y_2}{1 + r} + \frac{y_3}{(1 + r)^2} + \frac{y_4}{(1 + r)^3} + \ldots. \]

Therefore

\[ (1 + r)(W_i - c_i) = K_2 + y_2 + \frac{y_3}{1 + r} + \frac{y_4}{(1 + r)^2} + \ldots. \]

\[ = W_2 \]

The same argument holds for all \( t \). Then the wealth growth equation is

\[ W_{t+1} = (1 + r)(W_t - c_t) \]

Since the constraint functions are all linear they are concave. Since the utility function is concave the Lagrangian is concave and so the necessary conditions are also sufficient for a maximum.
Analysis of possible wealth consumption paths

For the finite horizon problem the consumption growth equation (derived from the FOC) is

\[
\frac{c_{t+1}}{c_t} = \theta .
\]

Therefore

\[
\frac{W_{t+1}}{c_t} = \frac{(1 + r)(W_t - c_t)}{c_t} = (1 + r) \frac{W_t}{c_t} - (1 + r)
\]

\[
= (1 + \frac{r}{\theta}) \frac{W_t}{c_{t-1}} - (1 + r)
\]

Given Assumption 2, the slope of this line is

Greater than 1.

(i) Choose \( \gamma \) and so \( \bar{c}_1 \), so that

\[
\frac{W_{t+1}}{c_t} = \frac{W_t}{c_{t-1}} = \gamma
\]
Then for all t, \( \frac{W_{t+1}}{c_t} = \gamma \)

Since consumption grows at the rate \( \theta - 1 \), so does total wealth.

(ii) Choose \( c_1 < \bar{c}_1 \) so that

\[
\frac{W_2}{c_1} = (1 + r) \frac{W_1 - c_1}{c_1} > \gamma
\]

From the Figure it should be clear that

\( \left\{ \frac{W_{t+1}}{c_t} \right\}_{t=1}^{\infty} \) is a strictly increasing function.

The total wealth grows faster than consumption.

But final wealth must be zero. Thus these path do not satisfy the terminal constraint

\( W_{T+1} = 0 \).

Thus no such path is optimal.
(ii) Choose \( c_1 > \bar{c}_1 \) so that

\[
\frac{W_2}{c_1} = (1 + r) \frac{W_1 - c_1}{c_1} < \gamma
\]

From the Figure it should be clear that

\[
\left\{ \frac{W_{t+1}}{c_t} \right\}_{t=1}^{\infty}
\]

is a strictly decreasing function.
**Solution when** \( T = 4 \)

With no bequest motive no capital will be left

After period \( T \) so \( \frac{W_{T+1}}{c_T} = \frac{W_5}{c_4} = 0 \)

Since this point lies on the purple line,
we can solve for \( \frac{W_4}{c_3} \).

This is also shown on the vertical axis.

(See the left hand marker on the 45° line.)

We can then repeat these steps solving “backwards” as depicted.

Note that the wealth consumption ratio for the last four periods must be exactly the same. Thus with \( T = 6 \) there is simply one more step needed to calculate \( \frac{W_2}{c_1} \).
It follows that as the number of periods increases the wealth consumption ratio in the early periods must be on the dotted 45° line very close to the point \((\gamma, \gamma)\).

Indeed for sufficiently large \(T\) the last few periods have almost no impact the early consumption to wealth ratio. Thus both rise at the same constant rate.
Exercise: Exchange economy

\[ U(x^h) = 5 \ln x_1^h + 5 \ln x_2^h + 4 \ln x_3^h + 4 \ln x_4^h \]

The aggregate endowment is (25,50,16,16).

Normalizing so that \( p_1 = 1 \), show that \( p = (1, \frac{1}{2}, \frac{5}{4}, \frac{5}{4}) \) is a WE price vector.

Class Exercise: Exchange economy with two periods

\[ U = u(c(1)) + \frac{4}{5} u(c(2)) \text{ where } u(c(t)) = \ln c_1(t) + \ln c_2(t) \]

The aggregate first period endowment is \( \omega_1 = (25,50) \). The aggregate second period endowment is \( \omega_2 = (16,16) \). The period 1 price of commodity 1 is 1.

Suppose that in period 1 there are both spot markets (markets for delivery “on the spot”) and futures markets (markets for future delivery of each commodity.)

(a) Explain why the spot price vector is \( p(1) = (1, \frac{1}{2}) \) and the futures price vector is \( p(2) = (\frac{5}{4}, \frac{5}{4}) \)

(b) Suppose that there are no futures markets. If the interest rate is zero what are the equilibrium spot prices and future spot prices?