5

General Equilibrium

5.1 The Robinson Crusoe Economy

Key ideas: Walrasian equilibrium allocation, optimal allocation, invisible hand at work

In Chapter 3 we studied equilibrium and efficiency in exchange economies. In this chapter we add firms to the economy and show how the welfare theorems generalize. In this introductory section we consider the simplest such economy – the one-person economy. To introduce trade, we assume that the single individual Robinson Crusoe is schizophrenic, making his decisions as a manager and a consumer separately. His decisions are guided by market prices. In Section 5.2 we examine a general equilibrium model with production and extend the two welfare theorems.

The power of the first theorem hinges critically on the assumption that a Walrasian equilibrium exists. This is addressed in Section 5.3. In the basic model all goods are private. In Sections 5.4 and 5.5 we show how the basic model can be extended to incorporate multiple time periods and public goods.

Although the two welfare theorems are both elegant and fundamental, they yield little insight into how underlying tastes and technology affect prices. For this we need to look at much simpler general equilibrium models. We illustrate this approach in Section 5.6 where we examine a two-commodity model with constant returns to scale.

As a first example, we consider an economy in which there are two commodities and each consumer has the same homothetic utility function $U(x^h), h = 1, \ldots, H$. Commodity 1 is consumption of hours (leisure time) and commodity 2 is corn. Each consumer has an endowment of hours and corn. Consumers must decide how many hours to consume and how many hours to sell in the labor market. From the discussion of consumers in Chapter 2, we know that with identical homothetic preferences market demand is

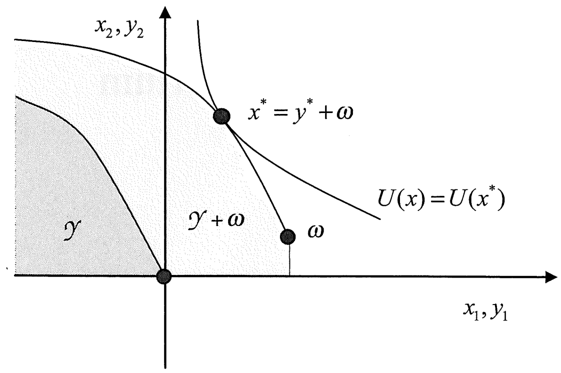

Figure 5.1-1. The optimum.

equal to the demand of a single consumer who holds the aggregate endowment $\omega = \sum_{h=1}^{H} \omega^h$. We call this representative agent Robinson Crusoe. There is a single firm in the economy and it has a convex production set $\mathcal{Y}$. Commodity 1 is an input in the production of commodity 2. Thus, as depicted in Figure 5.1-1, y_1 is negative and y_2 is positive.

At any point in time, Robinson Crusoe wears only one hat. As Robinson he wears his manager hat, deciding how much output to produce. As Crusoe he wears his consumer hat, deciding how many hours to work and how much of commodity 2 to consume, given his endowment and the dividends of the firm. Market prices then guide his consumption and production choices.

The Optimum

If Robinson Crusoe, with endowment vector ω, chooses the production plan y, his consumption vector of leisure and corn is $x = y + \omega$. The set of feasible consumption bundles is therefore the set $\mathcal{Y} + \omega$. This set is also depicted in Figure 5.1-1. Robinson Crusoe then chooses the consumption bundle x^* that maximizes his utility from the bundles in the set $\mathcal{Y} + \omega$. Formally, x^* solves the following maximization problem:

$$\underset{x}{\text{Max}}\{U(x) | x \in \mathcal{Y} + \omega\}.$$

Example:

$$\mathcal{Y} = \left\{(y_1, y_2) | y_1 \leq 0,\ y_2^2 + y_1 \leq 0\right\}, \quad U(x) = \ln x_1 + \ln x_2, \quad \omega = (144, 3).$$

We substitute for $x = y + \omega$ and write utility as $U(y + \omega) = \ln(\omega_1 + y_1) + \ln(\omega_2 + y_2)$. Because utility is increasing, the optimum must be on the boundary of the production set so that $y_1 = -y_2^2$. Substituting for ω and y_1, $U = \ln(144 - y_2^2) + \ln(3 + y_2)$.

FOC:

$$\frac{dU}{dy_2} = \frac{-2y_2}{144 - y_2^2} + \frac{1}{3 + y_2} = \frac{144 - 6y_2 - 3y_2^2}{(144 - y_2^2)(3 + y_2)} = \frac{3(6 - y_2)(8 + y_2)}{(144 - y_2^2)(3 + y_2)} = 0$$

Note that the slope is positive for $y_2 < 6$ and negative for $y_2 > 6$. Thus U takes on its maximum at $y_2 = 6$. Hence, $y^* = (-36, 6)$ and $x^* = y^* + \omega = (108, 9)$.

Walrasian Equilibrium

Robinson the Price-Taking Manager

Let us now consider what Robinson would do as a price-taking manager of the firm. Given the price vector p the profit of the firm is then $p \cdot y$. The profit is distributed to shareholders in proportion to their ownership shares. The larger the dividend, the better off are all the shareholders, so the owners' interests are best served if profit is maximized.

Robinson then chooses a production plan $y(p)$ such that

$$\Pi(p) = p \cdot y(p) = \underset{y}{\text{Max}}\{p \cdot y | y \in \mathcal{Y}^f\}.$$

The profit-maximizing production plan $y(p)$ is depicted in Figure 5.1-2.

Example (continued):
Robinson the manager solves the following maximization problem: $\underset{y}{\text{Max}}\{p \cdot y | y \in \mathcal{Y}\}$, that is,

$$\underset{y}{\text{Max}}\{p \cdot y | y_1 \leq 0, \; y_1 + (y_2)^2 \leq 0\}.$$

For an optimum the constraint must be binding. Then substituting for y_1, profit is

$$\pi(y_2) = -p_1(y_2)^2 + p_2 y_2.$$

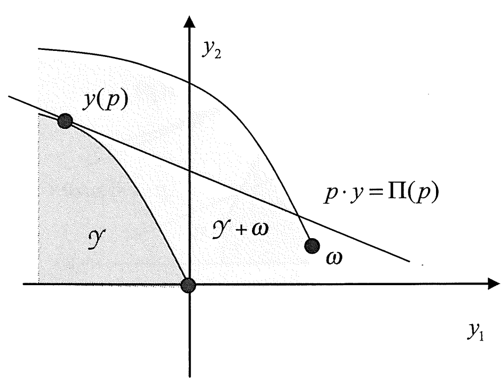

Figure 5.1-2. Profit maximization.

Solving for the profit-maximizing output we obtain, $y_2(p) = \frac{1}{2}\frac{p_2}{p_1}$. Thus $y_1(p) = -y_2(p)^2 = -\frac{1}{4}(\frac{p_2}{p_1})^2$ and maximized profit is $\Pi(p) = p \cdot y(p) = \frac{(p_2)^2}{4p_1}$.

Crusoe the Price-Taking Consumer

Next consider the choice of Crusoe the consumer. The value of his endowment is $p \cdot \omega$. In addition, as the single shareholder in the economy he collects all the dividends $\Pi(p) = p \cdot y(p)$. His spending on corn is therefore constrained as follows: $p \cdot x \leq p \cdot \omega + \Pi(p)$.

He therefore solves the following maximization problem:

$$\underset{x \geq 0}{\text{Max}}\{U(x) | p \cdot x \leq p \cdot \omega + \Pi(p)\}.$$

This is depicted in Figure 5.1-3 with Crusoe's optimal choice $x(p)$.

As shown, $x_1(p) > y_1(p) + \omega_1$ and $x_2(p) < y_2(p) + \omega_2$. That is, there is excess demand for leisure and excess supply of commodity 2. Thus the Walrasian auctioneer raises the price of commodity 1 and lowers the price of commodity 2. The relative price of commodity 1 rises so the profit and budget lines become steeper.

The Walrasian equilibrium (WE) is depicted in Figure 5.1-4.

Comparing Figure 5.1-1 and Figure 5.1-4 it is clear that the Walrasian equilibrium coincides with the optimum. Thus the price-taking behavior of the schizophrenic Robinson Crusoe, along with market clearing, guides him to the optimum. This is a simple illustration of Adam Smith's famous "invisible hand" at work.

Example (concluded):

Crusoe the consumer solves the following problem:

$$\underset{x}{\text{Max}}\{\ln x_1 + \ln x_2 | p \cdot x \leq \Pi(p) + \omega\}.$$

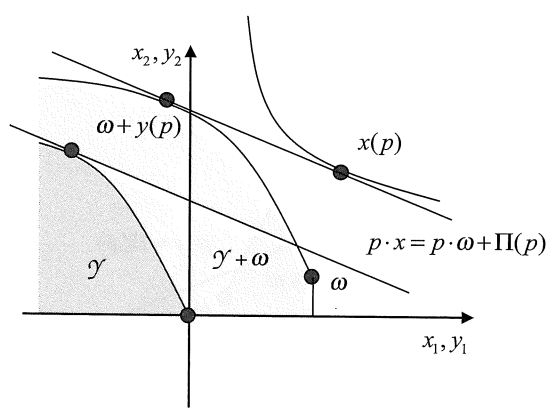

Figure 5.1-3. Utility maximization by Crusoe.

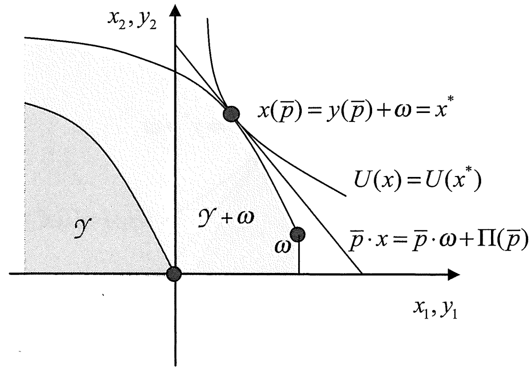

Figure 5.1-4. Walrasian equilibrium.

Because utility is strictly increasing, the budget constraint must be satisfied with equality at the maximum. From the FOC, and appealing to the Ratio Rule,

$$\frac{\frac{\partial U}{\partial x_1}}{p_1} = \frac{\frac{\partial U}{\partial x_2}}{p_2} \Rightarrow \frac{1}{p_1 x_1} = \frac{1}{p_2 x_2} = \frac{2}{p_1 x_1 + p_2 x_2} = \frac{2}{p \cdot \omega + \Pi(p)}.$$

Therefore $x_2(p) = \frac{1}{2}(\frac{\Pi(p)+p\cdot\omega}{p_2})$. We have already seen that

$$y_2^*(p) = \frac{1}{2}\frac{p_2}{p_1} \quad \text{and} \quad \Pi(p) = \frac{(p_2)^2}{4p_1}.$$

Therefore demand for commodity 2 is $x_2(p) = \frac{1}{2}(\frac{p_2}{4p_1} + 144\frac{p_1}{p_2} + 3)$.

It follows that excess demand for commodity 2 is

$$\begin{aligned} z_2(p) = x_2(p) - y_2(p) - 3 &= \frac{1}{2}\left(144\frac{p_1}{p_2} - \frac{3}{4}\frac{p_2}{p_1} - 3\right) \\ &= \frac{1}{2}\frac{p_1}{p_2}\left(12 - \frac{p_2}{p_1}\right)\left(12 + \frac{3}{4}\left(\frac{p_2}{p_1}\right)\right) \\ &= 0 \quad \text{at } \frac{p_2}{p_1} = 12. \end{aligned}$$

In Section 5.3 we ask what conditions are sufficient to guarantee existence of a Walrasian equilibrium. We conclude here by showing the problem that can arise if the production set is not convex. Consider the example depicted in Figure 5.1-5 in which there is a fixed cost that has to be incurred before any output is produced. For simplicity there is a zero endowment of commodity 2 so the endowment vector is on the horizontal axis.

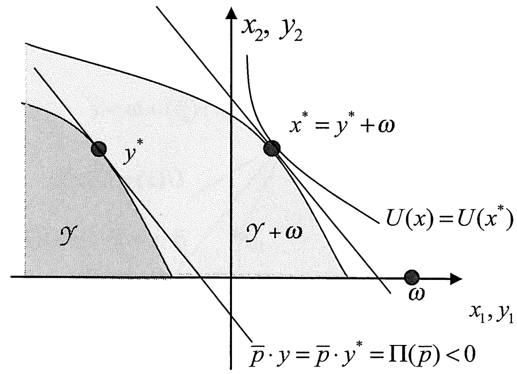

Figure 5.1-5. No Walrasian equilibrium with large fixed costs.

The optimal production plan is y^* and the optimal consumption plan is $x^* = y^* + \omega$. For this plan to be an equilibrium allocation, we draw the budget line through x^* tangential to the set $\mathcal{Y} + \omega$ and the parallel profit line through y^* tangential to the production set. Note that the profit line lies to the left of the origin. Yet then the optimum cannot be a Walrasian equilibrium because the firm is strictly better off producing nothing and making a profit of zero.

Exercises

Exercise 5.1-1: Equilibrium Robinson has a utility function $U(x_1, x_2) = x_1 x_2$. His endowment of hours is 147. He can produce coconuts according to the production function $y_2 = 2\sqrt{-y_1}$.

(a) Solve for his optimum.
(b) If the price of commodity 2 is one, what wage will induce Robinson, acting as a profit-maximizing manager, to demand the optimal labor input.
(c) Depict the production set, preferences, and separating plane in a neat figure.

Exercise 5.1-2: Walrasian Equilibrium with CRS Suppose that Robinson can produce coconuts using labor according to the production function $y_2 = -2y_1$. He has an endowment of β units of commodity 2 and an endowment of γ units of time. Preferences are represented by the utility function $U(x_1, x_2) = \ln x_1 + \ln(\alpha + x_2)$, where x_1 is leisure.

(a) Depict the production set $\mathcal{Y}$ in a neat figure and also the set $\mathcal{Y} + \omega$.
(b) Solve for the optimal number of working hours for all values of α, β and γ. Under what conditions is it optimal for Robinson not to work at all?
(c) Explain why equilibrium profit must be zero.
(d) If the optimum labor supply is zero, show that there is an interval of Walrasian equilibrium "real wage rates" p_1/p_2.

Exercise 5.1-3: Existence with a Non-Convexity Suppose that Robinson Crusoe can produce coconuts (y_2) using labor ($-y_1$). The production set $\mathcal{Y} = \hat{\mathcal{Y}} \cup \{0\}$ where $\hat{\mathcal{Y}} = \{(y_1, y_2) | y_1 \leq -\gamma,\ y_1 + \gamma + \frac{1}{8}y_2^2 \leq 0\}$. His utility function is $U(x) = x_1 + \ln x_2$. He has an endowment of 2γ units of time.

(a) Depict the production set in a neat figure.
(b) Show that the optimal output of coconuts is 2.
(c) Solve for the firm's supply curve for coconuts as a function of the "real wage" p_1/p_2.
(d) Solve for the consumer's demand curve for coconuts.
(e) Hence or otherwise show that if there is a WE, the WE price ratio $p_2/p_1 = \frac{1}{2}$.
(f) Compute the implied equilibrium profit as a function of γ.
(g) Hence obtain the values of γ for which there exists a WE.

5.2 Equilibrium and Efficiency with Production

Key ideas: Walrasian equilibrium, First and Second welfare theorems

We now consider an economy with an arbitrary number of commodities, consumers, and firms.

Firms

There are F firms in the economy. Firm f has a set of feasible production plans $\mathcal{Y}^f \subset \mathbb{R}^n$ and chooses a production vector $y^f = (y_1^f, \ldots, y_n^f) \in \mathcal{Y}^f$. A production plan for the economy $\{y^f\}_{f=1}^F$ is a plan for each of the firms. The aggregate production plan for the economy is the sum of all the individual plans

$$y = \sum_{f=1}^{F} y^f.$$

The set of all feasible aggregate production plans, $\mathcal{Y}$, is the aggregate production set.

Consumers

Commodities are private, that is, consumer h has preferences over her own consumption vector $x^h = (x_1^h, \ldots, x_n^h)$ and not those of other consumers. Let $X^h \subset \mathbb{R}^n$ be the consumption set of consumer h, $h = 1, \ldots, H$. That is, preferences are defined over X^h. We assume that consumer h has an endowment vector $\omega^h \in X^h$. A consumption allocation in this economy $\{x^h\}_{h=1}^H$ is an allocation of consumption bundles $x^h \in X^h$, $h = 1, \ldots, H$. The aggregate consumption in the economy is the sum of the individual consumption vectors $x = \sum_{h=1}^{H} x^h$. Similarly the aggregate endowment is $\omega = \sum_{h=1}^{H} \omega^h$.

Shareholdings

Firms are owned by consumers. Consumer h has an ownership share in firm f of θ^{hf}. Ownership shares must sum to 1, that is,

$$\sum_{h=1}^{H} \theta^{hf} = 1, \quad f = 1, \ldots, F.$$

Feasible Allocation

Given the aggregate demand x, endowment ω and supply y, define excess demand $z = x - \omega - y$. An allocation is feasible if aggregate excess demand is negative:

$$z = x - \omega - y \leq 0.$$

Pareto Efficient Allocation

A feasible plan for the economy $\{\hat{x}^h\}_{h=1}^{H}$, $\{\hat{y}^f\}_{f=1}^{F}$ is Pareto efficient (PE) if there is no other feasible plan $\{x^h\}_{h=1}^{H}$, $\{y^f\}_{f=1}^{F}$ that is strictly preferred by at least one consumer and weakly preferred by all consumers.

Price Taking

Let $p > 0$ be the price vector. Consumers and firms are price takers. Thus if firm f chooses the production plan y^f it has a profit of $p \cdot y^f$. Consumer h receives her share of the profit as a dividend payment. The total dividend payment received by consumer h is therefore $\sum_f \theta^{hf} p \cdot y^f$. Adding the value of her endowment, consumer h has a wealth

$$W^h = p \cdot \omega^h + \sum_f \theta^{hf} p \cdot y^f.$$

She then chooses a consumption bundle $\overline{x}^h$ in her budget set

$$\{x^h \in X^h | p \cdot x^h \leq W^h\}.$$

Note that her budget set is largest if wealth is maximized. Thus, as a shareholder, consumer h's interests are best served by firm managers who maximize profit.

Walrasian Equilibrium

Given the price vector p, let $\overline{y}^f$, $f = 1, \ldots, F$, be a production plan that maximizes the profit of firm f. That is,

$$p \cdot \overline{y}^f \geq p \cdot y^f, \quad \text{for all} \quad y^f \in \mathcal{Y}^f,\ f = 1, \ldots, F. \tag{*}$$

Also, let $\overline{x}^h$ be a most preferred consumption plan in consumer h's budget set. That is,

$$U^h(\overline{x}^h) \geq U^h(x^h), \quad \text{for all } x^h \text{ such that } p \cdot x^h \leq W^h. \tag{**}$$

The aggregate excess demand vector is then $\overline{z} = \overline{x} - \omega - \overline{y}$.

Definition: Walrasian Equilibrium Prices The price vector $p \geq 0$ is a WE price vector if for some $\{\overline{y}^f\}_{f=1}^F$ satisfying (*) and $\{\overline{x}^h\}_{h=1}^H$ satisfying (**), the excess demand vector is negative ($\overline{z} \leq 0$). Moreover $p_j = 0$ for any market in which excess demand is strictly negative ($\overline{z}_j < 0$).

The Adam Smith Theorem

The proof of the First welfare theorem follows closely that for the exchange economy. The proof of the Second welfare theorem parallels the proof for the two-person exchange economy. However there are some additional subtleties.

Proposition 5.2-1: First Welfare Theorem
If the preferences of each consumer satisfy the local no-satiation postulate, the Walrasian equilibrium allocation is Pareto efficient.

Proof: First we note that because $\overline{x}^h$ maximizes utility over consumer h's budget set, any strictly preferred bundle x^h must cost strictly more than the equilibrium allocation. That is,

$$U^h(x^h) > U^h(\overline{x}^h) \Rightarrow p \cdot x^h > p \cdot \omega^h + p \cdot \sum_{f=1}^{F} \theta^{hf} \overline{y}^f. \tag{5.2-1}$$

Moreover, appealing to the Duality Lemma (Lemma 2.2-3), any weakly preferred bundle x^h must cost at least as much. That is,

$$U^h(x^h) \geq U^h(\overline{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \omega^h + p \cdot \sum_{f=1}^{F} \theta^{hf} \overline{y}^f. \tag{5.2-2}$$

Suppose that the allocation $\{x^h\}_{h=1}^H$, $\{y^f\}_{f=1}^F$ is feasible and Pareto preferred to the WE allocation. We now show, by contradiction, that no such allocation exists.

For feasibility, excess demands must be negative so $z = x - \omega - y \leq 0$. Then, because the Walrasian equilibrium price vector is positive,

$$p \cdot (x - \omega - y) \leq 0. \tag{5.2-3}$$

Because all consumers must be at least as well off as in the PE allocation, inequality (5.2-2) must hold for all h. Moreover at least one consumer must be strictly better off so inequality (5.2-1) must hold for some h. Summing over consumers,

$$p \cdot \sum_h x^h > p \cdot \sum_h \omega^h + p \cdot \sum_h \sum_f \theta^{hf} \overline{y}^f = p \cdot \sum_h \omega^h + p \cdot \sum_f \overline{y}^f \sum_h \theta^{hf}.$$

Because shares sum to 1, this can be rewritten as follows:

$$p \cdot \sum_h x^h > p \cdot \omega + \sum_f p \cdot \overline{y}^f.$$

Also $\overline{y}^f$ is profit maximizing over Y^f. Hence $p \cdot \overline{y}^f \geq p \cdot y^f$. Therefore

$$p \cdot (x - \omega - y) \geq p \cdot \left(\sum_h x^h - \omega - \sum_f \overline{y}^f \right) > 0.$$

Yet this contradicts (5.2-3). Thus there is no Pareto preferred feasible allocation. □

Decentralization Theorem

Next we consider the Second welfare theorem with production. We follow the same line of argument as in the proof for the exchange economy. However we no longer assume that consumption bundles are necessarily positive. Thus consumers may supply commodities (e.g., labor services), and production vectors have both positive components (outputs) and negative components (inputs.)

Proposition 5.2-2: Second Welfare Theorem with Production
Let $\{\hat{x}^h\}_{h=1}^H, \{\hat{y}^f\}_{f=1}^F$ be a Pareto efficient allocation. Suppose

(a) consumption vectors are private,
(b) consumption sets $X^h, h = 1, \ldots, H$ are convex,
(c) utility functions are continuous, and quasi-concave and satisfy the local non-satiation property,
(d) for each h there is some $\underline{x}^h \in X^h$ such that $\underline{x}^h < \hat{x}^h$, and
(e) production sets $\mathcal{Y}^f, f = 1, \ldots, F$ are convex and satisfy the free disposal property.

Then there exists a price vector $p > 0$ such that

$$x^h \underset{h}{\succ} \hat{x}^h \Rightarrow p \cdot x^h > p \cdot \hat{x}^h, \quad h = 1, \ldots, H$$
$$y^f \in \mathcal{Y}^f \Rightarrow p \cdot y^f \le p \cdot \hat{y}^f.$$

Proof: Let $\{\hat{x}^h\}_{h=1}^H$, $\{\hat{y}^f\}_{f=1}^F$ be a PE allocation. As in the pure exchange economy we introduce the indirect utility function

$$V^1(x) = \underset{\{x^h\}_{h=1}^H}{\text{Max}} \left\{ U^1(x^1) | \sum_{h=1}^{H} x^h \le x, \quad U^h(x^h) \ge U^h(\hat{x}^h), \quad h = 2, \ldots, H \right\}.$$

As argued in Chapter 3, $V^1(x)$ is quasi-concave. Also because U^1 has the local non-satiation property, so does V^1.

We also define the set of feasible aggregate consumption vectors $\{\omega\} + \mathcal{Y}$. This is depicted in Figure 5.2-1 for a two-commodity example in which some of the initial endowment of commodity 1 is transformed into commodity 2. Note that because production sets are convex, so is the aggregate production set.

Consider the following optimization problem:

$$\underset{\{x^h\},\{y^f\}}{\text{Max}} \{V^1(x) | x \in \{\omega + \mathcal{Y}\}. \tag{5.2-4}$$

Because $\{\hat{x}^h\}_{h=1}^H$, $\{\hat{y}^f\}_{f=1}^F$ is Pareto efficient, $U^1(\hat{x}^1)$ is the highest feasible utility for consumer 1, given that no other consumer is made worse off than in the PE allocation. Thus $\{\hat{x}^h\}_{h=1}^H$, $\{\hat{y}^f\}_{f=1}^F$ solves this maximization problem and so $V^1(\hat{x}) = U^1(\hat{x}^1)$. The solution is depicted in Figure 5.2-1.

Given local non-satiation, the maximizing aggregate consumption vector $\hat{x} = \omega + \hat{y}$ must lie on the boundary of $\{\omega\} + \mathcal{Y}$. Because the preferred set

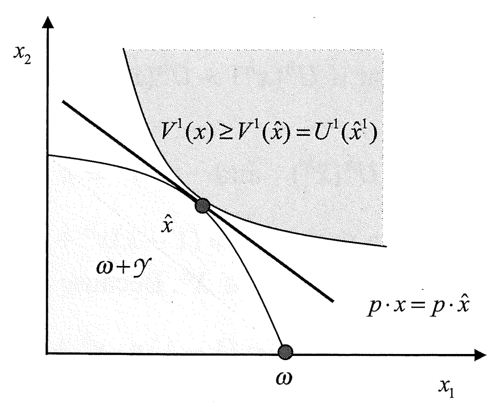

Figure 5.2-1. Supporting hyperplane.

$\hat{\mathcal{X}} \equiv \{x | V^1(x) \geq V^1(\hat{x})\}$ and $\mathcal{Y}$ are both convex, the set $\hat{\mathcal{X}} - \{\omega\} - \mathcal{Y}$ is convex. Moreover $\hat{x} - \omega - \hat{y} = 0$ is a boundary point of this set. Appealing to the Supporting Hyperplane Theorem there exists a vector $p \neq 0$ such that

$$p \cdot (x - \omega - y) \geq p \cdot (\hat{x} - \omega - \hat{y}) = 0, \quad \forall x \in \hat{\mathcal{X}} \text{ and } \forall y \in \mathcal{Y}.$$

Setting $x = \hat{x}$,

(i) $p \cdot (\hat{x} - \omega - y) = p \cdot (\hat{y} - y) \geq 0, \quad \forall y \in \mathcal{Y}.$

Setting $y = \hat{y}$,

(ii) $p \cdot (x - \omega - \hat{y}) = p \cdot (x - \hat{x}) \geq 0, \quad \forall x \in \hat{\mathcal{X}}.$

We now argue that the vector p must be positive. We suppose that some components of p are negative and show that this yields to a contradiction of (i). Define $\delta = (\delta_1, \ldots, \delta_n) > 0$ such that $\delta_j > 0$ if and only if $p_j < 0$. Then $p \cdot \delta < 0$. Consider an aggregate production vector $y = \hat{y} - \delta$. Then $p \cdot (\hat{y} - y) = p \cdot \delta < 0$. Because $\delta > 0$ it follows by free disposal that y is feasible. But then condition (i) is violated. Thus $p > 0$ after all.

Next note that we can rewrite conditions (i) and (ii) as follows:

(i) $y \in \mathcal{Y} \Rightarrow \sum_{f=1}^{F} p \cdot y^f \leq \sum_{f=1}^{F} p \cdot \hat{y}^f$

(ii) $V^1(x) \geq V^1(\hat{x}) \Rightarrow \sum_{h=1}^{H} p \cdot x^h \geq \sum_{h=1}^{H} p \cdot \hat{x}^h.$

Setting $y^j = \hat{y}^j$, $j \neq f$ in (i) it follows that

(i)′ $y^f \in \mathcal{Y}^f \Rightarrow p \cdot y^f \leq p \cdot \hat{y}^f, \quad f = 1, \ldots, F.$

Thus $\hat{y}^f$ is profit maximizing for firm f.

Similarly, setting $x^i = \hat{x}^i$, $i \neq h$ in (ii) it follows that[1]

(ii)′ $U^h(x^h) \geq U^h(\hat{x}^h) \Rightarrow p \cdot x^h \geq p \cdot \hat{x}^h.$

The final step is to show that if $U^h(x^h) > U^h(\hat{x}^h)$ then $p \cdot x^h > p \cdot \hat{x}^h$. Suppose instead that

$$U^h(x^h) > U^h(\hat{x}^h) \quad \text{and} \quad p \cdot x^h = p \cdot \hat{x}^h.$$

Consider the consumption vector $\breve{x}^h = (1 - \lambda)x^h + \lambda \underline{x}^h$, where $\lambda \in (0, 1)$. Because $\underline{x}^h \in X^h$ and X^h is convex, $\breve{x}^h \in X^h$. Because $\underline{x}^h < \hat{x}^h$,

$$p \cdot \breve{x}^h = (1 - \lambda)p \cdot x^h + \lambda p \cdot \underline{x}^h < (1 - \lambda)p \cdot x^h + \lambda p \cdot \hat{x}^h = p \cdot \hat{x}^h.$$

[1] From the definition of the indirect utility function $U^h(x^h) \geq U^h(\hat{x}^h)$, $h = 2, \ldots, H$. Also $U^1(x^1) = V^1(x) \geq V^1(\hat{x}) = U^1(\hat{x}^1)$.

Also, given the continuity of preferences, $U^h(\breve{x}^h) > U^h(\hat{x}^h)$ for λ sufficiently close to 1. However, this is impossible because condition (ii)$'$ is violated.

□

The two welfare theorems lie at the heart of arguments in favor of free trade as opposed to intervention by the state. They also lie at the heart of arguments for such intervention! At issue is the relevance of the assumptions that underpin the theorems. First there is the assumption of price-taking behavior. If economic agents are able to form cartels and exploit their monopoly power, they have an incentive to do so.[2] Second, and equally important, the theorems require that goods be private. That is, individuals care about their own consumption only. When this is not the case (whether the concern is noisy neighbors, industrial pollution, or a city park) the theorems fail.

However, whether or not intervention can be justified is a complex matter. The beauty of the First welfare theorem is that no economic agent has to know anything about anyone else's preferences. Any intervention requires an evaluation of costs and benefits that are often difficult to measure because agents have an incentive to misrepresent the truth.

Exercises

Exercise 5.2-1: Cobb-Douglas Economy All consumers have the same Cobb-Douglas utility function $U^h(x^h) = \sum_{j=1}^{2} a_j \ln x_j^h$, $h = 1, \ldots, H$. There are two inputs. The aggregate endowment of input i is ω_i, $i = 1, 2$. The production set of any firm producing commodity j is

$$\mathcal{Y}_j^f = \left\{ \left(z_j^f, q_j^f\right) | q_j^f \leq \left(z_{1j}^f\right)^{\alpha_j} \left(z_{2j}^f\right)^{1-\alpha_j} \right\}.$$

(a) Show that the industry production set is

$$\mathcal{Y}_j = \left\{ \left(z^f, q_j^f\right) | q_j \leq (z_{1j})^{\alpha_j} (z_{2j})^{1-\alpha_j} \right\}.$$

(b) Show that the optimal allocation of input 1 is

$$(z_{11}^*, z_{12}^*) = \left(\frac{a_1\alpha_1\omega_1}{a_1\alpha_1 + a_2\alpha_2}, \frac{a_2\alpha_2\omega_1}{a_1\alpha_1 + a_2\alpha_2} \right).$$

Solve also for the optimal allocation of input 2.

HINT: This is an economy with identical homothetic preferences.

(c) Appeal to part (a) to solve for WE input prices.

[2] It was for this reason that Adam Smith argued in favor of placing restrictions on the collusive activities of manufacturing associations, thus allowing the invisible hand to do its work.

(d) Show that the associated WE output price vector is

$$p_j = \left(\frac{r_1}{\alpha_j}\right)^{\alpha_j}\left(\frac{r_2}{1-\alpha_j}\right)^{1-\alpha_j}.$$

HINT: Marginal cost is independent of output in this economy. Why is this?

Exercise 5.2-2: Robinson Meets Friday Suppose that Robinson owns the firm but is unable to work. Friday has 32 units of potential labor hours and a utility function $U(x) = \ln x_1 + \ln x_2$. The production function is $y_2 = 4(-y_1)^{1/2}$, equivalently, $y_1 + y_2^2/16 \leq 0$.

(a) Solve for Friday's labor supply curve.
(b) Solve for the firm's demand for labor. Hence, solve for the Walrasian equilibrium.
(c) Draw the production set $\mathcal{Y}$ and the set $\mathcal{Y} + \omega$ in a neat figure. Show Friday's equilibrium budget constraint and the firm's profit-maximizing profit line.
(d) Suppose that Friday receives a lump-sum subsidy (and Robinson pays a lump-sum tax.) Show that at the no tax Walrasian equilibrium prices, Friday's supply of labor falls so that there is excess demand for labor. Hence draw a conclusion as to how the WE price ratio changes.
(e) Hence or otherwise comment on the price effect if Friday is given shareholdings in the firm.

Exercise 5.2-3: Robinson and Friday Suppose that Robinson owns the firm. Friday and Robinson have equal endowments of time to be allocated between leisure and work. The total endowment of time is ω_1 so each has an endowment of $\frac{1}{2}\omega_1$ units of time. Each has a utility function $U(x) = \ln x_1 + 2a \ln x_2$. The production function is $y_2 = (z_1)^{1/2}$.

(a) Treat this as a representative agent problem and show that the optimal total supply of labor is $z_1^* = \frac{a\omega_1}{1+a}$. Hence

$$x^* = \left(\frac{\omega_1}{1+a}, \left(\frac{a\omega_1}{1+a}\right)^{1/4}\right).$$

(b) Consider the consumer optimization problem (remembering that the representative agent has an endowment of time and dividend income) and hence solve for the WE prices.
(c) Show that Friday's demand for leisure is $x_1^F = \frac{\omega_1}{2(1+4a)}$. Market demand for leisure is given in part (a). Hence solve for Robinson's equilibrium demand for leisure. Show that this exceeds his time available if a is strictly positive and sufficiently small. Thus the simple representative agent approach needs modifying.
(d) What is the equilibrium in this case?