Walrasian Equilibrium in an exchange economy

Remark: If you prefer you may call a Walrasian Equilibrium a Price-taking Equilibrium

1. Homothetic preferences

Analysis of markets is greatly simplified if we are willing to make two strong assumptions

1. Identical strictly increasing utility functions
2. Utility is homothetic

Definition: Homothetic preferences

Homothetic preferences

Preferences are homothetic if for any consumption bundle x^1 and x^2 preferred to x^1, θx^2 is preferred to θx^1, for all $\theta > 0$.

(Scaling up the consumption bundles does not change the preference ranking).

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Homothetic utility function

A utility function is homothetic if for any pair of consumption bundles x^1 and x^2 ,

$U(x^2) \geq U(x^1)$ implies that $U(\theta x^2) \geq U(\theta x^1)$ for all $\theta > 0$

$U(x^2) = U(x^1)$ implies that $U(\theta x^2) = U(\theta x^1)$ for all $\theta > 0$

$U(x^2) > U(x^1)$ implies that $U(\theta x^2) > U(\theta x^1)$ for all $\theta > 0$

Remark: The second and third statements follow from the first so you only have to check the first.

Slide only for those interested (not covered in the lecture)

Lemma 1: If (1) $U(x^2) \geq U(x^1)$ implies that $U(\theta x^2) \geq U(\theta x^1)$ for all $\theta > 0$

then (2) $U(x^2) = U(x^1)$ implies that $U(\theta x^2) = U(\theta x^1)$ for all $\theta > 0$

Proof: $U(x^2) = U(x^1)$ implies that $U(x^2) \geq U(x^1)$. Appealing to (1), $U(\theta x^2) \geq U(\theta x^1)$ for all $\theta > 0$

$U(x^2) = U(x^1)$ implies that $U(x^1) \geq U(x^2)$. Appealing to (1), $U(\theta x^1) \geq U(\theta x^2)$ for all $\theta > 0$.

Combining these conclusions,

$U(\theta x^1) \geq U(\theta x^2) \geq U(\theta x^1)$ for all $\theta > 0$.

Therefore

$U(\theta x^1) = U(\theta x^2)$.

Lemma 2: If (1) $U(x^2) \geq U(x^1)$ implies that $U(\theta x^2) \geq U(\theta x^1)$ for all $\theta > 0$

then (3) $U(x^2) > U(x^1)$ implies that $U(\theta x^2) > U(\theta x^1)$ for all $\theta > 0$

Sketch of proof: Suppose that $U(x^2) > U(x^1)$ then $U(\theta x^2) \geq U(\theta x^1)$ for all $\theta > 0$

Suppose that for some θ, $U(\theta x^2) = U(\theta x^1)$. Then show that this contradicts Lemma 1.

Proposition: With identical homothetic preferences, market demand is the same as the demand of a single representative consumer with all of the income.

Proof by contradiction:

Let $\bar{x}$ be optimal for a consumer with income 1. i.e.

$\bar{x}$ solves $\underset{x\geq 0}{Max}\{U(x) \mid p\cdot x \leq 1\}$.

Since $I\bar{x}$ costs I it is a feasible consumption bundle

for a consumer with income I.

Suppose that the bundle is not optimal. Then

$\hat{x}$ solves $\underset{x\geq 0}{Max}\{U(x) \mid p\cdot x \leq I\}$ and $U(\hat{x}) > U(I\bar{x})$

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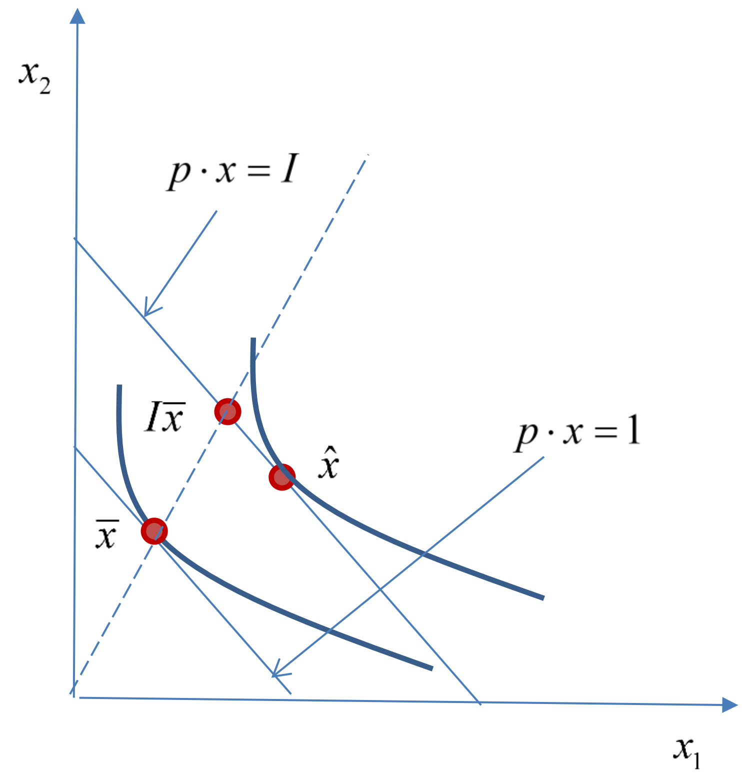

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By homotheticity, it follows that

$U(\theta\hat{x})>U(\theta I\bar{x})$ for all θ .

Setting $\theta=\dfrac{1}{I}$, $U(\frac{1}{\mathrm{I}}\hat{x})>U(\bar{x})$

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Setting $\theta = \frac{1}{I}$, $U(\frac{1}{\mathrm{I}}\hat{x}) > U(\bar{x})$

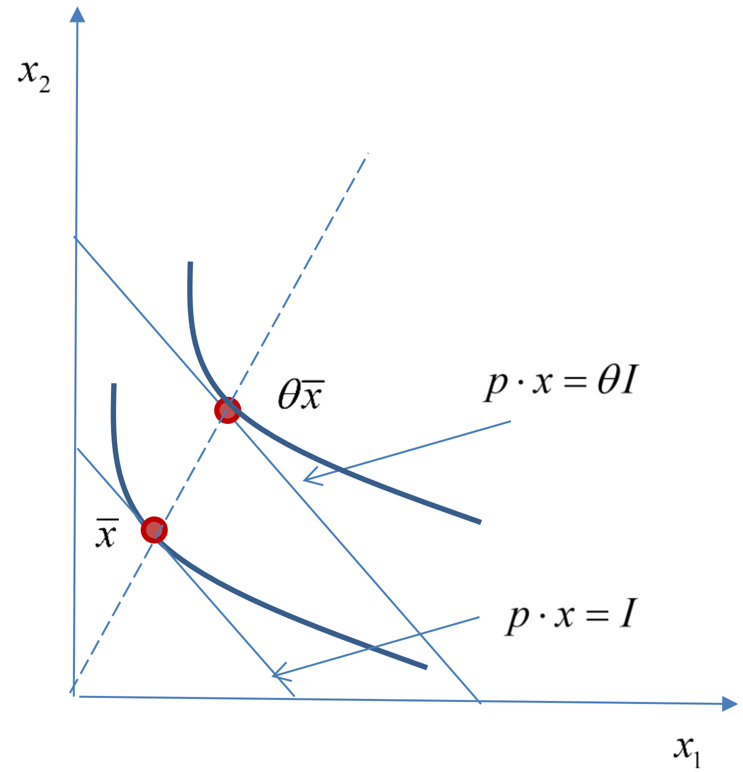
Since $\frac{1}{I}\hat{x}$ costs 1, it is a feasible consumption bundle for a consumer with income 1.

But then $\bar{x}$ is not optimal for the consumer with income 1, contradicting (*)

Homothetic preferences

For any $\bar{x} \gg 0$ and any $\theta > 0$ $MRS(\theta\bar{x}) = MRS(\bar{x})$

Why?



Examples of homothetic utility functions

(i) $U(x) = a_1 x_1 + a_2 x_2 = a \cdot x \text{ , } a >> 0$

(ii) $U(x) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \ \alpha >> 0$

(iii) $U(x) = (x_1^{1/2} + x_2^{1/2})^2$

(iv) $U(x) = -\frac{1}{x_1} - \frac{2}{x_2} - \frac{3}{x_3}$

(v) $U(x) = x_1^2 + x_2^2$

Definition: Market demand

If $x^h(p,I^h),\ h=1,...,H$ uniquely solves $\underset{x\geq 0}{Max}\{U^h(x)\,|\,p\cdot x\leq I^h\}$, then the market demand for H consumers with incomes $I^1,...,I^H$ is

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Consider a 2 consumer economy with incomes I^1 and I^2 .

Proposition: Market demand in a 2 person economy with identical homothetic preferences.

$$x(p,I^1)+x(p,I^2)=x(p,I^1+I^2)$$

Proof:

If $x(p,I)$ is the demand for a consumer with income I then $x(p,I^h)=I^h x(p,1)$ and so

$$x(p,I^1)+x(p,I^2)=I^1x(p,1)+I^2(x(p,1)=(I^1+I^2)x(p,1)$$

Also

$$x(p,I^1+I^2)=(I^1+I^2)x(p,1).$$

Corollary: Representative consumer

Suppose that consumers have identical strictly increasing homothetic preferences and that

$$\bar{x} \text{ solves } \underset{x \geq 0}{Max}\{U(x) \mid p \cdot x \leq I = \sum_{h=1}^{H} I^h\}$$

Then $\bar{x}$ is a market demand.

Proof: Follows almost immediately from the proposition

B. Walrasian equilibrium (WE) in an exchange economy

In a WE consumer h knows his own endowment and preferences but knows nothing about the economy except the vector of prices. Consumer h then solves for the set of Walrasian (utility maximizing) demands $x^h(p,\omega^h)$.

The price vector is a WE price vector if there is some WE demand $\bar{x}^h \in x^h(p,\omega^h)$, $h=1,...,H$ such that the sum of these demands (the market demand) is equal to the total endowment.

Walrasian equilibrium (WE) in an exchange economy with identical homothetic preferences

Consider the representative consumer with endowment $\omega = \sum_{h=1}^{H} \omega^h$. We assume $\omega \gg 0$.

Let $\bar{x}$ be a demand of the representative consumer. Then $\bar{x}$ solves $\underset{x}{Max}\{U(x) \mid p \cdot x \leq p \cdot \omega \equiv I\}$

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FOC for a maximum.

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1}(x) = \ldots = \frac{1}{p_n}\frac{\partial U}{\partial x_n}(x)$$

For p to be a WE price vector markets must clear. With only one consumer, $\bar{x} = \omega$.

Therefore the WE prices satisfy

$$\frac{1}{p_1}\frac{\partial U}{\partial x_1}(\omega) = \ldots = \frac{1}{p_n}\frac{\partial U}{\partial x_n}(\omega) \ .$$

Note that this only determines relative prices (i.e. price ratios.)

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Above we argued that if consumer h has an endowment of value $p \cdot \omega^h = I^h$ then

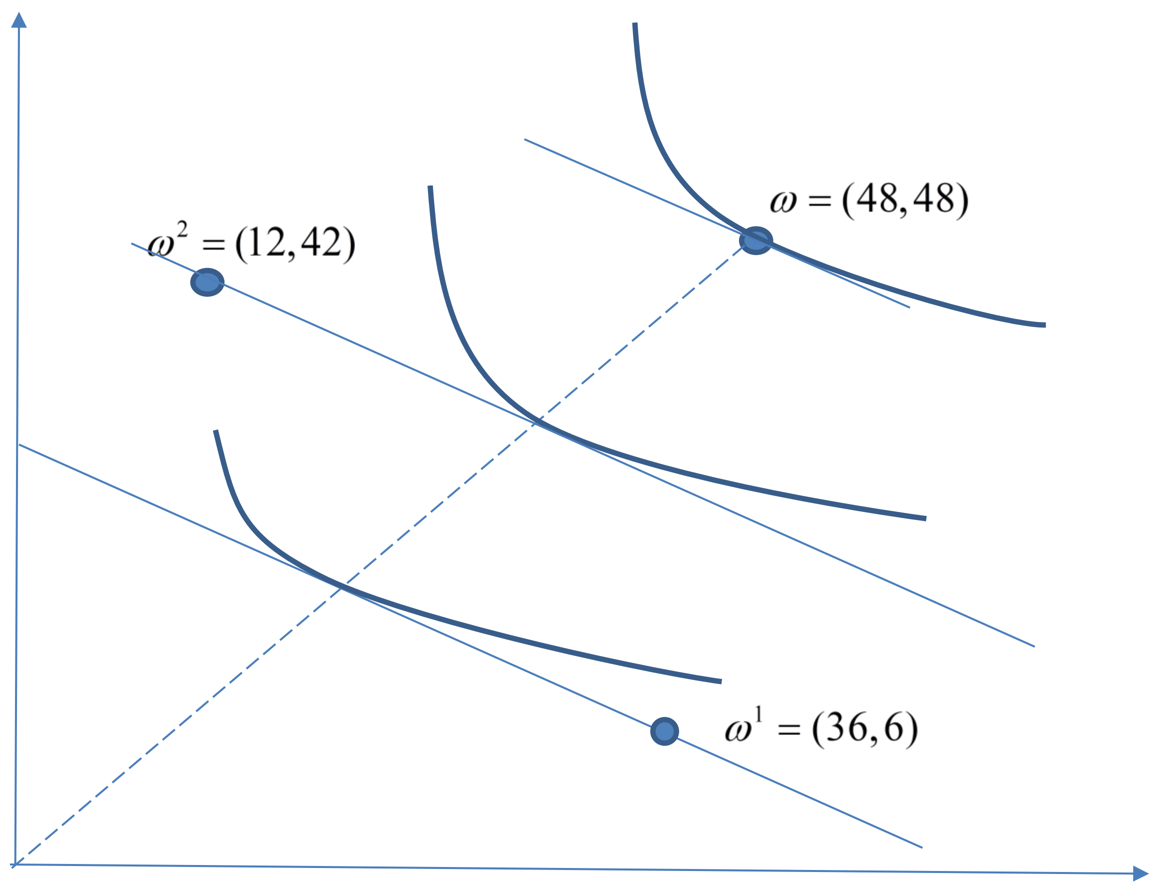
$\bar{x}^h = \frac{I^h}{I}\bar{x} = \frac{I^h}{I}\omega$, where I is the sum of all the incomes $I = I^1 + ... + I^H$

is a WE demand.

Therefore in the WE of the homothetic economy, consumer h consumes a fraction $\frac{I^h}{I}$ of the aggregate endowment.

Example: $U^h(x^h) = \ln x_1^h + 2\ln x_2^h \quad \omega^1 = (36,6) \;\; \omega^2 = (12,42)$

Exercise: Use the representative consumer to show that $p = (\frac{1}{3}, \frac{2}{3})$ is the unique WE price vector normalized so that the sum of the prices is 1.



We know that if income

Goes up by a factor of θ

Then so does consumption.

The value of consumer 1's

endowment is 48 and the

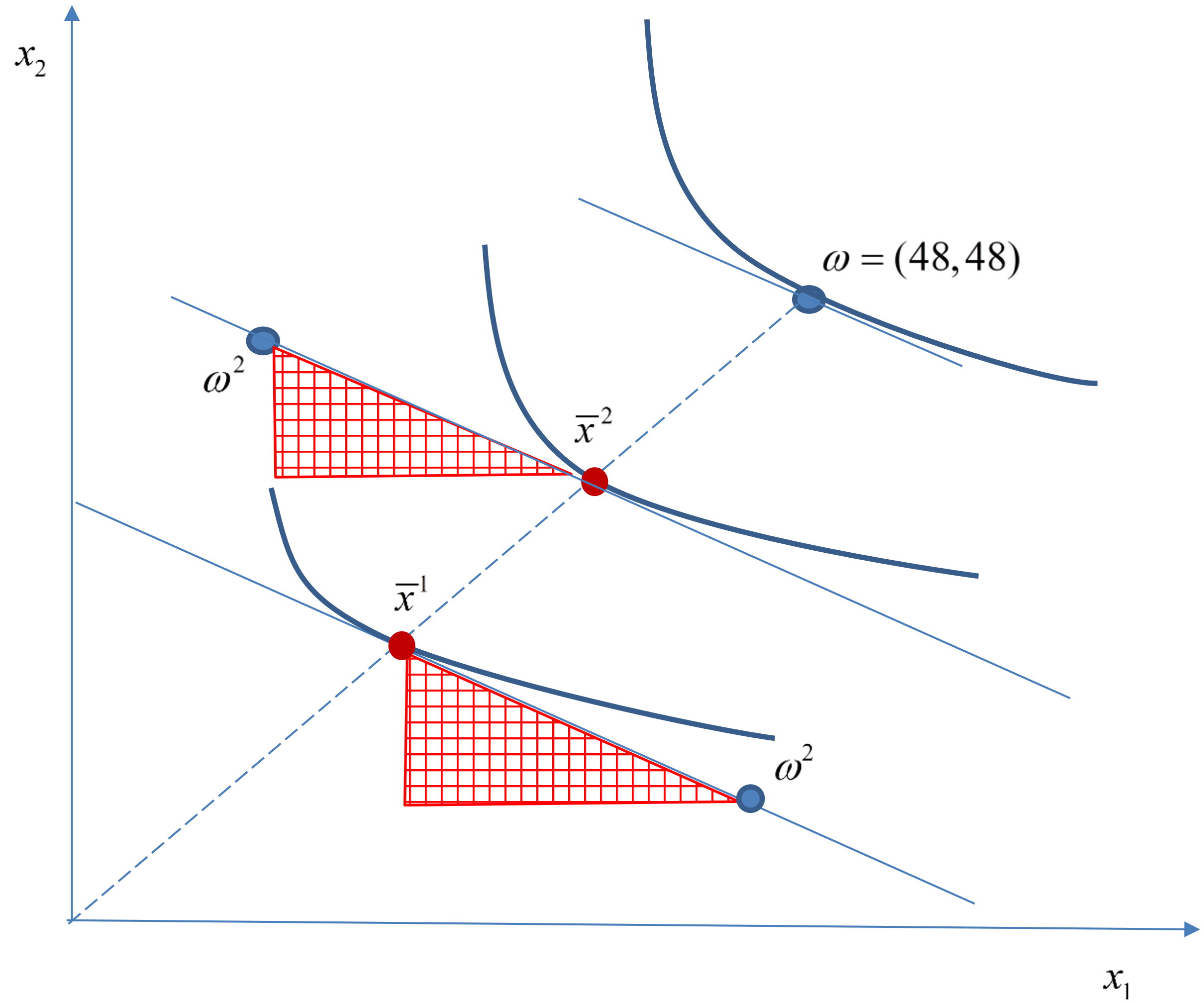
value of consumer 2's

endowment is 96 so they

consume respectively

1/3 and 2/3 of the aggregate

Endowment.



The trade triangles are depicted in the figure.

C. The market value of attributes

In studying industries like the airline industry economist often try to determine the implicit value of different attributes (for example, air travel: leg-room, percentage on-time arrival etc.)

We now consider a simple example to illustrate.

Each unit of commodity 1 and commodity 2 has different amounts of two attributes. For example, carbohydrates and protein. Quantities per unit are in grams.

(attribute A and B)

	commodity 1	commodity 2	total attribute endowment
Attribute A (carbohydrates)	2	1	100
Attribute B (protein)	1	3	100
Total commodity endowment	40	20	

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A consumer cares about the quantity of each attribute consumed. Let $(x_1, x_2, x_3, ...)$ be the consumption choice

$$a = 2x_1 + 1x_2 \text{ , } b = 1x_1 + 3x_2$$

$$U^h = U^h(a, b, x_3, ..., x_n) = \ln a + \ln b + \alpha_3 \ln x_3 .....$$

$$= \ln(2x_1 + x_2) + \ln(x_1 + 3x_2) + \alpha_3 \ln x_3 + ...$$

To keep the model simple we assume that every consumer has the same log utility function.

Exercise: Is the log utility function homothetic?

Exercise: Show that the WE price ratio for the first two commodities must be $\frac{p_2}{p_1} = \frac{4}{3}$ **.**

An alternative approach

Imagine a market for attributes. What would be the market clearing prices of each attribute?

	commodity 1	commodity 2	Total endowment of each attribute
Attribute A	2	1	$2 \times 40 + 1 \times 20 = 100$
Attribute B	1	3	$1 \times 40 + 3 \times 20 = 100$
Total commodity endowment	40	20	

Let $\lambda = (\lambda_a, \lambda_b)$ be the shadow (implicit) price vector for the two attributes.

Exercise: (a) Show that $\frac{\lambda_b}{\lambda_a} = 1$.

(b) Using these attribute prices, what is the value of each commodity?

Group Exercise

Each unit of commodity 1, 2 and 3 (flights on different airlines) have different amounts of two attributes

(attribute A and B)

	commodity 1	commodity 2	commodity 3
Attribute A	2	1	5
Attribute B	1	3	5
Total endowment	40	20	10

A consumer cares about the quantity of each attribute consumed.

$a = 2x_1 + 1x_2 + 5x_3$, $b = 1x_1 + 3x_2 + 5x_3$

$$U^h = U^h(a, b, x_3, ..., x_n) = \ln a + \ln b + ...$$

$$= \ln(2x_1 + x_2 + 5x_3) + \ln(x_1 + 3x_2 + 5x_3) + \alpha_4 \ln x_4 + ...$$

Left-hand groups: Solve for the equilibrium prices directly

Right-hand groups: Solve for the shadow prices of each attribute, (p_a, p_b) .

Solution for right hand groups

This is an economy with identical homothetic utility sop consider the representative consumer.
$U^R = U(a,b,x_3,...,x_n) = \ln a + \ln b + \alpha_4 \ln x_4 + ...$

The total endowments of attributes are as follows:

$\omega_a = 2*20 + 1*40 + 5*10 = 150$, $\omega_a = 3*20 + 1*40 + 5*10 = 150$.

Let p_a and p_b be the attribute prices. The budget constraint is then

$p_a a + p_b b + ... = I^h$.

The Necessary conditions can be written as follows:

$\frac{\partial U}{\partial a}(\bar{a},\bar{b}....) = \mu p_a$ and $\frac{\partial U}{\partial b}(\bar{a},\bar{b},...) = \mu p_b$, where μ is the shadow price.

The equilibrium demand for the representative consumer $(\bar{a},\bar{b})$ must equal supply (ω_a,ω_b).

Therefore

$$(\frac{\partial U}{\partial a}(\omega), \frac{\partial U}{\partial a}(\omega)) = \lambda(p_a, p_b)$$

i.e . $$(\frac{1}{150}, \frac{1}{150} = \lambda(p_a, p_b)$$

It follows that the shadow prices are equal.

Choose $(p_a, p_b) = (1,1)$.

Each unit of commodity 1 contains 2 units of attribute a and 1 unit of attribute b so the market value of this unit is $p_a(2) + p_b(1) = 3$. Similarly, the market value of a unit of commodity 2 is 4.

If you solve the left-side problem using these values as the prices you will find that demand is equal to supply in the commodity market.

Note that if you choose attribute prices of $(2,2)$, the equilibrium commodity prices must double as well. Only relative prices matter in micro models.