3

Equilibrium and Efficiency in an Exchange Economy

3.1 The 2 × 2 Exchange Economy

Key ideas: Pareto efficiency, Edgeworth Box diagram, first and second welfare theorems

In this chapter the focus shifts from the individual agent to the market. Specifically we examine the allocation that results if all economic agents are price takers and prices adjust until markets clear. Rather than attempt to bring firms and consumers into the analysis all at once, we focus here on equilibrium in which there is no production. Consumers have endowments of commodities that they may exchange. As we see later, the ideas developed here extend very directly to economies with production.

Even though this chapter focuses on equilibrium outcomes, it is helpful to keep in mind a possible adjustment process that might lead to equilibrium. Suppose that there is an auctioneer who calls out prices for each of the commodities. Consumers and firms respond with the demands that they would make at these prices. The auctioneer lowers prices in markets where there is excess supply and raises them in markets where there is excess demand. At a price-taking (Walrasian) equilibrium, all markets clear.

The central results are the two welfare theorems. The First welfare theorem formalizes Adam Smith's argument about the "invisible hand" of the market place. Under extremely weak assumptions we establish that a Walrasian equilibrium (WE) is efficient. The Second welfare theorem establishes that in a convex economy, any efficient allocation can be supported as a WE with appropriate lump-sum transfers.

In this first section we focus on a simple exchange economy in which there are two consumers, A (Alex) and B (Bev). The set $\mathsf{H} = \{A, B\}$ is then the set of consumers. Each consumer has an endowment of two commodities. Commodities are private. That is, each consumer cares only about his or her

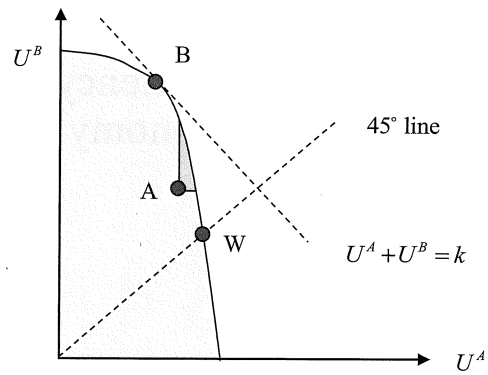

Figure 3.1-1. Utility possibility set.

own consumption. Consumer h, $h \in \mathsf{H}$ has an endowment ω^h, a consumption set $X^h = \mathbb{R}^2_+$ and a utility function $U^h(x^h)$ that is strictly monotonic.

Pareto Efficiency

With more than one consumer, the social ranking of allocations requires weighing the utility of one individual against that of another. Suppose that the set of possible utility pairs (the "utility possibility set") associated with all possible allocations of the two commodities is the shaded area depicted in Figure 3.1-1.

Setting aside the question of measuring utility, one philosophical approach to social choice places each individual behind a "veil of ignorance." Not knowing which consumer you are going to be, it is natural to assign a probability of $\frac{1}{2}$ to each possibility. If individuals are neutral toward risk while behind the veil of ignorance, they will prefer allocations with a higher expected utility

$$\tfrac{1}{2}U^A + \tfrac{1}{2}U^B.$$

This is equivalent to maximizing the sum of utilities, a proposal first put forth by Jeremy Bentham. In Figure 3.1-1, the Benthamite criterion picks the point B.

A more recent philosophical argument developed by John Rawls argues that behind the veil of ignorance consumers will be infinitely averse to risk and thus will place all weight on the worst possible outcome. The social criterion then become the Max Min criterion. Maximizing the minimum utility is achieved by moving out along the 45° line to the utility possibility frontier: the point W in the figure.

Economists tend to be agnostic when it comes to theorizing about social choice rankings. Instead they focus on minimizing unnecessary waste. The utility allocation A in Figure 3.1-1 is wasteful or "inefficient" because there are alternative allocations of goods that would make both individuals better off. Both the Benthamite and the Rawlsian allocations are said to be Pareto efficient (or simply "efficient") because the only way to raise the utility of one individual is by reducing the utility of the other. Generalizing to more than two individuals we have the following definition.

Pareto Efficient Allocation A feasible allocation of commodities is Pareto efficient (PE) if there is no other feasible allocation that is strictly preferred by at least one consumer and is weakly preferred by all consumers.

For the special 2 × 2 case, Alex and Bev must share the aggregate endowment $\omega = (\omega_1, \omega_2)$. Let $\hat{x}^B$ be the allocation to Bev and let $\hat{B}$ be the set of allocations that Bev prefers over $\hat{x}^B$. This is depicted in Figure 3.1-2. For any $x^B \in \hat{B}$, the allocation to Alex is $x^A = \omega - x^B$. Thus the best possible allocation to Alex that leaves Bev no worse off is Alex's utility maximizing allocation in $\hat{B}$.

Figure 3.1-3 shows Figure 3.1-2 rotated 180°. This depiction of the preferences of both consumers in a rectangle is called an *Edgeworth box* diagram. Note that at the bottom left corner Bev consumes the entire endowment. Therefore this is the zero consumption point for Alex. Also shown is the indifference curve for Alex through $\hat{x}^A = \omega - \hat{x}^B$. Any point in the

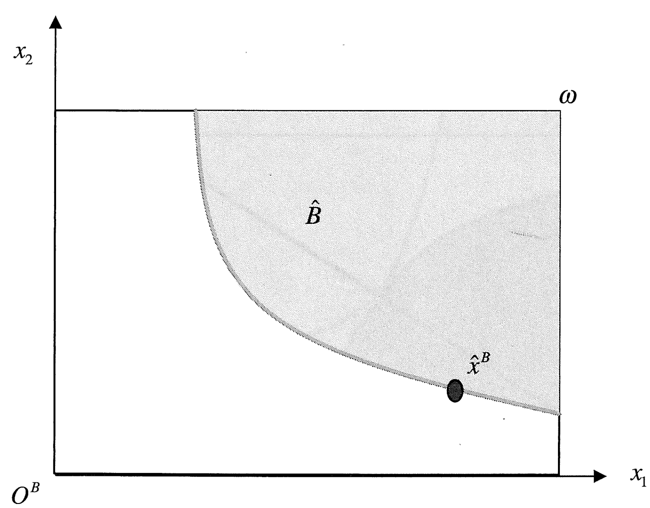

Figure 3.1-2. Bev's upper contour set.

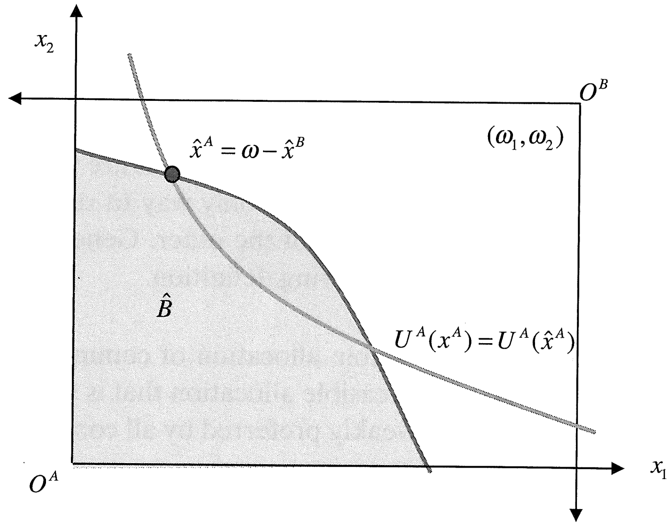

Figure 3.1-3. Edgeworth box diagram.

intersection of $\hat{B}$ and Alex's upper contour set $\{x|U^A(x^A) \geq U^A(\hat{x}^A)\}$ is strictly preferred by both Alex and Bev. Then the allocation depicted $\{\hat{x}^A, \hat{x}^B = \omega - \hat{x}^A\}$ is not PE.

For efficiency, there can be no such mutually preferred alternative. One such allocation is depicted in Figure 3.1-4. As long as an allocation $\{\hat{x}^A, \hat{x}^B = \omega - \hat{x}^A\}$ is in the interior of the Edgeworth box, a necessary condition for the allocation to be PE is that the slopes of the two indifference curves must be

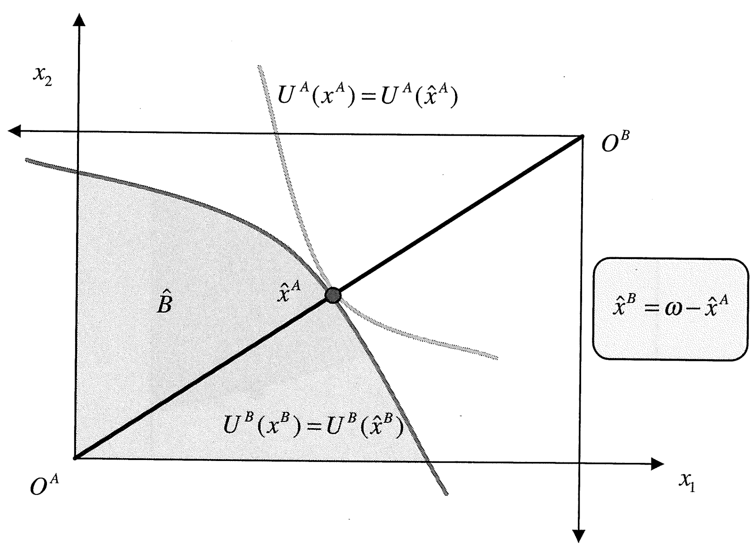

Figure 3.1-4. PE allocations with identical CES preferences.

equal. Thus the graph of the PE allocations is the set of allocations to Alex (and hence Bev) satisfying

$$\text{MRS}^A(\hat{x}^A) = \frac{\frac{\partial U^A}{\partial x_1}(\hat{x}^A)}{\frac{\partial U^A}{\partial x_2}(\hat{x}^A)} = \frac{\frac{\partial U^B}{\partial x_1}(\hat{x}^B)}{\frac{\partial U^B}{\partial x_2}(\hat{x}^B)}, \quad \text{where } \hat{x}^B = \omega - \hat{x}^A.$$

Mathematically, an allocation is Pareto efficient if

$$\hat{x}^A = \omega - \hat{x}^B \in \arg \underset{x^A, x^B}{\text{Max}} \left\{ U^A(x^A) | U^B(x^B) \geq U^B(\hat{x}^B), x^A + x^B \leq \omega \right\}.$$

Example: Identical Constant Elasticity of Substitution (CES) Preferences
If preferences are CES with elasticity of substitution σ, both consumers have a marginal rate of substitution, $\text{MRS}^h(x^h) = k(\frac{x_2^h}{x_1^h})^{1/\sigma}$. If a PE allocation is in the interior of the Edgeworth box, the indifference curves of the two consumers must have the same slope; that is,

$$\left(\frac{x_2^A}{x_1^A}\right)^{1/\sigma} = \left(\frac{x_2^B}{x_1^B}\right)^{1/\sigma} \quad \text{hence } \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}.$$

Appealing to the Ratio Rule[1] and then setting demand equal to supply,

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = \frac{x_2^A + x_2^B}{x_a^A + x_1^B} = \frac{\omega_2}{\omega_1}.$$

Thus, in a PE allocation each consumer is allocated a fraction of the aggregate endowment. It follows that for each consumer the marginal rate of substitution is

$$\text{MRS}^h(\hat{x}^h) = k\left(\frac{\omega_2}{\omega_1}\right)^{1/\sigma}. \tag{3.1-1}$$

The PE allocations are depicted in Figure 3.1-4.

Walrasian Equilibrium for an Exchange Economy

In a WE each consumer is a price taker. We write the set of consumers as H so in the two-person economy $\mathsf{H} = \{A, B\}$. Consumer $h \in \mathsf{H}$, with endowment ω^h has preferences represented by the utility function $U^h(x^h)$ where x^h is the private consumption of consumer h. The value of the consumer's endowment is $p \cdot \omega^h$ and so the consumer chooses a bundle of goods, $x^h(p, \omega^h)$, that solves

$$\underset{x}{\text{Max}}\{U^h(x) | p \cdot x \leq p \cdot \omega^h\}.$$

[1] If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$ then $a_1 = ka_2$ and $b_1 = kb_2$ and so $a_1 + b_1 = k(a_2 + b_2)$. Hence $\frac{a_1+b_1}{a_2+b_2} = k$.

Let $p \geq 0$ be a price vector of this exchange economy. Define $\omega = \sum_{h \in \mathsf{H}} \omega^h$ to be the vector of total endowments in the economy and $x(p) = \sum_{h \in \mathsf{H}} x^h(p, \omega^h)$ to be total (or "market") demand. Then the vector of excess demands is

$$z(p) = x(p) - \omega.$$

A market clears if either excess demand is zero or it is negative and the price of the commodity is zero.

Definition: Market-Clearing Prices Let $z_j(p)$ be the excess demand for commodity j at the price vector $p \geq 0$. The market for commodity j clears if $z_j(p) \leq 0$ and $p_j z_j(p) = 0$.

Walras' Law

We assume that the preferences of each consumer satisfy the local non-satiation axiom (discussed in Chapter 2). Given this axiom, each consumer must spend all of his income. To see this, we suppose instead that consumer h spends less than the value of his endowment and seek a contradiction. If the consumer's optimal consumption bundle x^h satisfies $p \cdot x^h < p \cdot \omega^h$, then for $\delta > 0$ and sufficiently small, the $\delta-$ neighborhood $N(x^h, \delta)$ lies in the budget set. Yet by local non-satiation, there must be some consumption bundle in $N(x^h, \delta)$ that is strictly preferred to x^h. However, if this is true, x^h cannot be optimal after all so we have a contradiction.

We now show that for any price vector p the market value of excess demands must be zero. First note that

$$p \cdot z(p) = p \cdot (x - \omega) = p \cdot \left(\sum_{h \in \mathsf{H}} (x^h - \omega^h) \right) = \sum_{h \in \mathsf{H}} (p \cdot x^h - p \cdot \omega^h).$$

Because all consumers spend their entire wealth the right-hand expression is zero. Hence

$$p_1 z_1(p) + p_2 z_2(p) = 0.$$

This is known as Walras' Law. Suppose that the first market clears, that is $p_1 z_1(p) = 0$. Then $p_2 z_2(p) = 0$ so market 2 must clear as well.[2] Note also that if one market does not clear, the other cannot clear either.

Definition: Walrasian Equilibrium The price vector $p \geq 0$ is a WE price vector if all markets clear.

For our two-commodity model, it follows from Walras' Law that we need to consider market clearing in only one market. Take any price vector

2 With H consumers and n commodities an identical argument establishes that $p \cdot z(p) = 0$. Thus if $n-1$ markets clear, then the remaining market must also clear.

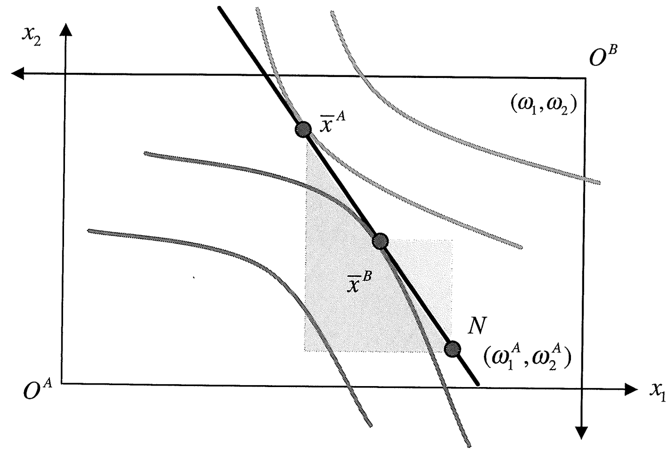

Figure 3.1-5. Excess supply of commodity 1.

$p = (p_1, p_2)$. Given an endowment allocation $\{\omega^A, \omega^B\}$, each consumer chooses his utility-maximizing consumption bundle.

It is helpful to depict trades in the Edgeworth box diagram. However there is one important caveat. From the viewpoint of each consumer, the budget set is the set of non-negative consumption bundles between the origin and the budget line. Consumers know nothing about the size of the Edgeworth box. In equilibrium all markets must clear. However, the demands of an individual consumer do not take into account aggregate supply constraints. For this reason, each side of the Edgeworth box is depicted as an axis for either Alex or Bev and indifference curves are depicted extending outside the box.

As depicted, in Figure 3.1-5, Alex wants to trade from the endowment point N to his most preferred desired consumption $\bar{x}^A$, whereas Bev wishes to trade from N to $\bar{x}^B$. Thus, there is excess supply of commodity 1.

By lowering the price of commodity 1 (relative to commodity 2) the budget line becomes less steep until eventually supply equals demand. The Walrasian equilibrium E is depicted in Figure 3.1-6.

Equilibrium and Efficiency

In Figure 3.1-6 the Walrasian equilibrium (WE) allocation is in the interior of the Edgeworth box. Thus the marginal rates of substitution must both be equal to the price ratio:

$$\mathrm{MRS}^A(\bar{x}^A) = \frac{\frac{\partial U^A}{\partial x_1}(\bar{x}^A)}{\frac{\partial U^A}{\partial x_2}(\bar{x}^A)} = \frac{p_1}{p_2} = \frac{\frac{\partial U^B}{\partial x_1}(\bar{x}^B)}{\frac{\partial U^B}{\partial x_2}(\bar{x}^B)} = \mathrm{MRS}^B(\bar{x}^B), \quad \text{where } \bar{x}^B = \omega - \bar{x}^A.$$

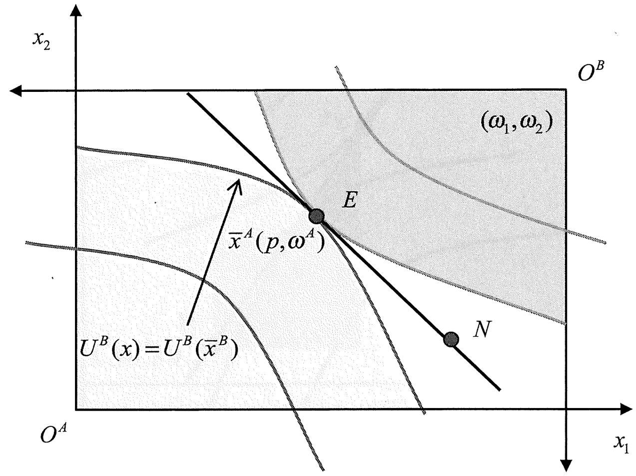

Figure 3.1-6. Walrasian equilibrium.

Comparing this condition with the necessary condition for an allocation to be PE, it follows that the WE allocation must be PE.

To prove that this result holds very generally, we appeal to the Duality Lemma. In Section 2.2 we argued that if the local non-satiation property holds, then the utility-maximizing bundle is cost minimizing among all preferred consumption bundles. Formally, if

$$\overline{x}^h = \arg \underset{x^h}{\text{Max}}\{ U^h(x^h) | p \cdot x^h \leq p \cdot \omega^h\},$$

then

$$p \cdot \overline{x}^h = \underset{x^h}{\text{Min}}\{ p \cdot x^h | U^h(x^h) \geq U^h(\overline{x}^h)\}.$$

With this observation, the proof that a WE is Pareto efficient is short and simple.

Proposition 3.1-2: First Welfare Theorem for an Exchange Economy
If preferences satisfy local non-satiation, a WE allocation in an exchange economy is PE.

Proof: Let $\{\overline{x}^h\}_{h \in \mathsf{H}}$ be a WE allocation for the exchange economy with endowments $\{\omega^h\}_{h \in \mathsf{H}}$. Let $p \geq 0$ be the WE price vector. Because $\overline{x}^h$ maximizes consumer h's budget constrained utility, any strictly preferred bundle x^h must cost strictly more, that is $p \cdot x^h > p \cdot \overline{x}^h$. Moreover, by the Duality Lemma (Lemma 2.2-3) any weakly preferred bundle x^h must cost at least as much as $\overline{x}^h$; that is $p \cdot x^h \geq p \cdot \overline{x}^h$.

Consider any allocation $\{x^h\}_{h\in\mathsf{H}}$ that is Pareto-preferred to $\{\overline{x}^h\}_{h\in\mathsf{H}}$. Because none of the consumers can be worse off in the Pareto-preferred allocation, it follows that

$$p \cdot x^h - p \cdot \overline{x}^h \geq 0, \quad h \in \mathsf{H}.$$

Moreover at least one consumer must be strictly better off. Thus

$$p \cdot x^h - p \cdot \overline{x}^h > 0 \quad \text{for some } h.$$

Summing over consumers,

$$p \cdot \left(\sum_{h\in\mathsf{H}} x^h - \sum_{h\in\mathsf{H}} \overline{x}^h \right) > 0.$$

Also all markets clear in a Walrasian equilibrium. Therefore

$$p \cdot \left(\sum_{h\in\mathsf{H}} \overline{x}^h - p \cdot \sum_{h\in\mathsf{H}} \omega^h \right) = 0.$$

Combining these results yields

$$p \cdot \left(\sum_{h\in\mathsf{H}} x^h - \sum_{h\in\mathsf{H}} \omega^h \right) > 0.$$

Because $p \geq 0$, it follows that there must be some commodity j such that $\sum_{h\in\mathsf{H}} x_j^h - \sum_{h\in\mathsf{H}} \omega_j^h > 0$. Thus all Pareto-preferred allocations are infeasible.[3] □

Clearly consumers' WE allocations depend on their initial endowments. Thus the outcome will change if a government intervenes and redistributes income. We now argue that, as long as preferences are convex, any PE allocation is also a WE allocation with the appropriate redistribution of resources.

We first sketch the argument for the two commodity case. Consider the PE allocation $\hat{x}^A, \hat{x}^B = \omega - \hat{x}^A$ in Figure 3.1-7. The indifference curves are tangential at $\hat{x}^A$, that is,

$$\frac{\frac{\partial U^A}{\partial x_1}(\hat{x}^A)}{\frac{\partial U^A}{\partial x_2}(\hat{x}^A)} = \frac{\frac{\partial U^B}{\partial x_1}(\hat{x}^B)}{\frac{\partial U^B}{\partial x_2}(\hat{x}^B)}.$$

Therefore, for some θ.

$$\frac{\partial U^A}{\partial x}(\hat{x}^A) = \theta \frac{\partial U^B}{\partial x}(\hat{x}^B).$$

[3] Note that the proof does not appeal to the assumption that the set of consumers is $\mathsf{H} = \{A, B\}$. It holds for any number of consumers.

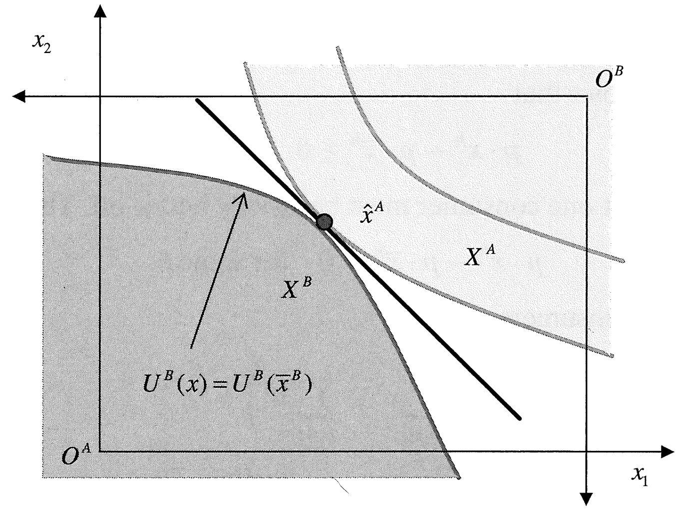

Figure 3.1-7. PE allocation $\{\hat{x}^A, \hat{x}^B = \omega - \hat{x}^A\}$.

Since the upper contour set $X^A = \{x^A | U^A(x^A) \geq U^A(\hat{x}^A)\}$ is convex, it follows from Lemma 1.1-2 that

$$U^A(x^A) \geq U^A(\hat{x}^A) \Rightarrow \frac{\partial U^A}{\partial x}(\hat{x}^A) \cdot (x^A - \hat{x}^A) \geq 0.$$

Making the same argument for Bev,

$$U^B(x^B) \geq U^B(\hat{x}^B) \Rightarrow \frac{\partial U^B}{\partial x}(\hat{x}^B) \cdot (x^B - \hat{x}^B) \geq 0.$$

Choose $p = \frac{\partial U^B}{\partial x}(\hat{x}^B)$. Then $\frac{\partial U^A}{\partial x}(\hat{x}^A) = \theta p$ and so

$$U^A(x^A) \geq U^A(\hat{x}^A) \Rightarrow p \cdot x^A \geq p \cdot \hat{x}^A$$

and

$$U^B(x^B) \geq U^B(\hat{x}^B) \Rightarrow p \cdot x^B \geq p \cdot \hat{x}^B.$$

Suppose that $U^A(x^A) > U^A(\hat{x}^A)$ and $p \cdot x^A = p \cdot \hat{x}^A$. Then, since U is strictly increasing and continuous, there exists $\delta \gg 0$ and sufficiently small such that $U^A(x^A - \delta) > U^A(\hat{x}^A)$ and $p \cdot (x^A - \delta) < p \cdot \hat{x}^A$. However, we have just argued that $U^A(x^A) \geq U^A(\hat{x}^A) \Rightarrow p \cdot x^A \geq p \cdot \hat{x}^A$, so this is impossible. Therefore

$$U^A(x^A) - U^A(\hat{x}^A) > 0 \Rightarrow p \cdot x^A > p \cdot \hat{x}^A.$$

Hence no allocation that Alex strictly prefers is in his budget set $\{x^A | p \cdot x^A \leq p \cdot \hat{x}^A\}$.

An identical argument holds for Bev. Because demand equals supply for each individual, all markets clear. Therefore, the vector p is a WE price vector.

Define the transfer payment $T^h = p \cdot (\hat{x}^h - \omega^h)$, $h \in \mathrm{H}$.

Because $\sum_{h\in\mathsf{H}} \hat{x}^h = \sum_{h\in\mathsf{H}} \omega^h$ the sum of these transfers is zero so this is a feasible redistribution of wealth. The budget constraint $p \cdot x^h \leq p \cdot \hat{x}^h$ can be rewritten as follows:

$$p \cdot x^h \leq p \cdot \omega^h + T^h.$$

Then given transfers T^h, $h \in \mathsf{H}$, the price vector p is a WE price vector.

This argument holds for PE allocations in which consumer $h \in \mathsf{H}$ has an allocation $\hat{x}^h \gg 0$. We now prove a more general result.

Proposition 3.1-3: Second Welfare Theorem for an Exchange Economy
In an exchange economy with endowments $\{\omega^h\}_{,h\in\mathsf{H}}$, suppose that $U^h(x)$, is continuously differentiable, quasi concave on $\mathbb{R}^2_+$ and that $\frac{\partial U^h}{\partial x^h}(x^h) \gg 0$, $h \in \mathsf{H}$. Then any PE allocation $\{\hat{x}^h\}_{h\in\mathsf{H}}$ where $\hat{x}^h \neq 0$, $h \in \mathsf{H}$, can be supported by a price vector $p \gg 0$.

Proof: We prove this for the two-person exchange economy.[4] If $\hat{x}^A, \hat{x}^B = \omega^A + \omega^B - \hat{x}^A$ is a PE allocation then

$$\hat{x}^A \in \arg\underset{x^A,x^B}{\text{Max}} \left\{U^A(x^A) | x^A + x^B \leq \omega^A + \omega^B,\ U^B(x^B) \geq U^B(\hat{x}^B)\right\}. \quad (3.3\text{-}2)$$

Let the aggregate endowment be ω. We begin by examining the feasible set of this optimization problem. Consider $\overline{x}^A = \delta \gg 0$ and $\overline{x}^B = \omega - \delta$. Because $U^B(\cdot)$ is strictly increasing and $0 < \hat{x}^B < \omega$, there exists some sufficiently small δ such that $U^B(\hat{x}^B) < U^B(\overline{x}^B) < U^B(\omega)$. Thus the feasible set has a non-empty interior. Because $U^B(\cdot)$ is quasi-concave the feasible set is convex. Also none of the constraint functions has a zero gradient. Thus by Proposition 1.2-4 the Constraint Qualification is satisfied. Then the Kuhn-Tucker conditions are necessary conditions.

The Lagrangian for the optimization problem (3.1-2) is

$$\mathfrak{L} = U^A(x^A) + \nu(\omega^A + \omega^B - x^A - x^B) + \mu(U^B(x^B) - U^B(\hat{x}^B)).$$

Differentiating we obtain the following Kuhn-Tucker conditions:

$$\frac{\partial \mathfrak{L}}{\partial x^A} = \frac{\partial U^A}{\partial x^A}(\hat{x}^A) - \nu \leq 0, \quad \text{where} \quad \hat{x}^A\left(\frac{\partial U^A}{\partial x^A}(\hat{x}^A) - \nu\right) = 0. \quad (3.1\text{-}3)$$

$$\frac{\partial \mathfrak{L}}{\partial x^B} = \mu\frac{\partial U^B}{\partial x^A}(\hat{x}^B) - \nu \leq 0, \quad \text{where} \quad \hat{x}^B\left(\mu\frac{\partial U^B}{\partial x^B}(\hat{x}^B) - \nu\right) = 0. \quad (3.1\text{-}4)$$

$$\frac{\partial \mathfrak{L}}{\partial \nu} = \omega^A + \omega^B - \hat{x}^A - \hat{x}^B \geq 0, \quad \text{where} \quad \nu(\omega^A + \omega^B - \hat{x}^A - \hat{x}^B) = 0. \quad (3.1\text{-}5)$$

[4] The reader is left to check that the proof extends directly to an exchange economy with more than two consumers and more than two commodities.

Because $\frac{\partial U^A}{\partial x^A} \gg 0$ it follows from (3.1-3) that $\nu \gg 0$. From (3.1-5) it then follows that

$$\omega^A + \omega^B - \hat{x}^A - \hat{x}^B = 0. \tag{3.1-6}$$

Because $\hat{x}^B > 0$ and $\frac{\partial U^B}{\partial x^B} \gg 0$ it follows from (3.1-4) that $\mu > 0$.

Now consider an economy with endowments $\hat{\omega}^h = \hat{x}^h$, $h \in \mathsf{H}$ and consider the price vector $p = \nu$. Consumer h chooses

$$\overline{x}^h = \arg \operatorname*{Max}_{x^h}\{U^h(x^h) | \nu \cdot x^h \leq \nu \cdot \hat{x}^h\}.$$

The FOC for this optimization problem is

$$\frac{\partial \mathfrak{L}}{\partial x^h} = \frac{\partial U^h}{\partial x^h}(\overline{x}^h) - \lambda^h \nu \leq 0, \quad \text{where} \quad \overline{x}^h \left(\frac{\partial U^h}{\partial x^h}(\overline{x}^h) - \lambda^h \nu \right) = 0.$$

Moreover, because $U^h(\cdot)$ is quasi-concave the FOC is also sufficient. Choose $\lambda^A = 1$ and $\lambda^B = 1/\mu$. Then, appealing to (3.1-3) and (3.1-4), the FOC hold at $\overline{x}^h = \hat{x}^h$, $h \in \mathsf{H}$.

Thus at the price $p = \nu$ no consumer wishes to trade. Therefore supply equals demand and so the price vector is a WE price vector.

Finally define transfers $T^h = \nu \cdot (\hat{x}^h - \omega^h)$. Appealing to (3.1-6), the sum of these transfers is zero. Consumer h's budget constraint with these transfers is

$$\nu \cdot x^h \leq \nu \cdot \omega^h + T^h = \nu \cdot \hat{x}^h.$$

Thus the PE allocation is achievable as a WE with the appropriate transfer payments among consumers. □

Example: Quasi-Linear Preferences

Alex has a utility function $U^A = x_1^A + \ln x_2^A$ whereas Bev has a utility function $U^B = x_1^B + 2 \ln x_2^B$.

At a PE allocation in the interior of the Edgeworth box the marginal rates of substitution must be equal. In this special case $\text{MRS}^A = x_2^A$ and $\text{MRS}^B = \frac{1}{2}x_2^B$. Thus, for an interior allocation in the Edgeworth box to be PE, it must be the case that $x_2^A = \frac{1}{2}x_2^B$.

Also total consumption $x_2^A + x_2^B = \omega_2$. Solving these two equations, it follows that for an allocation in the interior of the Edgeworth box to be PE, $x_2^A = \frac{1}{3}\omega_2$ and $x_2^B = \frac{2}{3}\omega_2$.

The Pareto efficient allocations are shown as heavy horizontal line segments in Figure 3.1-8. For any efficient allocation in the interior of the box, the marginal rate of substitution $\text{MRS}^A(x^A) = x_2^A = \frac{1}{3}\omega_2$. Thus the supporting prices are the same for all these allocations.

Consider the Pareto efficient allocation $(\hat{x}_1^A, \hat{x}_2^A)$ on the boundary of the Edgeworth box. Suppose that this is also the endowment point. From the

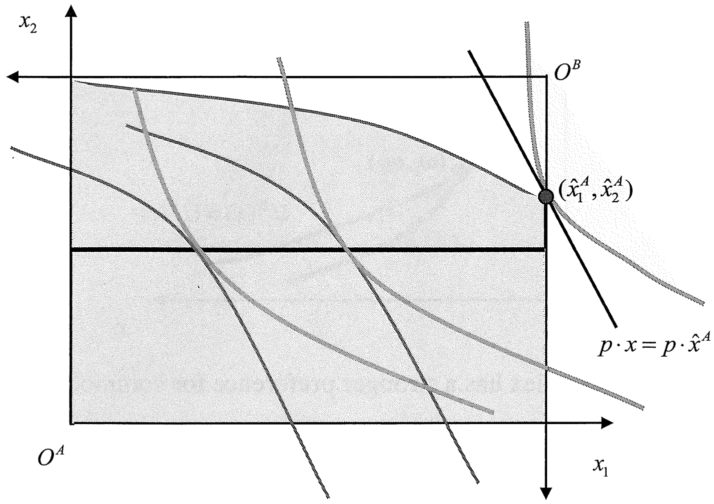

Figure 3.1-8. PE allocations with quasi-linear preferences.

First welfare theorem, if there is a Walrasian equilibrium then it must be PE. Thus the only possible equilibrium is a no-trade equilibrium. Consider the budget line through $\hat{x}^A = (\hat{x}_1^A, \hat{x}_2^A)$. This is in the interior of Alex's consumption set thus it must be tangential to his indifference curve. This determines the equilibrium price ratio.

The two welfare theorems offer a useful indirect way to analyze equilibrium. Often it is easier to characterize the PE allocations of an economy than to compute Walrasian equilibria. To illustrate this, we examine the effects on WE prices of a change in the distribution of wealth.

Homothetic Preferences

Suppose that the two individuals in the economy (Alex and Bev) have different convex and homothetic preferences. At the aggregate endowment, (ω_1, ω_2), Alex has a stronger preference for commodity 1 than Bev. That is, Alex is willing to give up more units of commodity 2 than Bev in exchange for an additional unit of commodity 1.

Assumption: Intensity of Preferences
At the aggregate endowment, Alex has a stronger preference for commodity 1 than Bev

$$\text{MRS}_A(\omega_1, \omega_2) = \frac{\partial U^A}{\partial x_1} \Big/ \frac{\partial U^A}{\partial x_2} > \frac{\partial U^B}{\partial x_1} \Big/ \frac{\partial U^B}{\partial x_2} = \text{MRS}_B(\omega_1, \omega_2). \quad (3.1\text{-}7)$$

This is depicted in Figure 3.1-9.

We now explore the implications of this assumption on the Pareto efficient allocations. Consider the Edgeworth box diagram shown in Figure 3.1-10.

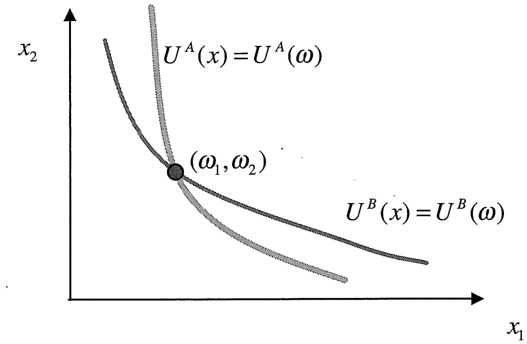

Figure 3.1-9. Alex has a stronger preference for commodity 1.

First note that along the dotted diagonal line $(x_1^h, x_2^h) = \theta^h \omega$. By hypothesis, preferences are homothetic. Because Alex places a higher value on commodity 1,

$$\mathrm{MRS}_A(\theta^A \omega) > \mathrm{MRS}_B(\theta^A \omega) = \mathrm{MRS}_B(\theta^B \omega).$$

It follows that the Pareto efficient allocations must lie below the diagonal.

Let C be an efficient allocation and C' be a second such allocation preferred by Alex. In Figure 3.1-10, C' must lie to the northeast of C. Intuitively, because Alex's consumption is higher and Bev's is lower at C', the marginal rate of substitution at C' will be higher, reflecting the greater influence of Alex's stronger preference for commodity 1.

To confirm this, let the MRS at C be m. Given homothetic preferences, for any point above the line $O_A D$, $\mathrm{MRS}^A(x_1, x_2) > m$. Also, for any point above the line $O_B F$, $\mathrm{MRS}^B(x_1, x_2) < m$. Hence in the upper shaded region

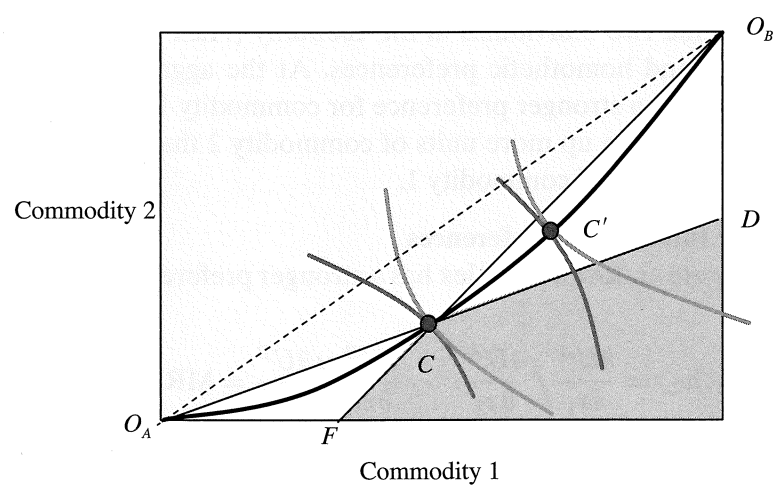

Figure 3.1-10. Pareto efficient.

Alex has a MRS exceeding m, while Bev has a MRS less than m. It follows that no such point can be Pareto efficient. A symmetric argument establishes that allocations in the lower shaded region are also not Pareto efficient.

Consider the efficient allocation C' to the northeast of C. Because this point lies below $O_A D$ and above $O_B F$, it follows that

$$\left.\frac{x_2^h}{x_1^h}\right|_C < \left.\frac{x_2^h}{x_1^h}\right|_{C'}, \quad h \in \mathsf{H}, \tag{3.1-8}$$

and

$$\text{MRS}^h(C') > m = \text{MRS}^h(C), \quad h \in \mathsf{H}. \tag{3.1-9}$$

It follows that, on the map of Pareto efficient allocations between C to C', the marginal rate of substitution increases. Hence the supporting prices must have the property that the relative price of commodity 1 rises. Thus the greater the wealth of Alex relative to Bev, the higher will be the relative price of commodity 1. Because we refer to these results later, we summarize them below.

Proposition 3.1-4: Pareto Efficient Allocations with Homothetic Preferences In the 2×2 exchange economy, suppose both consumer A and B have homothetic preferences. Suppose also that at the aggregate endowment, consumer A has a stronger preference for commodity 1. Then at any interior efficient allocation,

$$\frac{x_2^A}{x_1^A} < \frac{x_2^B}{x_1^B}.$$

Moreover, along the locus of efficient allocations, as consumer A's utility rises, the consumption ratio x_2^h/x_1^h and marginal rate of substitution of x_1 for x_2 of both consumers rise.

Exercises

Exercise 3.1–1: Prices with Quasi-Linear Preferences Consider a two-person economy in which the aggregate endowment is $(\omega_1, \omega_2) = (100, 200)$ and both have the same quasi-linear utility function $U(x^h) = x_1^h + \sqrt{x_2^h}$.

(a) Solve for the Walrasian equilibrium price ratio under the assumption that the equilibrium consumption of commodity 1 is positive for both individuals.
(b) What is the range of possible equilibrium price ratios in this economy?

Exercise 3.1-2: Pareto Efficient Allocations

(a) If U^A and U^B are strictly increasing, explain why the allocation $\{\hat{x}^A, \hat{x}^B\} = \{\omega^A + \omega^B, 0\}$ is a PE and WE allocation.

Suppose that $U^A = x_1^A + 10\ln x_2^A$ and $U^B = \ln x_1^B + x_2^B$. The aggregate endowment is $\omega = (20, 10)$.

(b) Show that the PE allocations in the interior of the Edgeworth box can be expressed in the form $\hat{x}_2^A = f(\hat{x}_1^A)$.
(c) Suppose that $\omega_2^A = f(\omega_1^A)$. How does the equilibrium price ratio change as ω_1^A increases along this curve?
(d) Which allocations on the boundary of the Edgeworth box are PE allocations?

Exercise 3.1-3: Walrasian equilibrium Suppose half the population (the Biggs) each have an endowment (24, 8) and the other half (the Littles) each have an endowment (20, 10). Each Mr. Bigg has a utility function

$$U^B = \ln x_1^B + \ln x_2^B.$$

Each Ms. Little has a utility function

$$U^L = \ln\left(4 + x_1^L\right) + \ln\left(6 + x_2^L\right).$$

(a) By solving for and then aggregating individual demands, solve for the equilibrium price ratio.
(b) Solve also for the contract curve and depict it in a neat Edgeworth box diagram showing the bottom left-hand corner as the zero consumption point for a representative Ms. Little and the top right-hand corner as the zero consumption point for a representative Mr. Bigg.
(c) Explain carefully why the equilibrium price will not change if endowments are reallocated in favor of the Biggs.
(d) What will be the equilibrium price ratio if the Littles have an endowment of (8, 0) while the Biggs have an endowment of (36, 18)?
(e) What if the Littles have an endowment of (2, 0) and the Biggs have an endowment of (42, 18)?

Exercise 3.1-4: Linear Preferences Consider a 2 person economy in which Alex's preferences are represented by the utility function $U^A(x) = 2x_1 + x_2$, while Bev's preferences are represented by the utility function $U^B(x) = x_1 + 2x_2$. The total endowment is (30, 20).

(a) Characterize the PE allocations and depict them in an Edgeworth box.

Show that if Alex has a sufficiently large fraction of the total endowment the equilibrium price ratio $p_1/p_2 = 2$. What if Bev has a large fraction of the total endowment?

(b) For what endowments will the price ratio lie between these two extremes? Characterize the Walrasian equilibrium.
(c) Show that for some endowments a transfer of wealth from Alex to Bev has no effect on prices. Also show that for other endowments there is no effect on the Walrasian equilibrium allocation.

Exercise 3.1-5: More on the Biggs and Littles Suppose that the Biggs only like commodity 1, whereas the Littles only like commodity 2. Depict the PE allocations in a neat Edgeworth box diagram. Is there a Walrasian equilibrium? If so depict it.

Exercise 3.1-6: Market Excess Demand Consider the following two-person exchange economy. Alex and Bev each has a consumption set $X = \{x | x \geq (2, 2)\}$. Alex has a consumption utility function $U^A = (x_1^A - 2)^5(x_2^A - 2)$ and endowment $\omega^A = (7 + \alpha, 1 - \alpha)$. Bev has a utility function $U^B = (x_1^B - 2)(x_2^B - 2)^5$ and endowment $\omega^B = (1 - \alpha, 7 + \alpha)$. The parameter $\alpha \in [0, 1]$.

(a) Show that for both consumers to be able to purchase a bundle in their consumption sets the price ratio must satisfy

$$\frac{5+\alpha}{1+\alpha} \geq \frac{p_1}{p_2} \geq \frac{1+\alpha}{5+\alpha}.$$

(b) Solve for each consumer's demand for commodity 1.

(c) Hence show that the market excess demand function for commodity 1 is

$$z_1(p) = -\frac{\alpha}{3}\left(\frac{p_2}{p_1} - 1\right).$$

(d) For what values of α does the excess demand for commodity 1 increase as the price of commodity 1 increases? Provide some intuition for this paradoxical result. HINT: Consider the income effects of an increase in p_1.

(e) If $\alpha = 0$ characterize the Walrasian equilibrium prices and allocations.

3.2 The Fundamental Welfare Theorems

Key ideas: Walrasian equilibrium, First and Second welfare theorems

We now consider an exchange economy with an arbitrary number of commodities and consumers. As we see, the insights gleaned from the two-person two-commodity economy generalize. Commodities are private, that is, consumer $h \in \mathsf{H} = \{1, \ldots, H\}$ has preferences over his own consumption vector $x^h = (x_1^h, \ldots, x_n^h)$ and not over those of other consumers. Let $X^h \subset \mathbb{R}^n$ be the consumption set of consumer h. That is, preferences are defined over X^h. We assume that consumer h has an endowment vector $\omega^h \in X^h$. A consumption allocation in this economy $\{x^h\}_{h \in \mathsf{H}}$ is an allocation of consumption bundles $x^h \in X^h$, $h \in \mathsf{H}$. The aggregate consumption in the economy is the sum of the individual consumption vectors $x = \sum_{h \in H} x^h$. Similarly the aggregate endowment is $\omega = \sum_{h \in H} \omega^h$.

Feasible Allocation

An allocation is feasible if the sum of the consumption vectors of all the consumers does not exceed the aggregate endowment

$$x - \omega \leq 0.$$

Pareto Efficient Allocation

A feasible plan for the economy, $\{\hat{x}^h\}_{h \in H}$, is Pareto efficient if there is no other feasible plan that is strictly preferred by at least one consumer and weakly preferred by all consumers.

Price Taking

Let $p \geq 0$ be the price vector. Consumers are price takers. Consumer h has an endowment ω^h.

She chooses a consumption bundle $\overline{x}^h$ in her budget set $\{x^h \in X^h | p \cdot x^h \leq p \cdot \omega^h\}$.

Walrasian Equilibrium

Consumer h chooses a most preferred consumption plan $\overline{x}^h$ in her budget set. That is,

$$U^h(\overline{x}^h) \geq U^h(x^h), \quad \text{for all} \quad x^h \text{ such that } p \cdot x^h \leq p \cdot \omega^h.$$

Let $\overline{x} = \sum \overline{x}^h$ be the total consumption of the consumers. Excess demand is then

$$\overline{z} = \overline{x} - \omega.$$

Definition: Walrasian Equilibrium Prices The price vector $p \geq 0$ is a Walrasian equilibrium price vector if there is no market in excess demand ($\overline{z} \leq 0$) and $p_j = 0$ for any market in excess supply ($\overline{z}_j < 0$).

Welfare Theorems

The proof of the First welfare theorem is exactly the same as for the two-commodity case examined in Section 3.1. In the proof of the Second welfare theorem for the two-person exchange economy we assumed quasi-concavity and differentiability of utility functions and appealed to the Kuhn-Tucker conditions. Here we drop the differentiability assumption and appeal directly to the Supporting Hyperplane Theorem.[5]

The allocation $\{\hat{x}^h\}_{h\in\mathsf{H}}$ is PE if there is no alternative allocation that increases the utility of consumer 1 without lowering the utility of at least one other agent. Thus $\{\hat{x}^h\}_{h\in\mathsf{H}}$ must solve the following optimization problem:

$$\underset{\{x^h\}_{h\in\mathsf{H}}}{\text{Max}}\left\{U^1(x^1)|U^h(x^h)\geq U^h(\hat{x}^h),\, h=2,\ldots,H, \sum_{h\in\mathsf{H}}(\omega^h-x^h)\geq 0,\, x^h\in\mathbb{R}^n_+\right\}.$$

Define $V^1(x)$ to be the maximum utility for consumer 1 given an aggregate supply of x, that is,

$$V^1(x)=\underset{\{x^h\}_{h=1}^H}{\text{Max}}\left\{U^1(x^1)|U^h(x^h)\geq U^h(\hat{x}^h),\right.$$
$$\left. h=2,\ldots,H,\, x-\sum_{h=1}^H x^h\geq 0\right\}. \tag{3.2-1}$$

Note that $\{\hat{x}^h\}_{h\in\mathsf{H}}$ solves this optimization problem if $x=\omega$.

Lemma 3.2-1: Quasi-Concavity of $V^1(\cdot)$ If U^h, $h\in\mathsf{H}$ is quasi-concave then so is the indirect utility function $V^1(\cdot)$.

Proof: Consider the aggregate endowments a and b. We must show that for any convex combination $c=(1-\lambda)a+\lambda b$, $V^1(c)\geq \text{Min}\{V^1(a), V^1(b)\}$. Suppose that the allocation$\{a^h\}_{h\in\mathsf{H}}$ solves the optimization problem (3.2-1), with aggregate endowment a and that $\{b^h\}_{h\in\mathsf{H}}$ solves the optimization problem when the aggregate endowment is b. Then $V^1(a)=U^1(a^1)$ and $V^1(b^1)=U^1(b^1)$. We must then show that $V^1(c)\geq\text{Min}\{V^1(a), V^1(b)\}$.

Because $\{a^h\}_{h\in\mathsf{H}}$ is feasible with a and $\{b^h\}_{h=1}^H$ is feasible with b, the convex combination $\{c^h\}_{h\in\mathsf{H}}$ is feasible with aggregate endowment $c=(1-\lambda)a+\lambda b$. Then $V^1(c)\geq U^1(c^1)$.

Appealing to quasi-concavity, $U^1(c^1)\geq\text{Min}\{U^1(a^1), U^1(b^1)\}$. Combining the last two inequalities, $V^1(c)\geq\text{Min}\{U^1(a^1), U^1(b^1)\}=\text{Min}\{V^1(a), V^1(b)\}$. □

Proposition 3.2-2: Second Welfare Theorem for an Exchange Economy Consumer $h\in\mathsf{H}$ has an endowment $\omega^h\in\mathbb{R}^n_+$. The consumption set for each individual X^h is the positive orthant $\mathbb{R}^n_+$. Suppose also that utility functions $U^h(\cdot)$, $h\in\mathsf{H}$ are continuous, quasi-concave and strictly increasing. If $\{\hat{x}^h\}_{h\in\mathsf{H}}$

[5] As we see in Chapter 5, this proof can be easily modified to include production.

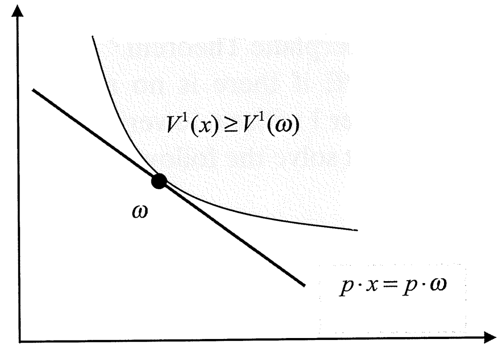

Figure 3.2-1. Supporting hyperplane.

where $\hat{x}^h \neq 0$,[6] $h \in \mathsf{H}$ is a PE allocation, then there exists a price vector $p > 0$ such that

$$U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h, \quad h \in \mathsf{H}.$$

Proof: Define

$$V^1(x) = \underset{\{x^h\}_{h=1}^H}{\mathrm{Max}} \left\{ U^1(x^1) | U^h(x^h) \geq U^h(\hat{x}^h), h = 2, \ldots, H, x - \sum_{h=1}^{H} x^h \geq 0 \right\}.$$

Appealing to Lemma 3.2-1, $V^1(\cdot)$ is quasi-concave. Also $V^1(\cdot)$ is strictly increasing because $U^1(\cdot)$ is strictly increasing and any increment in the aggregate supply can be allocated to the first consumer.

We have already noted that $\{\hat{x}^h\}_{h \in \mathsf{H}}$ is the solution of this optimization problem if $x = \omega$.

Moreover, because $U^1(\cdot)$ is strictly increasing,

$$\sum_{h=1}^{H} \hat{x}^h = \omega. \tag{3.2-2}$$

Because ω is on the boundary of the set $\{x | V^1(x) \geq V^1(\omega)\}$, it follows from the Supporting Hyperplane Theorem that there is a vector $p \neq 0$, such that

$$V^1(x) > V^1(\omega) \Rightarrow p \cdot x > p \cdot \omega \quad \text{and} \quad V^1(x) \geq V^1(\omega) \Rightarrow p \cdot x \geq p \cdot \omega. \tag{3.2-3}$$

The supporting line through the aggregate endowment vector is depicted in Figure 3.2-1.

[6] If there are M consumers with a zero allocation, we set these aside and appeal to the theorem for the H-M consumers with non-zero allocations. Because all feasible allocations are strictly preferred to the zero allocation, the theorem then extends immediately.

We now argue that the vector p must be positive. If not, define $\delta = (\delta_1, \ldots, \delta_n) > 0$ such that $\delta_j > 0$ if and only if $p_j < 0$. Then $V^1(\omega + \delta) > V^1(\omega)$ and $p \cdot (\omega + \delta) < p \cdot \omega$. But this contradicts (3.2-3) so p must be positive after all.

From (3.2-3) and the definition of the indirect utility function,

$$U^h(x^h) \geq U^h(\hat{x}^h), \quad h = 1, \ldots, H \Rightarrow p \cdot x = p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \omega. \quad (3.2\text{-}4)$$

Substituting for ω from (3.2-2) it follows that

$$U^h(x^h) \geq U^h(\hat{x}^h), \quad h = 1, \ldots, H \Rightarrow p \cdot \sum_{h=1}^{H} x^h \geq p \cdot \sum_{h=1}^{H} \hat{x}^h.$$

Setting $x^k = \hat{x}^k$, $k \neq h$, we may then conclude that for consumer h,

$$U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h. \quad (3.2\text{-}5)$$

It remains to show that any strictly preferred bundle costs strictly more. Suppose instead that $U^h(x^h) > U^h(\hat{x}^h)$ and $p \cdot x^h = p \cdot \hat{x}^h$. Then for all $\lambda \in (0, 1)$, $p \cdot \lambda x^h < p \cdot \hat{x}^h$. Also because $U^h(\cdot)$ is continuous, for all λ sufficiently close to 1, $U^h(\lambda x^h) > U^h(\hat{x}^h)$. But these two inequalities contradict (3.2-5). Hence

$$U^h(x^h) > U^h(\hat{x}^h) \Rightarrow p \cdot x^h > p \cdot \hat{x}^h. \qquad \square$$

Exercise 3.2-1: Walrasian Equilibrium with Identical Homothetic Preferences Suppose each consumer has a consumption set $\mathbb{R}^n_+$ and the same strictly increasing, quasi-concave, homothetic utility function $U \in \mathbb{C}^1$. Characterize Walrasian equilibrium prices.

References and Selected Historical Reading

Edgeworth, Francis Y. (1881). *Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences*. London: C. K. Paul.

Pareto, Vilfredo. (1909). *Manuele di economia politica*. Trans. Augustus M. Kelley, 1971.

Rawls, John. (1971). *A Theory of Justice*. Cambridge, MA: Belknap Press.

Walras, Leon. (1874). *Éléments d'économie politique pure, ou théorie de la richesse sociale*. Lausanne: F. Rouge Editeur.