2

Consumers

2.1 Theory of Choice

Key ideas: ordering consumption bundles, axioms of choice, utility representation of preferences

Anyone who has taken more than an introductory class in economics has become thoroughly familiar with the idea that consumers maximize "utility." In mathematical terms, preferences are represented by the contour sets of a function U defined over some set of consumption bundles, X. Take any consumption bundle x^0 in this set. The upper contour set of this function $\{x|U(x) \geq U(x^0), x \in X\}$ is the set of consumption vectors for which the consumer would be willing to trade away x^0. Equivalently, the upper contour set is the set of bundles that the consumer weakly prefers to x^0.

At heart, a theory of choice is a theory of how alternatives are ordered. Thus a more fundamental theory of choice begins with ordering axioms. Formally, for consumer (or household) h we define the binary relation $\underset{h}{\succsim}$. If the consumer orders bundle x^1 at least as highly as bundle x^2 we write $x^1 \underset{h}{\succsim} x^2$.

In general orders are not defined over all consumption vectors. Consider for example the weak inequality order ($\geq$). That is, a consumer prefers x^1 over x^2 if and only if $x^1 \geq x^2$. In this case the probability that any two randomly chosen bundles of dimension n can be ranked is $1/2^{n-1}$ so such an order is clearly unsatisfactory. For a strong theory we require that a consumer can compare all bundles in the consumption set X.

Order is Complete[1] For any consumption bundles $x^1, x^2 \in X$, either $x^1 \underset{i}{\succsim} x^2$ or $x^2 \underset{h}{\succsim} x^1$.

Next we require that there is a consistency across pair-wise rankings.

[1] Note that this includes the statement that $x \underset{h}{\succsim} x$. Mathematicians call such an ordering reflexive.

Order Is Transitive For any bundles $x^1, x^2, x^3 \in X$, if $x^1 \underset{h}{\succsim} x^2$ and $x^2 \underset{h}{\succsim} x^3$ then $x^1 \underset{h}{\succsim} x^3$.

Given the first axiom, consider any two consumption bundles x^1 and x^2. We can distinguish three cases:

(i) $x^1 \underset{h}{\succsim} x^2$ and $x^2 \underset{h}{\succsim} x^1$ (ii) $x^1 \underset{h}{\succsim} x^2$ and $x^2 \underset{h}{\not\succsim} x^1$ (iii) $x^2 \underset{h}{\succsim} x^1$ and $x^1 \underset{h}{\not\succsim} x^2$.

In case (i) where both bundles are ordered weakly ahead of the other we say that the consumer is indifferent between the two bundles and write $x^1 \underset{h}{\sim} x^2$. In case (ii) where the weak ordering goes one way but not the other, we say that x^1 is strictly preferred to x^2 and write $x^1 \underset{h}{\succ} x^2$. Case (iii) is the reverse of case (ii) so $x^2 \underset{h}{\succ} x^1$.

Note that both the indifference order and the strict preference order satisfy the transitivity axiom. However neither is a total order.

We now argue that these two axioms taken together are not enough. Consider the following example.

Example: The "Not-Less-Than" Order

$x^1 \underset{h}{\succsim} x^2$ if and only if $x^1 \not< x^2$.

This order is both complete and transitive. Consider the two consumption bundles x^a and x^b on the horizontal line in Figure 2.1-1. We first argue that $x^a \underset{h}{\succ} x^b$. For if not, then $x^b \underset{h}{\succsim} x^a$ and so $x^b \not< x^a$. But this is false. Next consider any consumption bundle y^t in the interior of the shaded region. Because $y_2^t > x_2^a$, it follows that $y^t \underset{h}{\succsim} x^a$. Since $x_1^a > y_1^t$, it also follows that $x^a \underset{h}{\succsim} y^t$. Then $y^t \underset{h}{\sim} x^a$.

To see why there is something rather unsatisfactory about such preferences, consider the sequence of consumption bundles $\{y^t\}_{t=1}^{\infty}$ in the interior of the shaded region that converges to x^b. For each y^t in the sequence

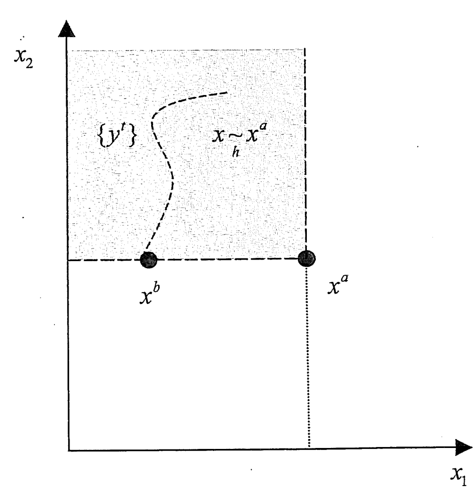

Figure 2.1-1. Discontinuous preference ordering.

$y^t \underset{h}{\sim} x^a$. However, for the limiting consumption x^b, we have $x^a \underset{h}{\succ} x^b$. Suppose there were a utility representation of these preferences. Then for all y^t in the sequence approaching x^b, $U(y^t) = U(x^a) > U(x^b)$. Thus the utility function could not be continuous.

To rule such oddities out we introduce a third axiom.

Order Is Continuous Let x^0 be the limit point of a sequence of consumption bundles $\{x^t\}_{t=1,2,\ldots}$.

$$\text{If for all } t, x^t \underset{i}{\succsim} y \text{ then } x^0 \underset{i}{\succsim} y. \text{ If for all } t, y \underset{i}{\succsim} x^t \text{ then } y \underset{i}{\succsim} x^0.$$

These postulates are entirely uncontroversial for frequent purchases. Suppose shoppers are evaluating two large shopping baskets full of goods. Although they may hesitate for a while, surely each shopper would eventually choose between them. And for anyone who strictly prefers the first shopping basket, making a tiny change in items in the basket should not change the ranking.

However, the supermarket is a familiar place. A customer has likely walked the aisles dozens of times and has a good idea about the vast majority of products available for purchase. Yet take that same person into an open-air market in a faraway country and fill up two shopping carts with exotic produce. Then the consumer is likely to be much less confident about making a choice. The postulates are therefore most reasonable when the choices are between familiar commodities.

Further complicating the selection of the shopping basket is the reality that the current choice is not independent of prior choices or of choices planned in the future. Major decisions like choosing a job, or where to live, or family size, or how much to invest for retirement depend very much on future consumption plans. Suppose that our consumer will live for T periods. Let $x_t = (x_{1t}, \ldots, x_{nt})$ be her period t consumption bundle. Then she must rank each $T \times n$ dimensional consumption bundle $x = (x_{11}, \ldots, x_{nT})$. The higher dimensionality of the choice problem adds to its complexity. Moreover, just as in the open-air market in a faraway country, there is considerable uncertainty about the complete consumption bundle.

As long as the consumer is able to assign probabilities to different outcomes then this uncertainty can be incorporated as well. To illustrate, suppose that there are just two periods and the consumer's health will be either good or bad in the next period. Let (x_1, x_2^g) be consumption over the two periods if her health is good and let (x_1, x_2^b) be consumption if her health is bad. (In the extreme, bad health might result in her death before she can enjoy any second-period consumption.) Let π^g be the probability that her health will be good and let π^b be the probability that her health will be bad. Then we can characterize the consumer's consumption plan as a "prospect" $(x_1, x_2^g, x_2^b; \pi^g, \pi^b)$. The preference ranking now involves different prospects.

Much of the controversy about this extended consumer choice model hinges on whether or not consumers assess probabilities consistently.

In later chapters we consider choice under uncertainty, but initially we focus on a world of full information. The ordering postulates alone say nothing about the desirability of commodities. Note, in particular, that an individual who is indifferent between all consumption bundles satisfies all the postulates. To incorporate unbounded "wants" into the choice model, we add the assumption that whatever bundle a consumer receives, there is always another similar bundle that is strictly preferred.

Local Non-Satiation For any consumption bundle $x \in X \subset \mathbb{R}^n$ and any $\delta-$ neighborhood $N(x, \delta)$ of x, there is some bundle $y \in N(x, \delta)$ such that $y \underset{h}{\succ} x$.

Especially when analyzing consumption of commodity groups (food, clothing, housing, etc.) it is natural to make the stronger assumption that more is always strictly preferred.

Strict Monotonicity If $y > x$ then $y \underset{h}{\succ} x$.

The final postulate reflects the idea that individuals have a preference for variety.

Preferences Are Convex Let X be a convex subset of $\mathbb{R}^n$. Preferences are convex on X if, for any consumption bundle $y \in X$, the set of bundles (weakly) preferred to y is convex. That is, for any $x^0, x^1 \in X$, if $x^0 \underset{h}{\succsim} y$ and $x^1 \underset{h}{\succsim} y$ then the convex combination $x^\lambda = (1-\lambda)x^0 + \lambda x^1 \underset{h}{\succsim} y, 0 < \lambda < 1$.

Two examples of convex preferences are illustrated in Figure 2.1-2. The shaded set in each diagram depicts the sets of bundles preferred over y, holding constant the consumption of all the other commodities. The first case

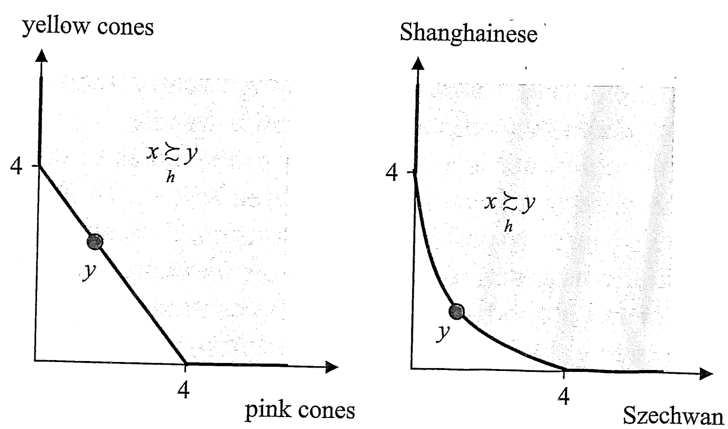

Figure 2.1-2. Convex preferences.

shows the limiting case where there is no strict preference for diversity. The consumer is indifferent between four pink cones of ice-cream per week and four yellow cones of ice-cream or any convex combination. In the second case the individual is indifferent between eating out an average of four times a month at a Szechwan restaurant and four times a month at a Shanghainese restaurant but strictly prefers convex combinations of these two bundles.

If this strict preference for diversity holds for all bundles we can strengthen the convexity axiom as follows.

Preferences Are Strictly Convex Suppose that x^0, x^1 and $y \in X$, where X is a convex subset of $\mathbb{R}^n$.

$$\text{If } x^0 \underset{h}{\succsim} y \quad \text{and} \quad x^1 \underset{h}{\succsim} y \quad \text{then} \quad x^\lambda \underset{h}{\succ} y, \quad 0 < \lambda < 1.$$

These basic postulates are a sufficiently strong foundation for a very general theory of market equilibrium. However for the study of specific markets, we typically add further structure to the preference rankings of consumers. To do so we give a consumer a ranking function (or utility function) $U(x)$ and impose restrictions on the nature of this function. The following theorem establishes that there is no inconsistency between this approach and the general axiomatic approach.

Proposition 2.1-1: Utility Function Representation of Preferences
If preferences are complete, reflexive, transitive and continuous, on the consumption set $X \subset \mathbb{R}^n$, they can be represented by a function $U(x)$ that is continuous over X.[2]

Although we do not provide a general proof, it is reasonably easy to see why this proposition must be true if preferences are strictly monotonic. Pick any consumption bundles x^0 and $x^1 \in X$ such that $x^1 > x^0$. If preferences are strictly monotonic, then $x^1 \underset{h}{\succ} x^0$. Consider all the consumption bundles in the set $T = \{x \in X | x^1 \underset{h}{\succsim} x \underset{h}{\succsim} x^0\}$. Define the convex combination $x^\lambda = (1-\lambda)x^0 + \lambda x^1$. We will argue that for any $y \in T$ there must be some weighting factor λ such that $y \underset{h}{\sim} x^\lambda$ and that the mapping $\lambda(y)$ from T onto the unit interval must be continuous.

The intuition is straightforward. Consider Figure 2.1-3. Along the line joining x^0 and x^1, $x^\lambda = x^0 + \lambda(x^1 - x^0)$ is a strictly increasing function of λ, because $x^1 > x^0$. Thus $x^\lambda \underset{h}{\succ} x^\mu$ if and only if $\lambda > \mu$. Because preferences are continuous and $x^1 \underset{h}{\succsim} y \underset{h}{\succsim} x^0$ it is intuitively clear that there must be some weight $\hat{\lambda}$ such that $y \underset{h}{\sim} x^{\hat{\lambda}}$. To find this weight we can follow a step-by-step approach. First, if either $y \underset{h}{\sim} x^0$ or $y \underset{h}{\sim} x^1$ then we are finished. If not,

[2] The converse is also true. See Exercise 2.1-7.

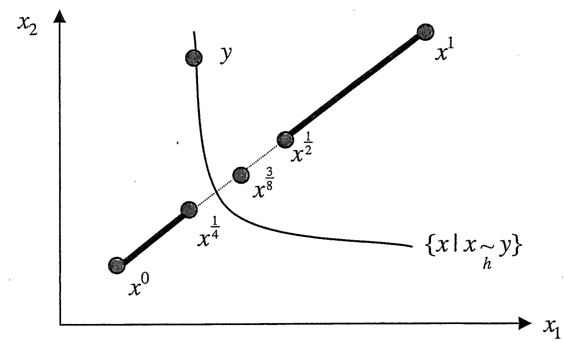

Figure 2.1-3. Constructing a utility function.

consider the convex combination $x^{\frac{1}{2}}$. If $y \underset{h}{\sim} x^{\frac{1}{2}}$, then we are done. If, as in Figure 2.1-3, $x^{\frac{1}{2}} \underset{h}{\succ} y$ then we can rule out any $\lambda \geq \frac{1}{2}$.

Now split the remaining interval $[0, \frac{1}{2}]$ and try $\lambda = \frac{1}{4}$. The figure shows $y \underset{h}{\succ} x^{\frac{1}{4}}$ so we can rule out any weight $\lambda \leq \frac{1}{4}$. Again split the remaining interval $[\frac{1}{4}, \frac{1}{2}]$ and try $\lambda = \frac{3}{8}$. Continuing in this way, either there is some step at which we find a weight $\lambda(y)$ such that $y \underset{h}{\sim} x^{\lambda(y)}$, or the weights converge to some limit point $\lambda(y)$.

We now formalize this intuition to prove the existence of such a utility function.[3] Define decreasing and increasing sequences $\{\nu_t\}_{t=0,1,\ldots}$ and $\{\mu_t\}_{t=0,1,\ldots}$ as follows:

$$\lambda_{t+1} = \frac{1}{2}(\nu_t + \mu_t) \quad \text{where} \quad (\nu_0, \mu_0) = (1, 0).$$

From this equation it follows that $\lambda_1 = \frac{1}{2}$. At step t, if $y \underset{h}{\sim} x^{\lambda_t}$ we are done. If $x^{\lambda_t} \underset{i}{\succ} y$, then define $(\nu_{t+1}, \mu_{t+1}) = (\lambda_{t+1}, \mu_t)$. If $y \underset{i}{\succ} x^{\lambda_t}$, then define $(\nu_{t+1}, \mu_{t+1}) = (\nu_t, \lambda_{t+1})$.

The first three steps of this process are depicted in Figure 2.1-4 for the preferences shown in Figure 2.1-3. Note that the sequences are constructed so that, at each step,

$$x^{\nu_t} \underset{h}{\succsim} x \underset{h}{\succsim} x^{\mu_t}.$$

If the process does not stop, we define $\hat{\lambda}$ to be the limit point of the decreasing sequence $\{\nu_t\}_{t=0}^{\infty}$. At each step the difference between the decreasing and increasing sequences is halved. Thus $\hat{\lambda}$ must also be the limit point of the increasing sequence $\{\mu_t\}_{t=0}^{\infty}$. Because for all t, $y \underset{h}{\succ} x^{\mu_t}$ it follows from the continuity of preferences that $y \underset{h}{\succsim} x^{\hat{\lambda}}$. Moreover, because for all t, $x^{\nu_t} \succ y$. It follows from the continuity axiom that $x^{\hat{\lambda}} \underset{h}{\succsim} y$ and so $x^{\hat{\lambda}} \underset{h}{\sim} y$.

[3] See Exercise 2.1-2 for an outline of how to prove continuity.

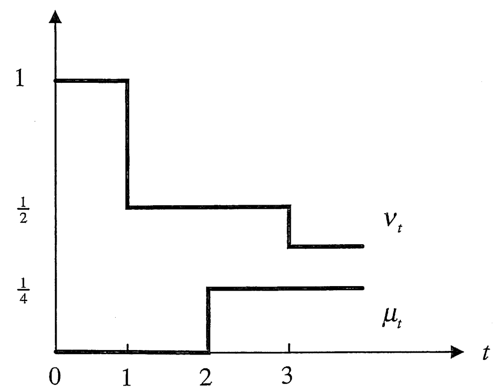

Figure 2.1-4. Monotonic sequences.

We now show that the assumption that preferences are convex is equivalent to the assumption that the utility representation of these preferences is quasi-concave.

Definition: Quasi-Concave Function[4] U is quasi-concave on X if, for any $x^0, x^1 \in X$ and convex combination $x^\lambda = (1-\lambda)x^0 + \lambda x^1, 0 < \lambda < 1$, $U(x^\lambda) \geq \text{Min}\{U(x^0), U(x^1)\}$.

Suppose that preferences are convex on X and can be represented by a utility function U. For any $x^0, x^1 \in X$, either (i) $x^1 \succsim x^0$ or (ii) $x^0 \succ x^1$. If (i) holds then $U(x^1) \geq U(x^0)$. Because preferences are convex, for any convex combination x^λ, $U(x^\lambda) \geq U(x^0)$. Combining these results it follows that $U(x^\lambda) \geq \text{Min}\{U(x^0), U(x^1)\}$.

If (ii) holds, an almost identical argument leads to this same conclusion. Thus if preferences are convex, the utility function is quasi-concave.

To prove the converse, suppose that $x^0 \succsim y$ and $x^1 \succsim y$. Then $U(x^0) \geq U(y)$ and $U(x^1) \geq U(y)$. If U is quasi-concave, it follows that $U(x^\lambda) \geq \text{Min}\,\{U(x^0), U(x^1)\} \geq U(y)$. Therefore $x^\lambda \succsim y$.

Exercises

Exercise 2.1-1: Transitivity

(a) Show that the transitivity axiom implies that if $x \underset{h}{\succ} y$ and $y \underset{h}{\succ} z$, then $x \underset{h}{\succ} z$. HINT: Appealing to the transitivity axiom, $x \underset{h}{\succsim} z$. Suppose that $x \underset{h}{\sim} z$ then appeal to transitivity to establish that $z \underset{h}{\succsim} y$.

(b) Is it also the case that if $x \underset{h}{\succ} y$ and $y \underset{h}{\succsim} z$, then $x \underset{h}{\succ} z$? HINT: Again assume that $x \underset{h}{\sim} z$ and seek a contradiction.

[4] U is strictly quasi-concave if the inequality is always strict.

Exercise 2.1-2: Continuity of the Utility Function Let $\lambda(x)$ be the monotonic utility function defined in the proof of existence of a utility function.

(a) Show that if $\lambda(\cdot)$ is discontinuous at y there must be some $\hat{\varepsilon}$ and sequence of consumption bundles $\{y^t\}_{t=1}^{\infty}$ approaching y such that $|\lambda(y^t) - \lambda(y)| > \hat{\varepsilon}$.
(b) Consider the subsequence $\{y_+^t\}_{t=1}^{\infty}$ such that $\lambda(y^t) - \lambda(y) > \hat{\varepsilon}$ and a second subsequence $\{y_-^t\}_{t=1}^{\infty}$ consisting of all the remaining bundles in the sequence$\{y^t\}_{t=1}^{\infty}$ such that $\lambda(y^t) - \lambda(y) < -\hat{\varepsilon}$. At least one must be an infinite sequence. Suppose that it is the former. Appeal to the continuity of preferences to establish a contradiction.

Exercise 2.1-3: Sufficient Condition for Convex Preferences Let $U(x)$ be a utility function and let $f(\cdot)$ be an increasing function.

(a) If $u(x) = f(U(x))$ is concave, show that preferences are convex.
(b) If u is strictly concave show that preferences are strictly convex.
(c) If f is strictly concave are preferences strictly convex?

Exercise 2.1-4: Sufficient Condition for Convex Preferences Let $U(x)$ be a utility function. If there is some increasing function f such that

$$f(U(x)) = \sum_{j=1}^{m} u_j(x), \quad \text{where } u_j(x),\ j = 1, \ldots, m \text{ is concave}$$

show that preferences are convex.

Exercise 2.1-5: Strictly Convex Preferences Consider strictly convex preferences defined on the consumption set $X = \mathbb{R}_+^2$.

(a) Suppose Alex has a utility function $U(x) = (1 + x_1)(1 + x_2)$. Show that his preferences are convex. Are his preferences strictly convex?
(b) Bev has a utility function $U(x) = x_1 x_2$. Are her preferences (i) convex or (ii) strictly convex?

Exercise 2.1-6: Strictly Convex Preferences and Strict Quasi-Concave Utility

(a) Show that if U is strictly quasi-concave on X, preferences are strictly convex on X.
(b) Prove the converse.

Exercise 2.1-7: Quasi-Linear Preferences Write the $n + 1$-dimensional consumption vector x as (y, z) where y is a scalar and z is an n-dimensional consumption vector. A utility function $U(x)$ is quasi-linear if it can be written as follows $U(x) = \alpha y + V(z)$. The consumption set $X = \mathbb{R}_+^{n+1}$.

(a) Show that if V is concave, U is quasi-concave.

(b) Show that if U is quasi-concave, V is concave. HINT: Suppose that for some x^0, x^1, x^λ, concavity fails; that is, $V(x^\lambda) < (1-\lambda)V(x^0) + \lambda V(x^1)$. Choose y^0, y^1 such that $U(x^0) = U(x^1)$ and show that $U(x^\lambda) < U(x^0)$.

2.2 Budget-Constrained Choice with Two Commodities

Key ideas: continuity of demand, quasi-linear, Cobb-Douglas and CES preferences, expenditure function, income and substitution effects, elasticity of substitution, determinants of demand elasticity

In this section we focus on the effects of income and price changes on a consumer's choice. Consider the simplest two-commodity choice problem of a consumer with income I facing prices p_1 and p_2,

$$\underset{x}{\text{Max}}\{U(x)|p \cdot x \leq I, x \in \mathbb{R}^2_+\}. \tag{2.2-1}$$

We assume that the local non-satiation assumption holds and that the utility function is continuous and strictly quasi-concave on $\mathbb{R}^2_+$. The local non-satiation assumption ensures that the consumer spends all his or her income and strict quasi-concavity ensures that there is a unique solution $x^0 = x(p, I)$.[5] Given uniqueness and continuity of preferences, it is intuitively plausible that the implied demand function $x(p, I)$ must be continuous. This is an important result and follows from the following mathematical theorem.[6]

Proposition 2.2-1: Theorem of the Maximum (I)
Consider the maximization problem $\text{Max}_x\{f(x,\alpha)|x \in X(\alpha), \alpha \in A\}$ where $X(\alpha) = \{x \in \mathbb{R}^n_+, h_i(x,\alpha) \geq 0, i = 1, \ldots, m\}$.

If f and $X(\alpha)$ are continuous[7] and, for all α, there is a unique solution $\bar{x}(\alpha)$, then $\bar{x}(\alpha)$ is continuous.

For the most general propositions about consumers (and firms as well) continuity is all that we need. However, to simplify modeling, it is convenient to assume some degree of differentiability as well. Then the necessary conditions for a maximum are restrictions on the gradient vector of the maximand and constraint functions.

For the rest of this section we assume that the utility function is continuously differentiable on $\mathbb{R}^2_+$. Initially we also assume that for all $x \in \mathbb{R}^2_+$, $\frac{\partial U}{\partial x}(x) \gg 0$ so that preferences are strictly increasing. Finally, whenever

[5] Suppose x^0 and x^1 are both solutions. Then $U(x^0) = U(x^1)$ and because U is strictly quasi-concave it follows that for any convex combination x^λ, $U(x^\lambda) > U(x^0)$ Finally, because $p \cdot x^0 \leq I$ and $p \cdot x^1 \leq I$, then $p \cdot x^\lambda \leq I$ Thus x^λ is feasible and strictly preferred.

[6] See Appendix C.

[7] The mapping from a parameter to a set is called a correspondence. For a formal definition of a continuous correspondence see Appendix A.

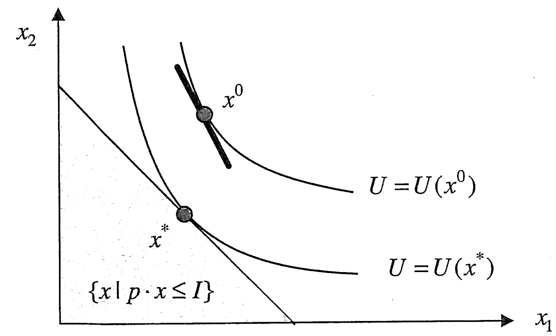

Figure 2.2-1. Budget-constrained choice.

we wish to avoid the possibility of corner solutions we also assume that $\lim_{x_j \to 0} \frac{\partial U}{\partial x_j} = \infty$, $j = 1, 2$.

Forming the Lagrangian for the maximization problem,

$$\mathcal{L} = U + \lambda(I - p \cdot x).$$

The first-order conditions (FOC) for a maximum are then as follows:

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial U}{\partial x_j}(x^*) - \lambda p_j = 0, \quad j = 1, 2.$$

Note that because marginal utility is strictly positive, the shadow price (or marginal utility of income) must be strictly positive.

Moreover, rearranging the FOC,

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} = \lambda. \tag{2.2-2}$$

Intuitively, a maximizing consumer will equate the marginal value of spending on each commodity. One extra dollar spent on commodity 1 yields $\frac{1}{p_1}$ additional units thus the marginal value per dollar spent on commodity 1 is $\frac{1}{p_1}\frac{\partial U}{\partial x_1}$. Spending on commodity 1 is increased until this is equal to the marginal value of spending an additional dollar on commodity 2.

We can also rewrite the FOC as follows:

$$\text{MRS}(x^*) = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{p_1}{p_2}.$$

That is, the slope of the indifference curve must equal the slope of the budget line in Figure 2.2-1.

Income Effects

Figure 2.2-2 shows the path of the consumer's choice $x^* = x(p, I)$ as income increases. As shown, this Income Expansion Path is initially positively

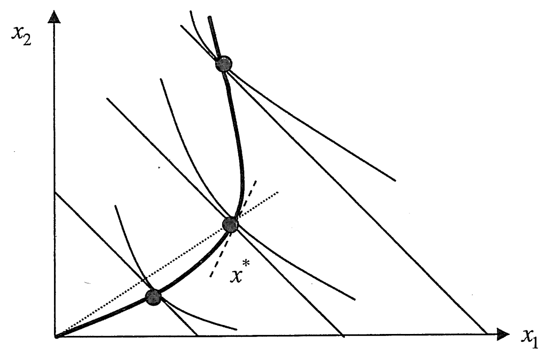

Figure 2.2-2. Income expansion path.

sloped (i.e., $\frac{\partial x_1}{\partial I}$ and $\frac{\partial x_2}{\partial I}$ are both positive). In this case the commodities are said to be normal in the neighborhood of the optimum. However, as depicted, for higher incomes consumption of commodity 1 declines as income increases. In this case commodity 1 is "inferior" in the neighborhood of the optimum.

To facilitate comparisons across commodities, it is helpful to consider the proportional effects on demand as income changes; in other words, the income elasticity of demand:

$$\mathcal{E}(x_j, I) = \frac{I}{x_j}\frac{\partial x_j}{\partial I}.$$

In Figure 2.2-2 the slope of the Income Expansion Path at $x^* = x(p, I)$ is steeper than the line joining x^* and the origin; that is,

$$\left.\frac{dx_2}{dx_1}\right|_{IEP} = \frac{\frac{\partial x_2}{\partial I}}{\frac{\partial x_1}{\partial I}} > \frac{x_2^*}{x_1^*}.$$

Rearranging this inequality,

$$\mathcal{E}(x_2, I) = \frac{I}{x_2^*}\frac{\partial x_2}{\partial I} > \frac{I}{x_1^*}\frac{\partial x_1}{\partial I} = \mathcal{E}(x_1, I).$$

Appealing to the following lemma, the income elasticities weighted by their expenditure shares must sum to 1. Thus, in Figure 2.2-2, the income elasticity of demand for commodity 2 exceeds 1 and for commodity 1 is less than 1.

Lemma 2.2-2: Income Elasticities Weighted by Expenditure Shares Sum to 1

$$k_1\mathcal{E}(x_1^*, I) + k_2\mathcal{E}(x_2^*, I) = 1,$$

where $k_j = \frac{p_j x_j^*}{I}$ is the expenditure share for commodity j.

Proof: To establish this proposition, we substitute the consumer's choice into the budget constraint and differentiate by I.

$$p_1\frac{\partial x_1^*}{\partial I}+p_2\frac{\partial x_2^*}{\partial I}=1.$$

Rearranging the left-hand side,

$$\left(\frac{p_1x_1^*}{I}\right)\frac{I}{x_1^*}\frac{\partial x_1^*}{\partial I}+\left(\frac{p_2x_2^*}{I}\right)\frac{I}{x_2^*}\frac{\partial x_2^*}{\partial I}=k_1\mathcal{E}(x_1^*,I)+k_2\mathcal{E}(x_2^*,I)=1. \qquad \square$$

We now examine income elasticities for three commonly used utility functions.

Example 1: Quasi-Linear Convex Preferences

If preferences are quasi-linear so that $U(x)=v(x_1)+\alpha x_2$, the marginal rate of substitution (MRS) at x^* is

$$\left.\frac{dx_2}{dx_1}\right|_{U=U(x^*)}=\frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}}=\frac{v'(x_1^*)}{\alpha}.$$

The MRS is independent of commodity 2 which means that the indifference curves are vertically parallel. As depicted in Figure 2.2-3, it follows that the Income Expansion Path is first horizontal and then vertical.

Over the range in which both commodities are consumed, it follows that the income elasticity of commodity 1 is zero. Given Lemma 2.2-2, the income elasticity of commodity 2 is the inverse of the expenditure share.

Example 2: Cobb-Douglas Preferences

$$U(x)=x_1^{\alpha_1}x_2^{\alpha_2},\ \alpha_1,\alpha_2>0.$$

Differentiating by x_1, $\frac{\partial U}{\partial x_1}=\alpha_1x_1^{\alpha_1-1}x_2^{\alpha_2}=\frac{\alpha_1U}{x_1}$. Similarly, $\frac{\partial U}{\partial x_2}=\frac{\alpha_2U}{x_2}$.

At the maximum the FOC must be satisfied, hence

$$\frac{\frac{\partial U}{\partial x_1}}{p_1}=\frac{\frac{\partial U}{\partial x_2}}{p_2}=\lambda.$$

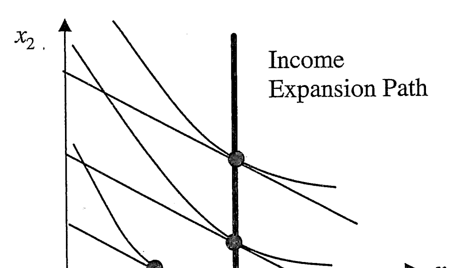

Figure 2.2-3. Quasi-linear preferences.

Substituting and then dividing by U,

$$\frac{\alpha_1}{p_1 x_1} = \frac{\alpha_2}{p_2 x_2} = \frac{\lambda}{U} \equiv \mu, \quad \text{hence} \quad p_j x_j = \frac{\alpha_j}{\mu}.$$

We then solve for μ by substituting back into the budget constraint.

$$p_1 x_1 + p_2 x_2 = \frac{\alpha_1 + \alpha_2}{\mu} = I, \quad \text{hence} \quad \mu = \frac{\alpha_1 + \alpha_2}{I}.$$

Demand for commodity j is therefore

$$x_j(p, I) = \frac{I}{p_j} \frac{\alpha_j}{\alpha_1 + \alpha_2}.$$

Finally, substituting back into the utility function, maximized utility is

$$U(x(p, I)) = \left(\frac{\alpha_1}{p_1}\right)^{\alpha_1} \left(\frac{\alpha_2}{p_2}\right)^{\alpha_2} \left(\frac{I}{\alpha_1 + \alpha_2}\right)^{\alpha_1 + \alpha_2}.$$

Example 3: CES Preferences

$$U(x) = \left(\alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}\right)^{\frac{1}{1-\frac{1}{\theta}}}, \alpha_1, \alpha_2, \theta > 0, \theta \neq 1.$$

From the definition of U,

$$U(x)^{1-\frac{1}{\theta}} = \alpha_1 x_1^{1-\frac{1}{\theta}} + \alpha_2 x_2^{1-\frac{1}{\theta}}.$$

Differentiating by x_j,

$$(1 - \tfrac{1}{\theta}) U^{-\frac{1}{\theta}} \frac{\partial U}{\partial x_j} = (1 - \tfrac{1}{\theta}) \alpha_j x_j^{-\frac{1}{\theta}}.$$

Hence $\frac{\partial U}{\partial x_j} = \frac{\alpha_j U^{\frac{1}{\theta}}}{x_j^{\frac{1}{\theta}}}$.

We follow the same steps as for Example 2.

Substituting into the FOC and then dividing by $U^{\frac{1}{\theta}}$,

$$\frac{\alpha_1}{p_1 x_1^{\frac{1}{\theta}}} = \frac{\alpha_2}{p_2 x_2^{\frac{1}{\theta}}} = \frac{\lambda}{U^{\frac{1}{\theta}}} \equiv \mu. \tag{2.2-3}$$

Hence $x_j = (\frac{\alpha_j}{p_j \mu})^\theta$ and so $p_j x_j = \frac{\alpha_j^\theta p_j^{1-\theta}}{\mu^\theta}$.

We then solve for μ^θ by substituting back into the budget constraint:

$$p_1 x_1 + p_2 x_2 = \frac{1}{\mu^\theta}\left(\alpha_1^\theta p_1^{1-\theta} + \alpha_2^\theta p_2^{1-\theta}\right) = I, \text{ hence } \frac{1}{\mu^\theta} = \frac{I}{\alpha_1^\theta p_1^{1-\theta} + \alpha_2^\theta p_2^{1-\theta}}.$$

Demand for commodity j is therefore

$$x_j(p, I) = \frac{I}{p_j}\left(\frac{\alpha_j^\theta p_j^{1-\theta}}{\alpha_1^\theta p_1^{1-\theta} + \alpha_2^\theta p_2^{1-\theta}}\right). \quad (2.2\text{-}4)$$

Substituting $x(p, I)$ into the utility function and collecting terms,

$$U(x(p, I)) = \frac{I}{(\alpha_1^\theta p_1^{1-\theta} + \alpha_2^\theta p_2^{1-\theta})^{\frac{1}{1-\theta}}}.$$

Note that the demand functions given by (2.2-4) reduce to the Cobb-Douglas demand functions if $\theta = 1$.

Dual Optimization Problem:[8] *Expenditure Minimization*

Given the very weak assumption of local non-satiation, for any budget-constrained utility maximization problem there is a "dual" optimization problem. As we shall see, this dual problem is very useful in understanding the determinants of demand.

Suppose that x^* is a solution to a consumer's maximization problem. That is,

$$x^* \in \arg\underset{x}{\mathrm{Max}}\{U(x)|x \geq 0,\ p \cdot x \leq I\}.$$

Such a consumption bundle is depicted in Figure 2.2-4.

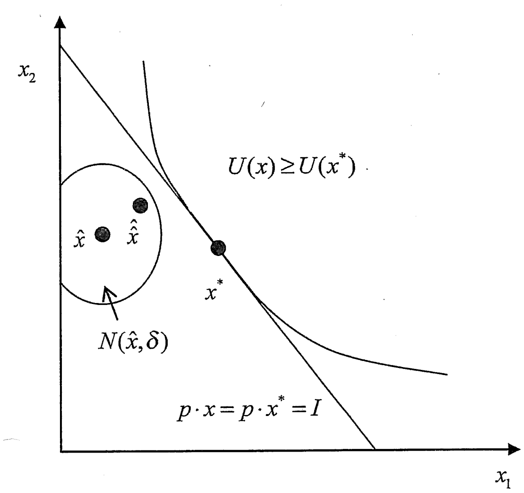

Figure 2.2-4. Dual optimization problem.

[8] The "dual" is the second member of a pair of items, in this case a pair of related optimization problems.

Consider any consumption bundle $\hat{x}$ such that $p \cdot \hat{x} < I$. If δ is sufficiently small the neighborhood $N(\hat{x}, \delta)$ lies in the budget set. If the local non-satiation property holds, then there exists some $\hat{\hat{x}}$ in this neighborhood that is strictly preferred to $\hat{x}$. Then $\hat{x}$ cannot be optimal. Hence

$$p \cdot x < I \Rightarrow U(x) < U(x^*).$$

Equivalently,

$$U(x) \geq U(x^*) \Rightarrow p \cdot x \geq p \cdot x^*.$$

Thus, x^* is expenditure minimizing, among all consumption bundles that are preferred to x^*. We summarize this result as follows.

Lemma 2.2-3: Duality Lemma If the local non-satiation assumption holds and $x^* \in \arg\text{Max}_x\{U(x)|x \geq 0,\ p \cdot x \leq I\}$, then $U(x) \geq U(x^*) \Rightarrow p \cdot x \geq p \cdot x^*$ and so $x^* \in \arg\text{Min}_x\{p \cdot x|x \geq 0,\ U(x) \geq U(x^*)\}$.

For any level of utility, $\overline{U}$ and price vector p we define the expenditure function $M(p, \overline{U})$ to be the minimum expenditure needed to achieve the utility level $\overline{U}$.

Definition: Expenditure Function

$$M(p, \overline{U}) = \underset{x}{\text{Min}}\{p \cdot x|U(x) \geq \overline{U}\}.$$

Although it is not difficult to solve for the expenditure function, it is often more convenient to solve first for the maximized utility

$$V(p, I) = \underset{x}{\text{Max}}\{U(x)|p \cdot x \leq I\}.$$

Given local non-satiation, the consumer spends his entire income. Moreover the higher his income the greater is his utility. Thus maximized utility $V(p, I)$ is a strictly increasing function of income. This is depicted in Figure 2.2-5.

For any utility $\overline{U}$ there is a unique income level M such that $\overline{U} = V(p, M)$. Note that for any lower income, maximized utility is less than $\overline{U}$. Thus the

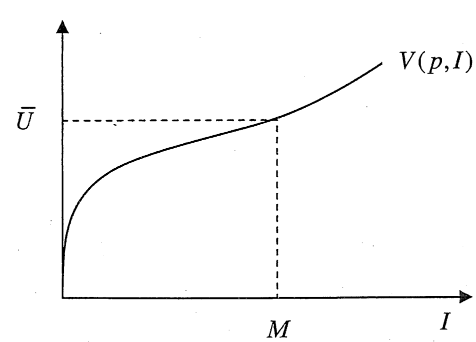

Figure 2.2-5. Maximized utility as a function of income.

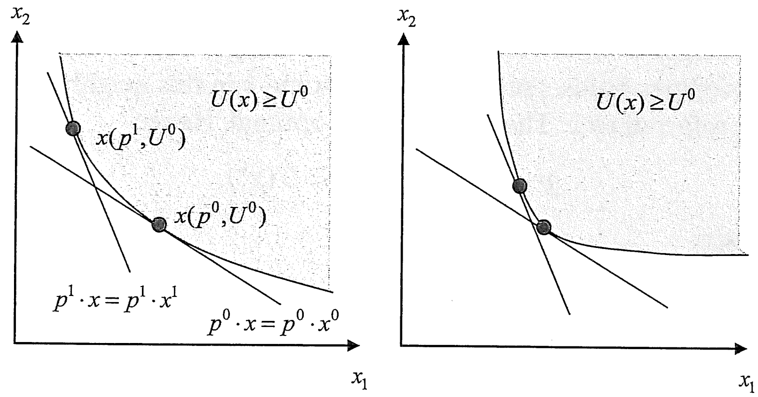

Figure 2.2-6. Substitution effect.

income level M is the minimized expenditure. For any utility level $\overline{U}$, we can therefore solve for $M(p, \overline{U})$ by inverting the equation $\overline{U} = V(p, M)$.

Compensated Demand

Let $x^c(p, \overline{U})$ be the solution to the dual problem, that is $x^c(p, \overline{U})$ solves

$$M(p, \overline{U}) = \underset{x}{\text{Min}}\{p \cdot x | U(x) \geq \overline{U}\}.$$

This is known as the consumer's compensated demand. Consider the effect on compensated demand of an increase in the price of commodity 1. This is depicted in Figure 2.2-6 for price vectors p^0 and p^1. As the price of commodity 1 rises, the consumer is compensated so that he is just able to maintain the utility level U^0. The following useful property of the expenditure function is an immediate implication of the Envelope Theorem[9]

$$\frac{\partial M}{\partial p} = x^c(p, U^0).$$

Informally, if the price of commodity j rises and the consumer maintains his consumption plan, his extra expenditure is x_j^*. This is the direct effect. The indirect effect associated with adjusting to the change in the price is of the second order.

Substitution Effect

The effect on demand of a compensated price change is called the substitution effect. As Figure 2.2-6 illustrates, the size of this effect depends critically on the curvature of the indifference curve. In the left diagram, as the price

[9] See Exercise 2.2-2. Converting the expenditure minimization problem to a maximization problem, $\mathfrak{L} = -p \cdot x + \lambda(U(x) - \overline{U})$ so $-\frac{\partial M}{\partial p_j} = \frac{\partial \mathfrak{L}}{\partial p_j} = -x_j$.

ratio changes, the consumption ratio $\frac{x_2^c(p,U^0)}{x_1^c(p,U^0)}$ changes a lot. That is, the substitution effect is large.

In the right diagram a price change has a small effect on the consumption ratio so the substitution effect is small. As we shall see, the elasticity of the consumption ratio with respect to the price ratio is a very useful measure of price sensitivity.

Definition: Elasticity of Substitution

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, \frac{p_1}{p_2}\right).$$

Example: CES Utility Function
From equation (2.2-3),

$$\frac{x_2^c}{x_1^c} = \frac{\alpha_2^\theta p_1^\theta}{\alpha_1^\theta p_2^\theta}.$$

Taking the logarithm,

$$\ln\left(\frac{x_2^c}{x_1^c}\right) = \theta \ln\left(\frac{p_1}{p_2}\right) + \text{constant}.$$

As is readily confirmed, $\mathcal{E}(y, x) = x\frac{d}{dx}\ln y$. Hence,

$$\mathcal{E}\left(\frac{x_2^c}{x_1^c}, \frac{p_1}{p_2}\right) = \theta.$$

Hence, for the CES utility function, the parameter θ is the elasticity of substitution.

We now show that there are several equivalent definitions of the elasticity of substitution. Lemma 2.2-4 is a direct implication of the following property of elasticity.

$$\mathcal{E}(\alpha z, \beta y) = \mathcal{E}(z, y) = y\frac{d}{dy}\ln z.$$

Lemma 2.2-4[10]

$$\sigma = \mathcal{E}\left(\frac{x_2^c}{x_1^c}, \frac{p_1}{p_2}\right) = \mathcal{E}(x_2^c, p_1) - \mathcal{E}(x_1^c, p_1)$$

The properties in Lemma 2.2-4 hold for any ratio. Two further useful results hold for compensated demand.

[10] In Exercise 2.2-3 you are asked to prove this Lemma.

Proposition 2.2-5: Elasticity of Substitution and Compensated Own Price Elasticity

$$\sigma = \frac{\mathcal{E}(x_2^c, p_1)}{k_1} \quad \text{where} \quad k_1 \equiv \frac{p_1 x_1}{p \cdot x}$$

and

$$\mathcal{E}(x_1^c, p_1) = -(1 - k_1)\sigma.$$

Proof: To demonstrate equivalence, first note that around the indifference curve as p_1 rises we have

$$\frac{\partial U}{\partial x_1}\frac{\partial x_1^c}{\partial p_1} + \frac{\partial U}{\partial x_2}\frac{\partial x_2^c}{\partial p_1} = 0.$$

Also, from the first-order condition, the marginal utility of each commodity is proportional to its price. Hence

$$p_1\frac{\partial x_1^c}{\partial p_1} + p_2\frac{\partial x_2^c}{\partial p_1} = 0. \tag{2.2-5}$$

Dividing by x_1^c and rearranging this equation,

$$\frac{p_1}{x_1^c}\frac{\partial x_1^c}{\partial p_1} = -\frac{p_2}{x_1^c}\frac{\partial x_2^c}{\partial p_1} = -\left(\frac{p_2 x_2^c}{p_1 x_1^c}\right)\frac{p_1}{x_2^c}\frac{\partial x_2^c}{\partial p_1}$$

$$= -\frac{k_2}{k_1}\frac{p_1}{x_2^c}\frac{\partial x_2^c}{\partial p_1}, \quad \text{where} \quad k_j = p_j x_j^c / p \cdot x^c.$$

Therefore

$$\mathcal{E}(x_1^c, p_1) = -\frac{k_2}{k_1}\mathcal{E}(x_2^c, p_1). \tag{2.2-6}$$

From Lemma 2.2-4,

$$\sigma = \mathcal{E}(x_2^c, p_1) - \mathcal{E}(x_1^c, p_1).$$

Substituting for the second term,

$$\sigma = \mathcal{E}(x_2^c, p_1) + \frac{k_2}{k_1}\mathcal{E}(x_2^c, p_1) = \frac{1}{k_1}\mathcal{E}(x_2^c, p_1).$$

Substituting this expression into (2.2-6),

$$\mathcal{E}(x_1^c, p_1) = -k_2\sigma = -(1 - k_1)\sigma. \qquad \square$$

Note that the compensated own price elasticity, $\mathcal{E}(x_1^c, p_1) = -(1 - k_1)\sigma$, is bounded from below by the elasticity of substitution. Moreover, if the expenditure share is small, the elasticity of substitution is a good approximation for the compensated own price elasticity.

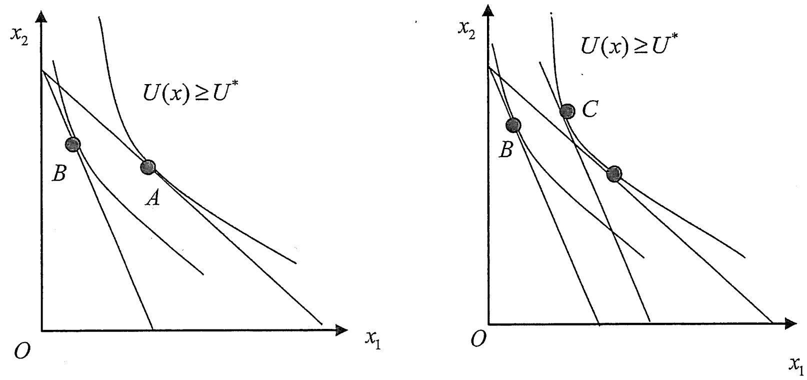

Figure 2.2-7. Decomposition of the price effect into income and substitution effects.

Decomposition of Price Effects

To understand the impact of a price change it proves helpful to decompose it into two parts: a compensated price effect and an income effect. Consider Figure 2.2-7. The left-hand diagram illustrates the effect of an increase in the price of commodity 1.

Suppose next that the individual is fully compensated as the price rises. In the right-hand diagram in Figure 2.2-7 the consumer moves along his indifference curve from A to C substituting commodity 2 for commodity 1. This is the substitution effect of the price increase. In the second step, the extra compensation is taken away and the budget line is pulled in toward the origin. The consumer then moves from C to B. Note that if, as depicted, commodity 1 is a normal good, both the substitution and income effects are negative. That is, the bigger (i.e., the more negative) the substitution effect and the bigger the income effect, the greater will be the total effect on demand for commodity 1.

For commodity 2, with constant prices, the income and substitution effects on demand are offsetting in the normal good case. Although there is substitution of x_2 into the consumption bundle around the indifference curve, the income effect is to reduce expenditure on all commodities. In Figure 2.2-7, the substitution effect dominates.

Returning to the own price effect, suppose that commodity 1 is an inferior good. Then the income effect offsets rather than reinforces the substitution effect. Thus the price sensitivity of inferior goods is typically lower for such commodities. Figure 2.2-8 illustrates the extreme situation in which the income effect dominates.

Although there is no convincing evidence of "Giffen goods" at the market level, it is easy to think of examples in which a commodity is a Giffen good for some consumers. Suppose Alex lives in the Arizona Desert and the price

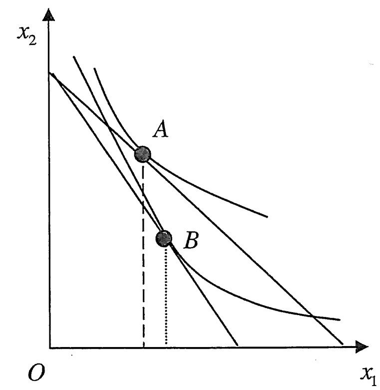

Figure 2.2-8. Giffen good (demand rises with price).

of electricity soars. The air-conditioning bill rises so much that Alex can no longer afford to spend his summer in New Zealand. Instead he stays home and, as a result, his demand for electricity rises.

Slutsky Equation

We now consider the decomposition of the price effect in mathematical terms. If $M(p, \overline{U})$ is minimized total expenditure at utility level $\overline{U}$, and $x_1(p, I)$ is the consumer's demand for commodity 1, the compensated demand is $x_1^c = x_1(p, M(p, \overline{U}))$. Differentiating by p_1, the slope of the compensated demand curve is

$$\frac{\partial x_1^c}{\partial p_1} = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial I}\frac{\partial M}{\partial p_1}.$$

Yet we have seen that $\frac{\partial M}{\partial p_1} = x_1$. Substituting into the above expression and rearranging,

$$\underbrace{\frac{\partial x_j}{\partial p_1}}_{\text{total price effect}} = \underbrace{\frac{\partial x_j^c}{\partial p_1}}_{\text{compensated price effect}} \underbrace{-x_1\frac{\partial x_j}{\partial I}}_{\text{income effect}}. \quad \text{Slutsky equation}$$

In particular, the Slutsky decomposition of the "own price effect" is as follows:

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^c}{\partial p_1} - x_1\frac{\partial x_1}{\partial I}. \tag{2.2-7}$$

Determinants of Demand Price Elasticity

Using the Slutsky equation, we can develop insights into the determinants of demand elasticity. Converting (2.2-7) into elasticity form,

$$\frac{p_1}{x_1}\frac{\partial x_1}{\partial p_1} = \frac{p_1}{x_1}\frac{\partial x_1^c}{\partial p_1} - \frac{p_1 x_1}{I}\frac{I}{x_1}\frac{\partial x_1}{\partial I},$$

where $x_1^c = x_1^c(p, \overline{U})$ is the compensated demand for commodity 1. Hence

$$\mathcal{E}(x_1, p_1) = \mathcal{E}(x_1^c, p_1) - k_1\mathcal{E}(x_1, I). \tag{2.2-8}$$

From Proposition 2.2-5, $\mathcal{E}(x_1^c, p_1) = -(1-k_1)\sigma$.

Substituting for $\mathcal{E}(x_1^c, p_1)$ in equation (2.2-8) we have the following proposition.

Proposition 2.2-6: Decomposition of Own Price Elasticity

$$\mathcal{E}(x_1, p_1) = -(1-k_1)\sigma - k_1\mathcal{E}(x_1, I).$$

We have therefore established that the own price elasticity must lie between the income elasticity and the elasticity of substitution. Holding the expenditure share constant, the higher the elasticity of substitution or the income elasticity, the more negative is the own price elasticity. Moreover, the higher the expenditure share of commodity 1, the greater the weight on the income elasticity. Intuitively, a higher share means that a price rise requires a bigger change in income for the individual to be compensated. Thus the income effect on the change in demand is greater.

Exercises

Exercise 2.2-1: Consumer Choice A consumer has a utility function $U(\cdot)$. He has an income of y and faces a price vector $p = (p_1, p_2)$. In each of the following cases, solve for the consumer's optimal choice under the assumption that he will consume a strictly positive amount of both commodities.

(a) $U(x) = \beta \ln x_1 + x_2$, (ii) $U(x) = \alpha \ln(1+x_1) + x_2$ and (iii) $U(x) = \ln(1+x_1) + \ln(1+x_2)$.

In each case, demand for both commodities $x(p, I)$, is strictly positive for some prices but not for others. Analyze each case in turn and find the shape in Figure 2.2-9 that depicts the set of price vectors for which $x(p, I) \gg 0$.

Exercise 2.2-2: Compensated Demand Show that the price effect on compensated demand $\frac{\partial M}{\partial p_j}(p, U^0) = x_j^c(p, U^0)$.

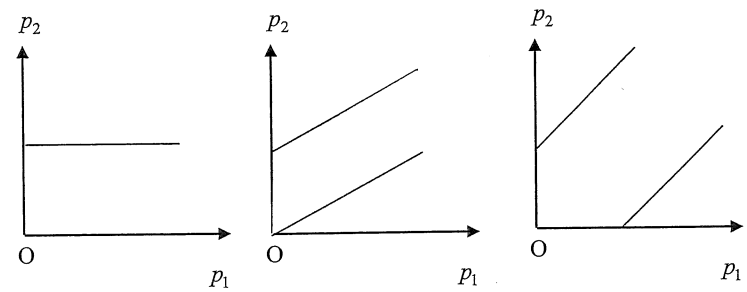

Figure 2.2-9. Price space.

HINT: Converting the minimization problem to the standard form, the compensated demand solves

$$-M(p, U^0) = \underset{x}{\text{Max}}\{-p \cdot x | U(x) - U^0 \geq 0\}.$$

Write down the Lagrangian and appeal to the Envelope Theorem.

Exercise 2.2-3: Elasticity of Substitution

(a) Show that $\mathcal{E}(y(x), z(x)) = \frac{\frac{d}{dx}\ln y}{\frac{d}{dx}\ln z}$.
(b) Use this to show that $\mathcal{E}(\frac{1}{y}, \frac{1}{x}) = \mathcal{E}(y, x)$ and that $\mathcal{E}(y_2/y_1, x) = \mathcal{E}(y_2, x) - \mathcal{E}(y_1, x)$.
(c) Use these results to prove Lemma 2.2-3.

Exercise 2.2-4: CES Preferences ($\sigma < 1$) A consumer has a CES utility function $U(x) = (x_1^{1-\frac{1}{\sigma}} + x_2^{1-\frac{1}{\sigma}})^{\frac{1}{1-\frac{1}{\sigma}}}$, $0 < \sigma < 1$.

(a) If $\sigma = \frac{1}{2}$ write down the indifference curve in the form $x_2 = f(x_1)$ and hence show that the indifference curve through (a, a) has an asymptote $x_2 = \frac{a}{2}$.
(b) What is the other asymptote?
(c) Repeat the analysis if $\sigma = \frac{1}{4}$.
(d) Show that as $\sigma \to 0$ the horizontal asymptote approaches $x_2 = a$.
(e) Hence show that in the limit the indifference curve becomes L-shaped.

Exercise 2.2-5: CES Preferences ($\sigma > 1$) An individual has a symmetric CES utility function $U(x) = (x_1^{1-\frac{1}{\sigma}} + x_2^{1-\frac{1}{\sigma}})^{\frac{1}{1-\frac{1}{\sigma}}}$, $\sigma > 1$.

(a) If $\sigma = 2$ write down the indifference curve in the form $x_2 = f(x_1)$ and hence show that the indifference curve through (a, a) has endpoints $(0, 4a)$ and $(4a, 0)$.
(b) What is the marginal rate of substitution at these points?
(c) Show that as σ increases the intercept with the axes $b(\sigma)$ decreases.
(d) Depict the limiting indifference curve as $\sigma \to \infty$.

Exercise 2.2-6: Parallel Income Expansion Paths A consumer has a utility function $U(x) = \sum_{i=1}^{2} -\alpha_i e^{-Ax_i}$. She has an income I and faces a price vector p.

(a) Show that her optimal consumption bundle satisfies a condition of the following form:

$$x_2 - x_1 = a + b \ln \frac{p_1}{p_2}.$$

(b) Depict her Income Expansion Paths in a neat figure.

2.3 Consumer Choice with *n* Commodities

Key ideas: indirect utility function, homothetic preferences, the representative agent, elasticity of substitution

For a budget-constrained consumer there are no completely general propositions about the effect of price changes on consumption. As we have seen in the previous section, the magnitude and direction of price effects depend upon the relative size of income and substitution effects. We can, however, say a lot about compensated price changes. We therefore begin by analyzing the dual consumer problem

$$M(p, U^*) = \underset{x}{\text{Min}}\{p \cdot x | U(x) \geq U^*\}.$$

Proposition 2.3-1:
For compensated price changes the product of the price and quantity changes is negative ($\Delta p \cdot \Delta x \leq 0$).

Proof: Let x^0 be expenditure minimizing when the price vector is p^0 and x^1 be expenditure minimizing when the price vector is p^1. Both x^0 and x^1 satisfy the constraint $U(x) \geq U^*$. Therefore

$$p^0 \cdot x^0 \leq p^0 \cdot x^1 \text{ (because } x^0 \text{ is cost minimizing at } p^0\text{)}$$

and

$$p^1 \cdot x^1 \leq p^1 \cdot x^0 \text{ (because } x^1 \text{ is cost minimizing at } p^1\text{).}$$

Rearranging, it follows that

$$-p^0 \cdot (x^1 - x^0) \leq 0 \quad \text{and} \quad p^1 \cdot (x^1 - x^0) \leq 0.$$

Adding these two inequalities,

$$\Delta p \cdot \Delta x = (p^1 - p^0) \cdot (x^1 - x^0) \leq 0. \qquad \square$$

Special Case of a Single Price Change

Note that if only the price of commodity j changes it follows that $\Delta p_j \Delta x_j \le 0$; therefore the compensated own price effect $\frac{\partial x_j^c}{\partial p_j}$ is negative.

Writing the dual problem as a maximization problem, we have

$$-M(p, U^*) = \underset{x}{\text{Max}}\{-p \cdot x | U(x) \ge U^*\}.$$

The Lagrangian of this maximization problem is $\mathfrak{L} = -p \cdot x + \lambda(U(x) - U^*)$. Appealing to the Envelope Theorem,

$$-\frac{\partial M}{\partial p_j} = \frac{\partial \mathfrak{L}}{\partial p_j} = -x_j^c.$$

Therefore

$$\frac{\partial}{\partial p_i}\frac{\partial M}{\partial p_j} = \frac{\partial x_j^c}{\partial p_i}. \qquad (2.3\text{-}1)$$

Setting $i = j$ in this equation, it follows that the expenditure function is a concave function of each price taken separately. We now prove a stronger result.

Proposition 2.3-2:
The expenditure function $M(p, U^*)$ is a concave function of the price vector, that is, for any p^0, p^1,

$$M(p^\lambda, U^*) \ge (1-\lambda)M(p^0, U^*) + \lambda M(p^1, U^*). \qquad (2.3\text{-}2)$$

Proof: The method of proof is similar to the proof of Proposition 2.3-1. For the three prices p^0, p^1 and p^λ let x^0, x^1 and x^λ be the expenditure-minimizing consumption vectors. Because $M(p, U^*)$ is minimized expenditure we start with the terms on the right-hand side of inequality (2.3-2). Because x^λ is feasible,

$$M(p^0, U^*) = p^0 \cdot x^0 \le p^0 \cdot x^\lambda \quad \text{and} \quad M(p^1, U^*) = p^1 \cdot x^1 \le p^1 \cdot x^\lambda.$$

Hence

$$(1-\lambda)M(p^0, U^*) \le (1-\lambda)p^0 \cdot x^\lambda \quad \text{and} \quad \lambda M(p^1, U^*) \le \lambda p^1 \cdot x^\lambda.$$

Adding these two inequalities,

$$(1-\lambda)M(p^0, U^*) + \lambda M(p^1, U^*) \le (1-\lambda)p^0 \cdot x^\lambda + \lambda p^1 \cdot x^\lambda$$
$$= p^\lambda \cdot x^\lambda = M(p^\lambda, U^*).$$

□

Special Case $M(p, U) \in \mathbb{C}^2$

Assuming $M(p, U^*)$ is twice differentiable, it is concave in p if and only if the matrix of second derivatives is negative semi-definite. Appealing to equation (2.3-1) it follows that

$$\left[\frac{\partial^2 M}{\partial p_i \partial p_j}\right] = \left[\frac{\partial x_j^c}{\partial p_i}\right]$$

is negative semi-definite.

Indirect Utility Function

Let $x^* = x(p, I)$ be the n commodity consumption bundle chosen by a consumer with utility function $U(\cdot)$ and income I facing a price vector p. Define $V(p, I)$ to be maximized utility, that is,

$$V(p, I) = \underset{x}{\text{Max}}\{U(x) | p \cdot x \leq I, x \geq 0\} = U(x^*(p, I)).$$

The maximized utility function $V(p, I)$ is known as the indirect utility function.

Forming the Lagrangian for this problem,

$$\mathfrak{L}(x, \lambda) = U(x) + \lambda(I - p \cdot x).$$

By the Envelope Theorem,

$$\frac{\partial V}{\partial I} = \frac{\partial \mathfrak{L}}{\partial I}(x^*, \lambda^*) = \lambda^*$$

and

$$\frac{\partial V}{\partial p_j} = \frac{\partial \mathfrak{L}}{\partial p_j}(x^*, \lambda^*) = -\lambda^* x_j^*(p, I).$$

Combining these results yields the following simple rule.

Proposition 2.3-3: Roy's Identity

$$x_j^*(p, I) = -\frac{\partial V}{\partial p_j} / \frac{\partial V}{\partial I}.$$

Thus we can always recover the consumer's demand functions from the indirect utility function.

Example: Cobb-Douglas Preferences

Consider the indirect utility function[11] $V(p, I) = \times_{i=1}^{n}(\frac{\alpha_i I}{p_i})^{\alpha_i}$, where $\sum_{i=1}^{n} \alpha_i = 1$.

[11] We use the shorthand $\times_{i=1}^{n} a_i$ to denote $a_1 \times \cdots \times a_n$, the product of the components of the n dimensional vector a. Mathematicians typically prefer to use the notation $\Pi_{i=1}^{n} a_i$ However, in economics, the Greek letter pi has a host of different uses.

Taking the logarithm,

$$\ln V = \ln I - \sum_{i=1}^{n} \alpha_i \ln p_i + \sum_{i=1}^{n} \alpha_i \ln \alpha_i.$$

$$\text{Then } \frac{\partial}{\partial I} \ln V = \frac{1}{V}\frac{\partial V}{\partial I} = \frac{1}{I} \quad \text{and} \quad \frac{\partial}{\partial p_i} \ln V = \frac{1}{V}\frac{\partial V}{\partial p_i} = -\frac{\alpha_i}{p_i}.$$

Appealing to Roy's Identity,

$$x_i(p, I) = \frac{\alpha_i I}{p_i}.$$

Finally, substituting this back into the indirect utility function,

$$U(x(p, I)) \equiv V(p, I) = \underset{i=1}{\overset{n}{\times}} x_i(p, I)^{\alpha_i}.$$

Therefore

$$U(x) = \underset{i=1}{\overset{n}{\times}} (x_i)^{\alpha_i}.$$

Thus we can retrieve not only the demand functions but also the original Cobb-Douglas utility function.

For this example demand is proportional to income. We now introduce a family of utility functions that have this property.

Definition: Homothetic Preferences A function $U(x)$ is homothetic if, for any x^0, x^1 and $\theta > 0$ $U(x^0) \geq U(x^1) \Rightarrow U(\theta x^0) \geq U(\theta x^1)$.

Note that if an individual with homothetic preferences is indifferent between the two bundles x^0 and x^1 then $U(x^0) \geq U(x^1)$ and $U(x^1) \geq U(x^0)$. Therefore, if preferences are homothetic, $U(\theta x^0) \geq U(\theta x^1)$ and $U(\theta x^1) \geq U(\theta x^0)$ so $U(\theta x^0) = U(\theta x^1)$. It follows also that if $U(x^0) > U(x^1)$, then $U(\theta x^0) > U(\theta x^1)$.

Proposition: 2.3-4:
If preferences are homothetic and x^* is optimal given income I, then θx^* is optimal given income θI.

Proof: Let x^{**} be optimal given income θI. Because $\frac{1}{\theta}x^{**}$ is feasible with budget I, and x^* is optimal,

$$U(x^*) \geq U\left(\frac{1}{\theta}x^{**}\right).$$

By homotheticity, it follows that

$$U(\theta x^*) \geq U(x^{**}).$$

Because x^{**} is optimal with income θI and θx^* is feasible,

$$U(x^{**}) \geq U(\theta x^*).$$

Together these last two inequalities imply that $U(x^{**}) = U(\theta x^*)$. Therefore θx^* is also optimal with income θI. □

Lemma 2.3-5: Homogeneous Functions Are Homothetic If $U(\lambda x) = \lambda^k U(x)$, then preferences are homothetic.

Proof: A function U is homogeneous of degree k if $U(\lambda x) = \lambda^k U(x)$. Suppose $U(x) \geq U(y)$. Then

$$U(\lambda x) = \lambda^k U(x) \geq \lambda^k U(y) = U(\lambda y).$$ □

In fact, as we now establish, if preferences are homothetic, there is no loss in generality if it is assumed that the utility function is homogeneous of degree 1.

Proposition 2.3-6:
If preferences are homothetic and strictly increasing, they can be represented by a utility function that is homogeneous of degree 1.

Proof: Let $\mathbf{e}$ be the unit vector $(1, \ldots, 1)$. Because preferences are strictly increasing, for any consumption vector $\hat{x}$, there must be some number $\hat{u}$ such that $\hat{x} \sim \hat{u}\mathbf{e}$. This is depicted in Figure 2.3-1 for the two-commodity case. The number $\hat{u}$ is chosen so that the indifference curve through $\hat{x}$ also passes through the point $\hat{u}\mathbf{e}$ on the 45° line.

Because for each $\hat{x}$ there exists a number $\hat{u}$, the mapping $g : x \to u$ is a utility function. In particular, $g(\hat{x}) = \hat{u}$. Appealing to homotheticity, because $\hat{x} \sim \hat{u}\mathbf{e}$, it follows that $\lambda\hat{x} \sim (\lambda\hat{u})\mathbf{e}$, that is $g(\lambda\hat{x}) = \lambda\hat{u}$. Because $g(\hat{x}) = \hat{u}$ it follows that $g(\lambda\hat{x}) = \lambda g(\hat{x})$. □

Example:
$U(x) = a_1 x_1^\alpha + a_2 x_2^\alpha$, $0 < \alpha \leq 1$. Note first that $U(\lambda x) = a_1(\lambda x_1)^\alpha + a_2(\lambda x_2)^\alpha = \lambda^\alpha U(x)$. Thus the utility function is homogeneous of degree α and

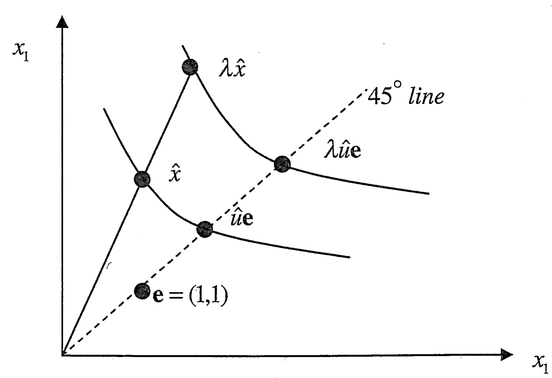

Figure 2.3-1. Homothetic preferences.

is therefore homothetic. Define $u(x) = U(x)^{\frac{1}{\alpha}}$. Then $u(\lambda x) = U(\lambda x)^{\frac{1}{\alpha}} = (\lambda^{\alpha} U(x))^{\frac{1}{\alpha}} = \lambda U(x)^{\frac{1}{\alpha}} = \lambda u(x)$. Thus preferences can be represented by the homogenous of degree 1 utility function $u(x)$.

Representative Preferences

If a group of consumers has identical homothetic preferences, it is especially easy to sum individual demands to solve for the aggregate demand. As the following proposition reveals, what matters is the total income of the group, and not its distribution among the members. Thus there is no loss in generality if it is assumed that all the income is in the hands of one "representative" member of the group.

Proposition 2.3-7: Representative Preferences
Define $x(p, I) = \arg\mathrm{Max}_x \{U(x) | p \cdot x \leq I\}$. If U is homothetic then

$$\sum_{h=1}^{H} x(p, I^h) = x(p, I^R) \quad \text{where} \quad I^R = \sum_{h=1}^{H} I^h.$$

Proof: Define $x(p, 1) = \arg\mathrm{Max}_x \{U(x) | p \cdot x \leq 1\}$. Because U is homothetic,

$$x(p, I^h) = I^h x(p, 1)$$

and so

$$\sum_{h=1}^{H} x(p, I^h) = \sum_{h=1}^{H} I^h x(p, 1) = I^R x(p, 1) = x(p, I^R).$$ □

Elasticity of Substitution with n *Commodities (a Two-Stage Approach)*

To analyze the effects of a price change, there is a second "indirect" utility function that will prove to be helpful. Suppose that the consumer breaks his choice problem down into two stages. At the first stage he fixes his consumption of commodity j and asks how he should best spend a budget of y on the other n-1 commodities. Let $u(x_j, y)$ be the resulting indirect utility, that is,

$$u(x_j, y) = \underset{x_{-j}}{\mathrm{Max}} \left\{ U(x) | \sum_{\substack{i=1 \\ i \neq j}}^{n} p_i x_i \leq y \right\}.$$

Because we are interested in the effect of a change in the price of commodity j we have suppressed the dependence of the indirect utility on $p_{-j} \equiv (p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n)$.

In stage 2 our consumer decides how to allocate his budget between commodity j and spending for the other goods,

$$\underset{x_j, y}{\text{Max}}\{u(x_j, y) | p_j x_j + y \leq I\}.$$

This second-stage problem is precisely the problem studied in the previous section. In particular we can define the elasticity of substitution for commodity j as follows.

Definition: Elasticity of Substitution

$$\sigma_j = \frac{\mathcal{E}(y^c, p_j)}{k_j}, \quad \text{where} \quad k_j = \frac{p_j x_j}{I}.$$

Having reduced the n commodity problem to an equivalent two-commodity problem, we can appeal to Section 2.2. In particular, from Proposition 2.2-2, the own price elasticity can be decomposed as follows:

$$\mathcal{E}(x_j, p_j) = -(1 - k_j)\sigma_j - k_j \mathcal{E}(x_j, I). \tag{2.3-3}$$

Thus again the own price elasticity is a weighted average of the elasticity of substitution and the income elasticity.

Remark: One popular model of imperfect substitutes assumes that preferences are in the CES family and that the elasticity of substitution is independent of the number of commodities. Suppose that the number of commodities is large and the share of any one commodity is small; that is $k_j \approx 0$, $j = 1, \ldots, n$. It follows from the decomposition formula that $\mathcal{E}(x_j, p_j) \approx -\sigma$. Thus the own price elasticity of each commodity is independent of the number of substitutes. For most markets this seems likely to be at odds with the facts!

The elasticity of substitution is a measure of the overall substitutability of a product. Intuitively it must be some average of the substitutability of the product with each of the other products. We define this pair-wise elasticity of substitution as follows.[12]

Definition: Elasticity of Substitution between Pairs of Products

$$\sigma_{ij} = \frac{\mathcal{E}(x_i^c, p_j)}{k_j} = \frac{\mathcal{E}(x_j^c, p_i)}{k_i} = \sigma_{ji}$$

At first sight it is surprising that for any pair of commodities we can pick any one commodity and the price of the other commodity to compute this

[12] Note that this definition is only equal to the elasticity of the consumption ratio $\mathcal{E}(x_i/x_j, p_j)$ in the two-commodity case.

pairwise elasticity of substitution. To see why this must be the case note that from equation (2.3-1)

$$\frac{\partial x_j^c}{\partial p_i} = \frac{\partial}{\partial p_i}\frac{\partial M}{\partial p_j} \quad \text{and} \quad \frac{\partial x_i^c}{\partial p_j} = \frac{\partial}{\partial p_j}\frac{\partial M}{\partial p_i}.$$

Because the order in which M is differentiated does not matter, the cross partial derivatives are equal. Multiplying both sides by the two prices yields the following result.

$$p_j x_j^c \left(\frac{p_i}{x_j^c}\frac{\partial x_j^c}{\partial p_i}\right) = p_i p_j \frac{\partial x_j^c}{\partial p_i} = p_j p_i \frac{\partial x_i^c}{\partial p_j} = p_i x_i^c \left(\frac{p_j}{x_i^c}\frac{\partial x_i^c}{\partial p_j}\right).$$

Rearranging, it follows that $\sigma_{ij} = \sigma_{ji}$.

The next result follows directly from the elasticity of substitution definitions.

Proposition 2.3-8:

$$(1-k_j)\sigma_j = \sum_{\substack{i=1\\ i\neq j}}^{n} k_i \sigma_{ij}, \quad \text{where} \quad k_i = \frac{p_i x_i}{p\cdot x}.$$

Because $\sum_{\substack{i=1\\ i\neq j}}^{n} k_i = 1 - k_j$ it follows that the elasticity of substitution is the expenditure share weighted average of the pair-wise elasticities of substitution.

Exercises

Exercise 2.3-1: Expenditure Function with Cobb-Douglas Preferences Suppose that $U(x) = \times_{j=1}^{n} x_j^{\alpha_j} \equiv x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $\sum_{j=1}^{n} \alpha_j = 1$.

(a) Solve for the indirect utility function $V(p, I)$.
(b) Explain why you can "invert" your result to obtain the expenditure function.
(c) Hence solve for the expenditure function.

Exercise 2.3-2: Quasi-Linear Concave Preferences and Multi-Valued Demand Bev has a utility function $U(x) = \sqrt{x_1 x_2} + x_3$.

(a) Suppose that she allocates y toward the purchase of commodities 1 and 2 and purchases x_3 units of commodity 3. Show that her resulting utility will be $U^*(x_3, y) = \frac{y}{2\sqrt{p_1 p_2}} + x_3$.
(b) Given this preliminary optimization problem has been solved, her budget constraint is $p_3 x_3 + y \leq I$. Solve for her optimizing values of x_3 and y.
(c) Under what conditions, if any, is she strictly worse off if she is told that she can consume at most two of the three available commodities?

(d) Is this two-stage approach to optimization equivalent to solving directly for the optimal consumption bundle? Explain.

Remark: Note that for one price ratio, the individual is indifferent between a range of possible consumption bundles. Thus there is no demand function. Such multi-valued mappings are known as demand correspondences.

Exercise 2.3-3: Consuming Pairs of Commodities A consumer has concave utility function $U(x) = \sqrt{x_1 x_2} + \sqrt{x_3 x_4}$. Solve for the consumer's choice as a function of his income and prices. HINT: You may wish to try breaking the problem down into two smaller problems, as in the previous question.

Exercise 2.3-4: Slutsky Equation with Endowments Suppose that an individual has an endowment of commodities $\overline{x}$ as well as an income of I.

(a) Show that the Slutsky equation for this individual is
(b) $\frac{\partial x_i}{\partial p_j} = \frac{\partial x_i^c}{\partial p_j} - (x_j - \overline{x}_j)\frac{\partial x_i}{\partial I}$.
(c) If all h individuals in an economy have an endowment and no other income and each has the same marginal propensity to consume out of income ($\frac{\partial x_i^h}{\partial I} = \alpha, h = 1, \ldots, H$), show that if markets clear, the slope of the market demand curve and the slope of the compensated market demand curves are identical.
(d) Returning to part (a) can you interpret one commodity as leisure time? If so examine the effect of a wage increase on demand for leisure and hence on the supply of labor.

Exercise 2.3-5: Compensated Price Elasticities Appeal to the convexity of the expenditure function to show that for any pair of commodities x_i and x_j compensated price elasticities must satisfy the following inequality

$$\mathcal{E}(x_i^c, p_i)\mathcal{E}(x_j^c, p_j) \geq \mathcal{E}(x_i^c, p_j)\mathcal{E}(x_j^c, p_i).$$

2.4 Consumer Surplus and Willingness to Pay

Key ideas: willingness to pay, compensating variation, equivalent variation

Suppose the price of a commodity falls. How much better off is a consumer? A simplified answer is as follows. Let $B(q)$ be the benefit measured in dollars if consumption is q. If the unit price is p the net benefit is $B(q) - pq$. The consumer then makes purchases until the marginal benefit $B'(q)$ is equal to the price. Because this is true for each price, the mapping from quantities to market-clearing prices (or "demand price function") is

$$p(q) = B'(q).$$

Then the total benefit of q units is $B(q) = \int_0^q p(x)dx$.

The marginal benefit function is depicted in Figure 2.4-1.

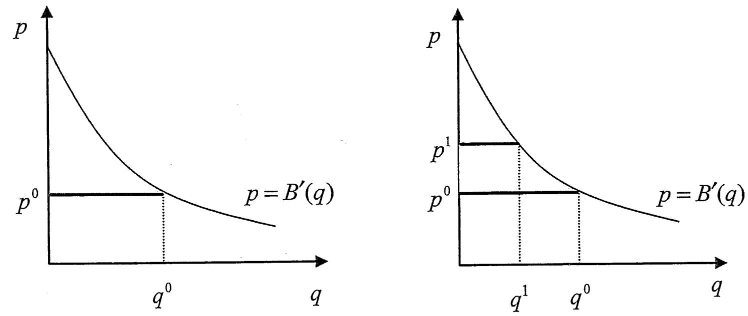

Figure 2.4-1. Consumer surplus.

At the price p^0 the consumer purchases q^0 units. The gross consumer benefit

$$B(q^0) = \int_0^{q^0} B'(q)dq$$

is the area under the demand price function. The net consumer benefit or consumer surplus is

$$B(q^0) - p^0q^0 = \int_0^{q^0} (B'(q) - p^0)dq.$$

This is the shaded area in the left-hand diagram in Figure 2.4-1. It follows that if the price rises to p^1, the change in the consumer surplus is the shaded area in the right-hand diagram in Figure 2.4-1; that is, the area to the left of the demand price function.

Left unexplained in this argument is the origin of the consumer's "benefit function." In general benefit it is not well defined because the marginal benefit of consuming a particular commodity depends on how much of the other commodities are consumed. However there is another way to proceed. We can ask how much an individual would need to be compensated to be no worse off after the price increase. For a price decrease, we can ask how much the consumer would be willing to pay for the change.

Compensating Variation

Consider a consumer with income I facing a price vector p^0. Let U^0 be maximized utility. That is, $U^0 = \text{Max}_x\{U(x)|p \cdot x \leq I\}$. Then I is the minimized income of the dual problem,

$$I = M(p^0, U^0) = \underset{x}{\text{Min}}\{p \cdot x|U(x) \geq U^0\}.$$

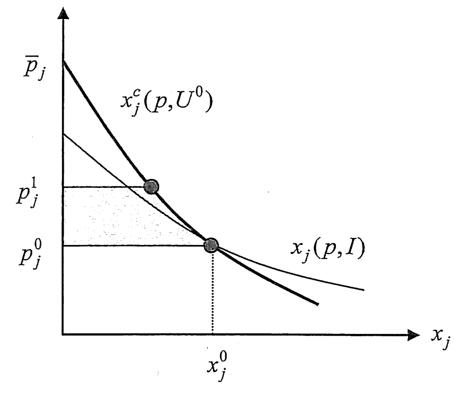

Figure 2.4-2. Compensating variation.

The compensating variation (CV) in income is defined as the extra income that the consumer needs to be fully compensated when the price changes from p^0 to p^1.

$$CV = M(p^1, U^0) - I = M(p^1, U^0) - M(p^0, U^0).$$

Suppose that the price of commodity j rises. By the Envelope Theorem, the rate at which income must rise with p_j is the compensated demand. That is,

$$\frac{\partial M}{\partial p_j} = x_j^c(p, U^0).$$

Integrating this expression,

$$CV \equiv M(p^1, U^0) - M(p^0, U^0) = \int_{p_j^0}^{p_j^1} \frac{\partial M}{\partial p_j} dp_j = \int_{p_j^0}^{p_j^1} x_j^c(p, U^0) dp_j.$$

Thus the total compensation needed to maintain the initial utility is the area behind the compensated demand curve. Figure 2.4-2 depicts the ordinary demand curve $x_j(p, I)$ and the compensated demand curve $x_j^c = x_j^c(p, U^0)$ under the assumption that commodity j is a normal good. Then, as the price of commodity j rises and the consumer is compensated with additional income, consumption of the commodity is greater. The compensated demand curve must therefore be steeper. The compensating variation for the price increase is the more heavily shaded region behind the compensated demand curve.

The entire shaded area is the compensation required if the price rises above $\overline{p}_j$ so that the consumer is squeezed out of the market for commodity j completely. Thus consumer surplus, as measured by the area to the left

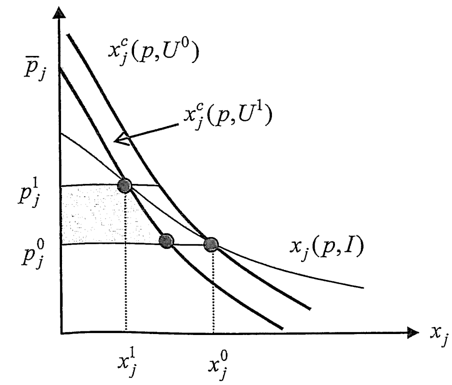

Figure 2.4-3. Equivalent variation.

of the ordinary (uncompensated) demand curve, is an underestimate of the compensating variation in income.

Equivalent Variation

Another way of measuring the loss to the consumer due to a price increase is to evaluate compensation *after* the price increase has taken place. With the price increasing to p^1, the consumer has a new maximized utility $U^1 = \text{Max}_x\{U(x)|p^1 \cdot x \leq I\}$. The question now becomes, how much income would the consumer be willing to give up to be offered the old price vector?

Let $x_j^1 = x_j(p^1, I)$ be the consumption of commodity j after the price increase. This is depicted in Figure 2.4-3.

The equivalent variation (EV) is the amount of income that must be taken away to leave the consumer's utility constant as the price is returned to its initial level, that is,

$$EV = M(p^1, U^1) - M(p^0, U^1).$$

Again appealing to the Envelope Theorem we can rewrite this as follows:

$$EV = \int_{p_j^0}^{p_j^1} \frac{\partial M}{\partial p_j} dp_j = \int_{p_j^0}^{p_j^1} x_j^c(p, U^1) dp_j.$$

Thus the equivalent variation is the area to the left of the demand curve $x_j^c(p, U^1)$. As long it is a normal good, demand for commodity j is lower at the lower utility level U^1. Thus this measure is smaller than the compensating variation.

Note also that the area to the left of the ordinary (uncompensated) demand curve must lie between CV and EV. As Figure 2.4-3 suggests, the difference between the two "true measures" of compensation and consumer surplus will be small, unless the price change is very large. Thus in practice, estimates of welfare gains are typically based on estimates of ordinary rather than compensated demand curves.

Applied economists have tried to use estimates of compensating and equivalent variation to determine upper and lower bounds on the social cost of un-priced commodities such as pollution. For example, the EV group of individuals might be shown two photos depicting different levels of pollution and asked how much they would be willing to pay to live in the less polluted environment. The CV group might be asked how much money they would need to be paid to live in the more polluted environment. Unfortunately the answers are several orders of magnitude apart.[13] Thus any such estimate is very imprecise. For this reason economists typically try to find some way of indirectly inferring estimating costs using market demand analysis.

Measuring Benefits with Multiple Price Changes

Suppose that we wish to measure the cost associated with raising the price of two commodities. What is the generalization of the area behind the demand curve depicted in Figure 2.4-3? Consider the case of changes in the prices of commodities 1 and 2. Let p_1^0 and p_2^0 be current prices and let (p_1^1, p_2^1) be a pair of higher prices.

Arguing as before, the compensating variation is

$$\begin{aligned} CV &= M(p^1, U^0) - M(p^0, U^0) \\ &= M(p_1^1, p_2^1, U^0) - M(p_1^1, p_2^0, U^0) + M(p_1^1, p_2^0, U^0) - M(p_1^0, p_2^0, U^0) \\ &= \int_{p_2^0}^{p_2^1} \frac{\partial}{\partial p_2} M(p_1^1, p_2, U^0)dp_2 + \int_{p_1^0}^{p_1^1} \frac{\partial}{\partial p_1} M(p_1, p_2^0, U^0)dp_1 \\ &= \int_{p_2^0}^{p_2^1} x_2^c(p_1^1, p_2, U^0)dp_2 + \int_{p_1^0}^{p_1^1} x_1^c(p_1, p_2^0, U^0)dp_1 \end{aligned}$$

This is the sum of the areas to the left of the two compensated inverse demand curves.

13 Presumably those surveyed are suspicious and fear that if they say they are willing to pay a lot might also be charged a lot. And if the government is going to give money out, why not get back some of those taxes you paid but felt were wasted!

Exercises

Exercise 2.4-1: Consumer Surplus, Compensated Variation, and Equivalent Variation

(a) Depict indifference curves for the derived utility function $u(x_j, y)$. Show the individual's choice before and after an increase in the price of commodity j from p_j to p'_j. Draw the budget line that keeps the individual at his initial utility level.
(b) Mark the two points where the budget lines meet the vertical axis as A and B. Explain why AB is the compensating variation.
(c) Either in a second figure (or in the same one if it is not too messy!) show the equivalent variation on the vertical axis.

Exercise 2.4-2: Compensating Variation for Consumers with Different Incomes A consumer with utility function $U(x, y) = x^{\alpha} y^{1-\alpha}$ has a budget constraint $p^0 x + y \leq I$.

(a) Solve for his maximized utility $U^0(p, I)$ and hence show that if the price of x rises to p^1, his income must rise to $M = I(\frac{p^1}{p^0})^{\alpha}$ to be fully compensated.
(b) Obtain expressions for both the change in the consumer surplus and the compensating variation as a function of price and income.
(c) Use a spreadsheet to compute the percentage difference between the consumer surplus and CV for different price ratios p^1/p^0.
(d) How would your answer to part (a) change if the utility function was $U(x, y) = \alpha \ln x + (1 - \alpha) \ln y$?

2.5 Choice over Time

Key ideas: interest rate, lifetime budget constraint, futures prices, and future spot prices

The one-period model can be readily adapted to analyze the savings decision of a consumer. To focus on intertemporal choice, we begin by assuming that there is a single consumption good in each period. A consumer has an income in period t of y_t and consumption is c_t. Her utility is $U(c) = U(c_1, \ldots, c_T)$. The consumer can borrow or save in period t at the interest rate r. Assuming period 1 assets are zero, the consumer can accumulate assets by saving some of her period 1 income and earning the period 1 interest rate. Her period 2 asset level is

$$K_2 = (1 + r)(y_1 - c_1).$$

In period 2 the consumer has the assets she has previously accumulated plus his current income. She consumes some part of her wealth and earns interest on the remainder. Her period 3 asset level is therefore

$$K_3 = (1 + r)(K_2 + y_2 - c_2).$$

Similarly in period t, the consumer must decide how much to consume and how much to save. The consumer then chooses a consumption stream $\{c_t\}_{t=1}^{T}$ to maximize lifetime utility subject to T asset accumulation constraints.

Lifetime Budget Constraint

To analyze this choice problem we first show that the T constraints can be reduced to a single lifetime budget constraint. Consider the period 1 asset accumulation constraint. Dividing by $1 + r$ to put everything in "present value" terms,

$$\frac{K_2}{1+r} = y_1 - c_1.$$

Similarly expressing the period 2 constraint in present value terms,

$$\frac{K_3}{(1+r)^2} = \frac{K_2}{(1+r)} + \frac{y_2 - c_2}{(1+r)}.$$

Substituting for K_2,

$$\frac{K_3}{(1+r)^2} = y_1 - c_1 + \frac{y_2 - c_2}{1+r}.$$

Repeating this for all T periods,

$$\frac{K_{T+1}}{(1+r)^T} = y_1 - c_1 + \cdots + \frac{y_T - c_T}{(1+r)^{T-1}}.$$

Because the consumer cares only about her consumption vector, she will never want to leave assets behind so $K_{T+1} \leq 0$. Moreover, no financial intermediary will agree to a loan that will never be repaid; thus $K_{T+1} \geq 0$. To satisfy both inequalities $K_{T+1} = 0$. Rearranging the previous equation yields the following lifetime budget constraint:

$$PV\{c\} = c_1 + \frac{c_2}{1+r} \cdots + \frac{c_T}{(1+r)^{T-1}} = y_1 + \frac{y_2}{1+r} \cdots + \frac{y_T}{(1+r)^{T-1}} = PV\{y\}.$$

The consumer's choice problem is therefore to maximize lifetime utility $U(c_1, \ldots, c_T)$, subject to the constraint that the present value of consumption cannot exceed the present value of income.

Two Periods

We analyze this problem in detail in Chapter 6. Here we focus on the simplest two-period case. The solution to the two-period optimization problem

$$\underset{c}{\text{Max}} \left\{ U(c) | c_1 + \frac{c_2}{1+r} \leq y_1 + \frac{y_2}{1+r} \right\}$$

is depicted in Figure 2.5-1 and Figure 2.5-2.

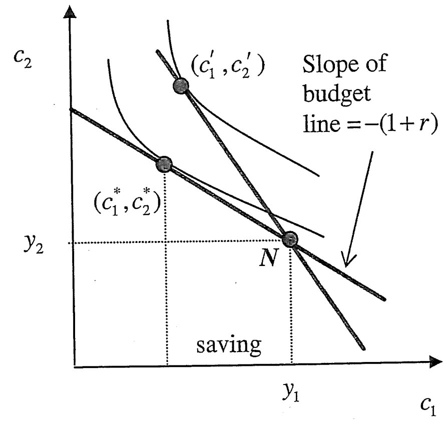

Figure 2.5-1. Optimal saving.

In Figure 2.5-1 the consumer saves part of her first-period income for period 2 when she is semi-retired and her income is lower. In Figure 2.5-2 a young consumer borrows in period 1 and plans to repay it in period 2 when her income will be higher.

From these figures we can glean some insights into the effects of an increase in the interest rate on aggregate saving and borrowing. In each case the substitution effect is around the indifference curve toward period 2 consumption. The higher interest rate lowers the present dollars needed to purchase a unit of future consumption. That is. it lowers the relative price of future consumption. However, the income effect is different for savers and borrowers. Holding period 1 consumption constant, the increased interest rate raises the future assets of savers so they achieve a higher utility level. Assuming that consumption goods are normal goods, the higher wealth leads to greater first-period consumption. That is, for first-period consumption, the income effect works against the substitution effect. This strongly suggests that the net effect on first-period consumption is likely to be small. But then the net effect on saving must also be small.

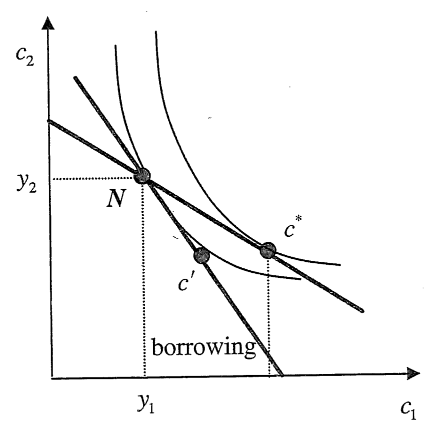

Figure 2.5-2. Optimal borrowing.

For borrowers, the higher interest rate makes it more costly to repay debt so they are worse off. The income effect on first-period consumption reinforces the consumption effect and so borrowing falls.

These observations fit the macro data well. Increases in interest rates do reduce consumer borrowing but have only small effects on aggregate consumer saving.

Futures and Future Spot Prices

We now extend the model to multiple commodities. For the key insight, it is enough to consider a two-period model. Let $x_t = (x_{t1}, \ldots, x_{tn})$, $t = 1, 2$ be the consumer's consumption bundle in period t. We consider an endowment economy. The consumer has a period t endowment of $\omega_t = (\omega_{t1}, \ldots, \omega_{tn})$. Her lifetime utility $U(x_1, x_2)$ is a function of her entire consumption bundle. To optimally trade in period 1, the consumer needs to know her trading opportunities in period 2.

Initially we assume that in period 1, there are markets both for commodities to be delivered in the current period and for commodities to be delivered in period 2. The latter are called "futures markets." (A wide variety of agricultural commodities are traded in such markets.) Given that there are prices for all $2n$ commodities, the consumer's endowment has a value of $W_1 = p_1 \cdot \omega_1 + p_2 \cdot \omega_2$. She then solves the following optimization problem:

$$\underset{x}{\text{Max}}\{U(x_1, x_2) | p_1 \cdot x_1 + p_2 \cdot x_2 \leq W_1\}.$$

From a mathematical perspective, this optimization problem is no different from a one-period problem in which the commodities have been partitioned into two disjoint subsets. However, from an economics perspective, clearly something is missing. All decisions are made in period 1 and all transactions are completed. Thus all stores can close their doors at the end of the first period!

Future Spot Markets and a Financial Intermediary

This is not the fatal flaw that it appears to be. Rather than introduce futures markets, suppose that there are no such markets. The consumer can trade in period 1 and knows that she can trade again in period 2. Her consumption choice is thus dependent upon the current prices (spot prices) and the prices that will prevail in the future (future spot prices). In the two-period model we let the future spot price vector be p_2^s. To move wealth between the two periods, the consumer utilizes a financial intermediary (the banking system). Let r be the interest rate offered by banks. As in the one-commodity case, if the value of the consumer's first period endowment $p_1 \cdot \omega_1$ exceeds the value of her consumption, she accumulates assets in period 2 of

$$K_2 = (1 + r)(p_1 \cdot \omega_1 - p_1 \cdot x_1).$$

Expressing this in present value terms,

$$\frac{K_2}{1+r} = p_1 \cdot \omega_1 - p_1 \cdot x_1.$$

In period 2, the consumer has both the value of her second period endowment and her assets to spend. Thus her future spot market budget constraint is

$$p_2^s \cdot x_2 \leq K_2 + p_2^s \cdot \omega_2.$$

Expressing this in present value terms and substituting for K_2,

$$p_1 \cdot x_1 + \frac{p_2^s}{1+r} \cdot x_2 \leq p_1 \cdot \omega_1 + \frac{p_2^s}{1+r} \cdot \omega_2$$

We now compare this with the budget constraint when it is possible to trade in spot and futures markets in period 1. This budget constraint is

$$p_1 \cdot x_1 + p_2 \cdot x_2 \leq p_1 \cdot \omega_1 + p_2 \cdot \omega_2.$$

Note that if $p_2^s = (1+r)p_2$, the two budget constraints are identical. Thus trading in futures markets can always be replicated by trading in spot and future spot markets. Resources are moved from one period to another by borrowing or lending.

There is one important difference between the two models. If all markets open in period 1, the equilibrating forces of the marketplace determine the prices at which commodities are traded. If only spot markets open, consumers (and firms) must form beliefs about the future spot prices in order to know how much to save or invest. If changes in a particular market are small, future spot prices will typically be similar to the current prices so forecasting is straightforward. However, if a market is subject to big shocks (such as drought or flood) the futures market plays an important role in price determination.

Exercises

Exercise 2.5-1: Choice over Time with Many Commodities A consumer has a quasi-concave utility function $U(x_1, \ldots, x_T)$, where x_t is the consumption vector in period t. Define

$$U^*(c_1, \ldots, c_T) = \underset{x}{\text{Max}}\{U(x) | p_t \cdot x_t \leq c_t, t = 1, \ldots, T\}.$$

(a) Show that $U^*(c_1, \ldots, c_T)$ is quasi-concave.
(b) Hence explain how the multi-commodity model can be analyzed as a one-commodity model.
(c) Suppose that $U(x) = \sum_{t=1}^{T} \delta^{t-1} u(x_t)$, where $u(x) = \sum_{j=1}^{n} \alpha_j \ln x_j$. Show that the indirect utility function $U^*(c) = \sum_{t=1}^{T} \delta^{t-1} \ln c_t +$ terms independent of c. Hence

explain why the individual's expenditure vector $(c_1^*, \ldots, c_T^*)$ is independent of future spot prices.

(d) Write down the life-cycle expenditure budget constraint. Then use the Lagrange method to show that $c_{t+1} = (1+r)^t \delta^t c_1$.

(e) Hence show that $c_1^* = \frac{1-\delta}{1-\delta^T} W_1$, where W_1 is the present value of the consumer's lifetime endowments.

Exercise 2.5-2: All-or-Nothing Consumer A consumer has utility function $U(x) = x_{11}^{\alpha} x_{12}^{1-\alpha} + \delta x_{21}^{\beta} x_{22}^{1-\beta}$, $0 < \alpha, \beta < 1$.

(a) Solve for the indirect utility function if the consumer must spend c_t in period t.
(b) Hence show that this consumer is always happiest only consuming in one period.
(c) Modify the utility function so ensure that it is strictly quasi-concave. Given your modification, would the consumer ever buy only one commodity in any period? Might the consumer still consume only in one period?

Exercise 2.5-3: Saving and Borrowing in an Exchange Economy with a Continuum of Consumers Consumers have Cobb-Douglas preferences $U(x, \alpha) = \alpha \ln x_1 + (1-\alpha) \ln x_2$, where x_t is consumption in period t. Consumers vary in their preference parameter α. We will refer to a consumer with parameter α as a type α consumer. There is a single commodity. Types are distributed continuously over the interval [0, 1]. Type α has density $f(\alpha) = 1$ over this interval so that the total mass of types is 1. Each type has the same income in each of two periods.

(a) If the interest rate is r, which types will be savers and which will be borrowers.
(b) Show that type α has a net saving of $s(\alpha) = \omega(1 - \frac{2+r}{1+r}\alpha)$.
(c) Solve for the total saving $S(r)$ by all types who save and the total borrowing $B(r)$ by all types who borrow. Depict $S(r)$ and $B(r)$ in a neat figure.

Exercise 2.5-4: Futures and Future Spot Prices A consumer has the two-period utility function

$$U(x_1, x_2) = \ln x_{11} + 2 \ln x_{21} + \delta(\ln x_{12} + 2 \ln x_{22}).$$

The period 1 (spot) market price vector is $p_1 = (p_{11}, p_{21})$ and the futures market price vector is $p_2 = (p_{12}, p_{22})$.

(a) Show that expenditure on each of the first- and second-period commodities is as follows.

$$(p_{11}x_{11}^*, p_{21}x_{21}^*, p_{12}x_{12}^*, p_{22}x_{22}^*) = \frac{p \cdot \omega}{3(1+\delta)}(1, 2, \delta, 2\delta)$$

(b) Obtain an expression for total spending in period 1.
(c) Hence show that the consumer will save if and only if $\frac{p_2 \cdot \omega_2}{p_1 \cdot \omega_1} < \delta$.

References and Selected Historical Reading

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