What is a Walrasian Equilibrium (WE)

1. Model of a private ownership economy

Commodities:

The set of commodities is $\mathcal{N} = \{1,...,n\}$

Endowments:

$\omega_j \geq 0$ the initial endowment of commodity j (land, labor, coconuts…)

Consumers:

The set of consumers is $\mathcal{H} = \{1,...,H\}$.

Each consumer's preferences can be represented by a continuously differentiable strictly increasing function $U(x^h),\ h=1,...,H$ where $x^h = (x_1^h,...,x_n^h)$

Notation: If $\overline{x}_i \geq x_i$ for $i=1,...,n$ and the inequality is strict for some i we write $\overline{x} > x$. If it is strict for all i we write $\overline{x}_i >> x_i$.

The function $U^h(x)$ is strictly increasing if $\overline{x} > x$ implies that $U^h(\overline{x}) > U^h(x)$.

Firms

The set of firms is $\mathcal{F} = \{1,...,F\}$.

Transformers of inputs into outputs:

Let $z^f = (z_1^f,...,z_n^f)$ be a vector of firm f 's inputs.

Each component is a quantity of one of the commodities.

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Production vector

$y^f \equiv (z^f, q^f)$ is an ordered list of all the inputs and outputs.

Each firm must choose among the production vectors that are feasible.

We write this set of feasible plans as Y^f .

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Private ownership:

Shareholdings: k^{hf} is the share-holding of consumer h in firm f .

Feasible consumption vectors

Sum over all the consumers

$$\sum_{h=1}^{h} x_j^h \leq \sum_{h=1}^{h} \omega_j^h - \sum_{f=1}^{F} z_j^f + \sum_{f=1}^{F} q_j^f \text{ ,} \qquad \text{where } (z^f, q^f) \in Y^f \qquad \text{(production plans are feasible)}$$

total consumption $\leq$ total endowment - total inputs + total outputs

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Price taking equilibrium allocation

An allocation to consumers, $\{\bar{x}^h\}_{h=1}^H$, feasible production plans $\{\bar{y}^f\}_{f=1}^F$ and a price vector $p \geq 0$

Such that

(i) No consumer has a strictly preferred point in his/her budget set.

(ii) There is no strictly more profitable feasible plan for any firm.

(iii) All markets clear.

$$\sum_{h=1}^{h} \bar{x}_j^h \leq \sum_{h=1}^{h} \omega_j^h - \sum_{f=1}^{F} \bar{z}_j^f + \sum_{f=1}^{F} \bar{q}_j^f \text{ and } p_j = 0 \text{ if the inequality is strict.}$$

We will look at simple economies to gain insights into how commodities are allocated via markets.

Example 1:

Alex and Bev only like bananas.

Alex has an endowment of 4 bananas and 6 coconuts $\omega^A = (4,6)$.

Bev has an endowment of 10 bananas and 7 coconuts $\omega^A = (10,7)$.

Class question: What vectors of prices (p_b, p_c) are equilibrium prices?

Example 2:

Alex likes only bananas. Bev likes only coconuts.

Alex has an endowment of 4 bananas and 6 coconuts $\omega^A = (4,6)$.

Bev has an endowment of 10 bananas and 7 coconuts $\omega^A = (10,7)$.

Group exercise:

Right-side groups: Study supply and demand in the market for coconuts.

Left–side groups: Study supply and demand in the market for coconuts.

Let the prices be $p = (p_b, p_c)$

Example 3: Cobb-Douglas utility function: $\underset{x\geq 0}{Max}\{U(x)=x_1^{\alpha_1}x_2^{\alpha_2} \mid p_1x_1+p_2x_2\leq I\}$

Necessary conditions for a maximum

Method 1: Equalize the marginal utility per dollar

To make differentiation simple, try to find an increasing function of the utility function that is simple.

Define the new utility function $u(x)=\ln U(x)$

The new maximization problem is

$$\underset{x\geq 0}{Max}\{u(x)=\ln U(x) \mid p_1x_1+p_2x_2\leq I\}$$

That is

$$\underset{x\geq 0}{Max}\{\alpha_1\ln x_1+\alpha_2\ln x_2 \mid p_1x_1+p_2x_2\leq I\}$$

Note that

$$\frac{\partial u}{\partial x_j}=\frac{\alpha_j}{x_j}\ .$$

Necessary conditions

$$\frac{1}{p_1}\frac{\partial u}{\partial x_1} = \frac{1}{p_2}\frac{\partial u}{\partial x_2} = \lambda$$. For the example it follows that $\frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \lambda$.

Also $p_1 x_1 + p_2 x_2 = I$.

Technical tip

Ratio Rule:

If $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ and $b_1 + b_2 \neq 0$ then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}$.

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Therefore

$$\frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \frac{\alpha_1 + \alpha_2}{p_1 x_1 + p_2 x_2} = \frac{\alpha_1 + \alpha_2}{I}$$

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Also $p_1 x_1 + p_2 x_2 = I$.

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$$\frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \frac{\alpha_1 + \alpha_2}{p_1 x_1 + p_2 x_2} = \frac{\alpha_1 + \alpha_2}{I}$$

We can then solve for x_1 and x_2

Cobb-Douglas demands

$$x_j = \frac{\alpha_j}{\alpha_1 + \alpha_2}\frac{I}{p_j} ,$$

Method 2: Equate the MRS and price ratio

$U(x) = x_1^{\alpha_1} x_2^{\alpha_2}$. Then $\dfrac{\partial U}{\partial x_1} = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}$ and $\dfrac{\partial U}{\partial x_2} = \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1}$

$$MRS(\bar{x}_1, \bar{x}_2) = \frac{\dfrac{\partial U}{\partial x_1}}{\dfrac{\partial U}{\partial x_2}} = \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2}}{\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1}} = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1} .$$

Then to be the maximizer,

$$MRS(\bar{x}_1, \bar{x}_2) = \frac{\alpha_1}{\alpha_2} \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

As we have seen, it is helpful to rewrite this as follows:

$$\frac{p_1 x_1}{\alpha_1} = \frac{p_2 x_2}{\alpha_2} .$$

Then proceed as before.

Data Analytics (Taking the model to the data)

$$x_1(p,I)=\frac{\alpha_1}{\alpha_1+\alpha_2}\frac{I}{p_j}.$$

Take the logarithm

$$\ln x_j = \ln(\frac{\alpha_1}{\alpha_1+\alpha_2})+\ln I-\ln p_j$$

The model is now linear. We can then use least squares estimation

$$\ln x_j = a_0 + a_1(\ln I - \ln p_j)$$

or

$$\ln x_j = a_0 + a_1 \ln I + a_2 \ln p_j$$

Exercise: If $U(x)=(a_1+x_1)^{\alpha_1}(a_2+x_2)^{\alpha_2}$, solve for the demand function $x_1(p,I)$

Market demand in a Cobb-Douglas economy with no production

Consumer $h \in \mathcal{H}$ has some initial endowment $\omega = (\omega_1, ..., \omega_n)$. With a price vector p, the market value of this endowment is $I^h = p \cdot \omega^h$. If the consumer sells his entire endowment he can then purchase any consumption bundle x^h satisfying

$$p \cdot x^h \leq I^h = p \cdot \omega^h .$$

Cobb-Douglas demands

$$x_1^h = \frac{\alpha_1^h}{\alpha_1^h + \alpha_2^h} I^h = \gamma^h I^h \text{ where } \gamma^h = \frac{\alpha_1^h}{\alpha_1^h + \alpha_2^h}$$

$$= \gamma^h (p_1 \omega_1^h + p_2 \omega_2^h)$$

Therefore

$$x_1^h - \omega_1^h = \gamma^h (p_1 \omega_1^h + p_2 \omega_2^h) - \omega_1^h = \gamma^h p_2 \omega_2^h - (1 - \gamma^h) p_1 \omega_1^h$$

Consider the two consumer case (Alex and Bev)

$$x_1^A - \omega_1^A = \gamma^A p_2 \omega_2^A - (1-\gamma^A) p_1 \omega_1^A$$

$$x_1^B - \omega_1^B = \gamma^B p_2 \omega_2^B - (1-\gamma^B) p_1 \omega_1^B$$

Summing over consumers

$$\sum x_1^h - \sum \omega_1^h = p_2 \sum \gamma^h \omega_2^h - p_1 \sum (1-\gamma^h)\omega_1^h$$

Market clears if

$$0 = \sum x_1^h - \sum \omega_1^h = p_2 \sum \gamma^h \omega_2^h - p_1 \sum (1-\gamma^h)\omega_1^h$$

Hence the WE price ratio is

$$\frac{p_1}{p_2} = \frac{\sum \gamma^h \omega_2^h}{\sum (1-\gamma^h)\omega_1^h} .$$