Maximization

The two foundation stones of economic modelling are (i) maximization and (ii) equilibrium.

In the first two lectures we will review both. It is assumed that you reviewed the on-line course on maximization over the summer (http://www.econ.ucla.edu/riley/CalculusOfEconomics/)[1]

1. Laws of supply and Demand

The first law of firm supply

As an output price p rises, the maximizing output $q(p)$ increases (at least weakly).

Consider a firm that sells a product at a price of p^1 . i.e. selling additional units has an insignificant effect on the market price. Thus the marginal revenue from selling an additional unit is $MR(q) = p^1$. The firm then maximizes profit by looking on the margin.

The two possibilities are depicted below.

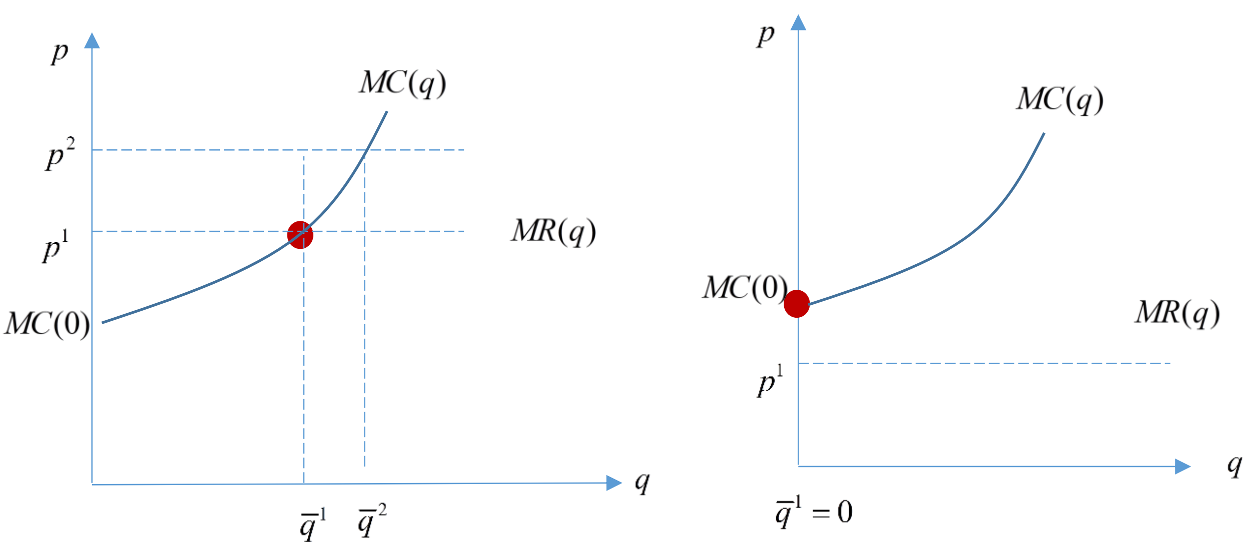


Fig. 1: Profit-maximizing output

[1] If you have questions about any of the modules please email me for an appointment. riley@econ.ucla.edu

In the left-hand figure $MC(0) < p^1$ so profit is maximized at $\bar{q}^1 > 0$ where

$$MR(\bar{q}^1) - MC(\bar{q}^1) = p^1 - MC(\bar{q}^1) = 0.$$

In the right-hand figure $MC(0) > p^1$ so profit is maximized by not producing.

Suppose that $\bar{q}^1 > 0$ as in the left figure, note that if the price of the output rises to p^2. Marginal revenue at $\bar{q}^1$ is greater than marginal cost. Thus the firm's new profit-maximizing output is $\bar{q}^2 > \bar{q}^1$. If $\bar{q}^1 = 0$ as in the right-hand figure, an almost identical argument establishes that $\bar{q}^2 \geq \bar{q}^1$.

The profit-maximizing response at every price is depicted in Figure 2. If the price is below $MC(0)$ the profit-maximizing output is $\bar{q}(p) = 0$. For all higher prices, output strictly increases with price as depicted.

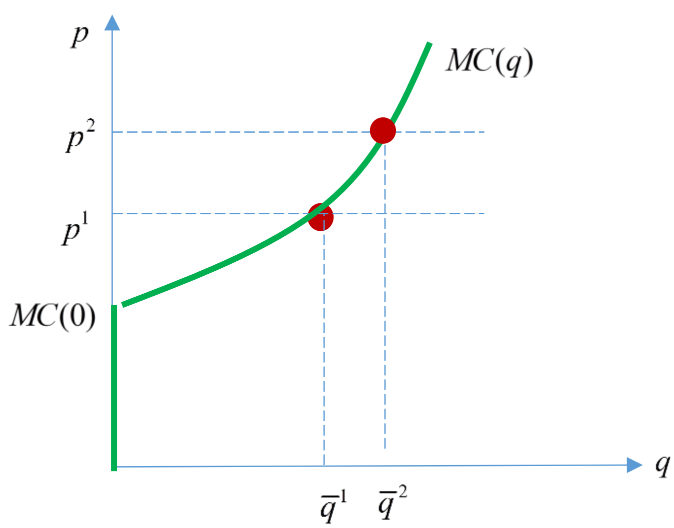


Fig. 2: Firm's supply curve

The first law of input demand

As an input price r rises, the maximizing input $z(r)$ decreases (at least weakly).

The argument is similar. Now the marginal input cost $MC(z) = r$. The input price-taking firm compares the marginal input cost with the "marginal revenue product", i.e. the rate at which revenue rises as input (and therefore output) increases. This is depicted in Figure 3.

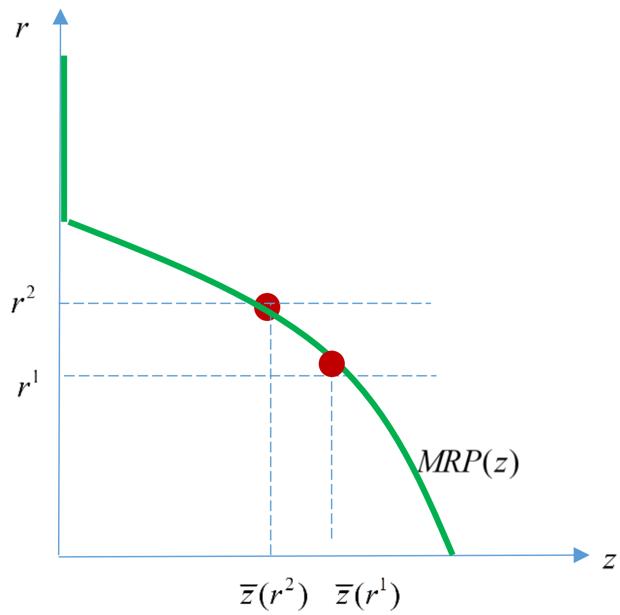


Fig. 3: Firm's input demand curve

If, as depicted , $\bar{z}(r^1) > 0$, then the firm responds to an increase in the input price to r^2 by lowering input demand to $\bar{z}(r^2)$.

2. Resource constrained maximization

Consider the following problem:

$$\underset{x \geq 0}{Max}\{f(x) \mid b - g(x) \geq 0\}.$$

We give it the following economic interpretation. If the firm chooses vector $x = (x_1,...,x_n)$ its output is $f(x)$. The market price is 1 so revenue is also $f(x)$. There is only one input. If the firm chooses x the total input requirement is $z = g(x)$. The firm owns b units of the resource and cannot buy additional units or sell units. Then the firm maximizes its revenue $f(x)$ subject to the resource constraint $b - g(x) \geq 0$.

To understand how to solve this problem we consider two "relaxed problems in which units of the input can be bought or sold at the price $r \geq 0$.

Problem (i): Profit maximization of a multi-product firm that uses a single input

If the firm chooses x so that output is $f(x)$, the input requirement is $z = g(x)$. Let r be the input price. Profit is therefore

$$\pi = 1q - rz = f(x) - rg(x) \ .$$

Let $\bar{x}(r)$ be the profit-maximizing output vector.

i.e. $\bar{x}(r)$ solves $\underset{x \geq 0}{Max}\{\pi = f(x) - rg(x)$ (2-1)

This is a standard maximization problem with non-negativity constraints.

Exercise: Suppose $q = f(x) = x^{1/4} x_2^{1/4}$, $z = g(x) = x_1 + x_2$.

Solve problem 1 and graph the firm's input demand as a function of the input price.

Suppose that we solve problem (i) for every input price r . Then input demand is $\bar{z}(r) = g(\bar{x}(r))$. By the Law of input demand, $\bar{z}(r)$, decreases as r increases as depicted in Figure 4.

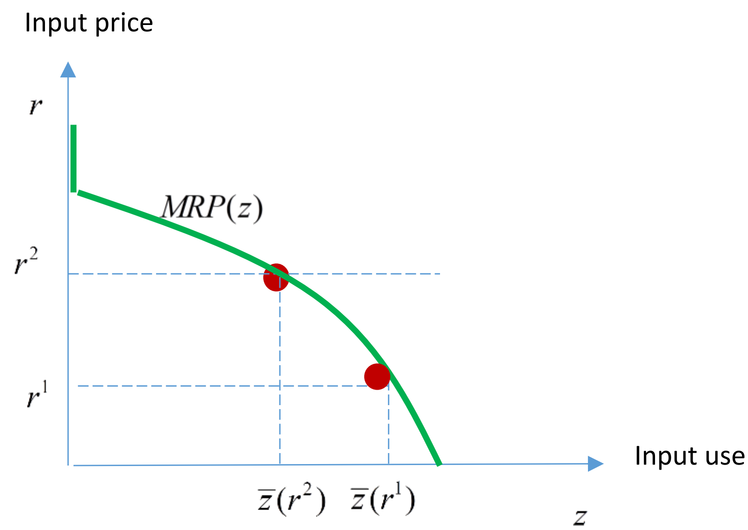


Fig. 4: Firm's input demand curve

Problem (ii): Profit maximization of a multi-product firm that owns b units of the input.

Now suppose that the firm owns b units of the input. If $g(x) > b$ the firm pays $r(g(x)-b)$ for the additional units. If the firm sells $b-g(x)$ the firm's additional revenue is

$$r(b-g(x) = -r(g(x)-b)$$

The profit of this firm is therefore

$$\pi = 1q - r(z-b) = f(x) - r(g(x)-b) \ .$$

Let $\bar{x}(r)$ be the profit-maximizing output vector.

i.e. $\bar{x}$ solves $\underset{x \geq 0}{Max}\{f(x) - rg(x) + rb\} = \underset{x \geq 0}{Max}\{f(x) - rg(x)\} + rb$.

The solution is depicted in Figure 5.

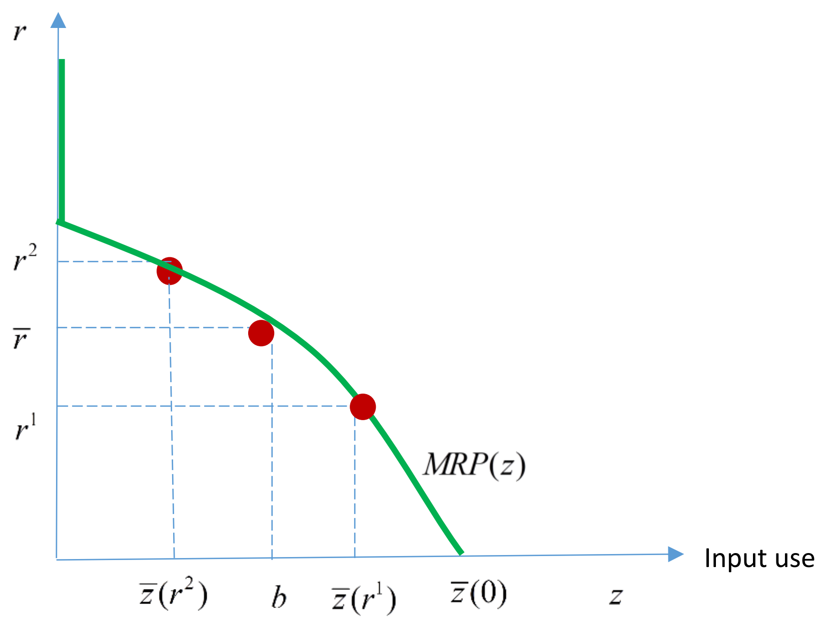


Fig. 5: Firm's input demand curve

At the low price r^1 the firm purchases $g(\bar{x}(r^1))-b$ additional inputs. At the high price the firm sells $b-g(\bar{x}(r^2))$ units.

Note that the two problems have the same maximizer, $\bar{x}$. The fact that the firm owns some of the input does not affect marginal decisions and therefore output decisions.

Case 1: $\bar{z}(0)=g(\bar{x}(0))<b$.

In this case even with an input price of zero, input use is less than the input available. Moreover with an input price of zero, problem (ii) reduces to

$$\underset{x\geq 0}{Max}\{f(x)-r(g(x)-b\}=\underset{x\geq 0}{Max}\{f(x)\}$$

Thus the constraint is not binding.

Case 2: $\bar{z}(0)=g(\bar{x}(0))\geq b$.

By the First Law of input demand, there must be some input price $\bar{r}$ such that $\bar{z}(\bar{r}) = b$ as depicted in Figure 5.

Suppose then that the price of the input is $\bar{r}$. At this price, maximized profit is

$$\Pi = 1\bar{q} - \bar{r}(\bar{z} - b) = f(\bar{x}) - \bar{r}(g(\bar{x}) - b) \quad . \tag{2-2}$$

But we defined $\bar{r}$ to be the input price such that $g(\bar{x}) = b$. Therefore, by (2-2), maximized profit is

$$\Pi = f(\bar{x}) \tag{2-3}$$

Consider any other x . SInce $\bar{x}$ is profit-maximizing, It follows from (2-2) and (2-3) that

$$\Pi = f(\bar{x}) \geq f(x) - \bar{r}(g(z) - b)$$

$$= f(x) + \bar{r}(b - g(z)) \, . \tag{2-4}$$

Resource constrained maximization

Now consider a firm that owns the b units of the resource and units of this resource cannot be purchased or sold. It has no variable costs so solves the following revenue maximization problem.

$$\underset{x \geq 0}{Max}\{f(x) \mid b - g(x) \geq 0\}$$

From the analysis of Problem (ii), if the input price is $\bar{r}$, then for all $x \geq 0$,

$$\Pi = f(\bar{x}) \geq f(x) + \bar{r}(b - g(x)) \, .$$

For the constrained firm, the second term on the right-hand side is positive. Therefore at the input price $\bar{r}$,

(i) $g(\bar{x}) = b$ and (ii) $f(\bar{x}) \geq f(x)$ for all $x \geq 0$ satisfying $b - g(x) \geq 0$

Thus the Necessary Conditions for the solution of problem 2 are the necessary conditions for the resource constrained optimization problem.

Conclusion: Necessary conditions for a maximum with a resource constraint, i.e.

$$\underset{x\geq 0}{Max}\{f(x)\,|\,b-g(x)\geq 0\}$$

Consider the relaxed problem in which there is a market for the resource and the firm owns b units of the resource. If the price of the resource is λ , then profit in the relaxed problem is

$$\mathfrak{L}=f(x)-\lambda(g(x)-b)=f(x)+\lambda(b-f(x)) \ . \qquad (2\text{-}5)$$

Since this market is a theoretical rather than an actual market we call the price a shadow price.

Suppose we find a shadow price $\bar{\lambda}\geq 0$ and $\bar{x}$ such that the Necessary First Order Conditions for the relaxed problem are satisfied and in addition,

(i) $b-g(\bar{x})>0 \Rightarrow \bar{\lambda}=0$ (ii) $\bar{\lambda}>0 \Rightarrow b-g(\bar{x})=0$.

Then these conditions are the necessary conditions for the resource constrained problem.

3. An Example

A division manager has a budget of B , the input price vector is $p >> 0$. Her job is to maximize output $q = f(z) = z_1^{\alpha_1} z_2^{\alpha_2}$, where $\alpha >> 0$ and z is the vector of inputs. She therefore chooses $\bar{z}$ to solve

$$\underset{x \geq 0}{Max}\{ f(z) \mid p \cdot z \leq B\} \ .$$

Three methods of solution

1. The Lagrange Method

Consider the profit for the relaxed problem

$$\mathfrak{L} = f(z) + \lambda(B - r \cdot z)$$

If $\lambda = 0$ there is no maximum for the profit as $f(x)$ is strictly increasing. Therefore $\lambda > 0$.

If follows that $r \cdot \bar{z} = B$

Note also that $f(z) = 0$ if either z_1 or $z_2 = 0$ and $f(z) > 0$ for any $z >> 0$. Therefore $\bar{z} >> 0$.

The necessary conditions for the relaxed problem (and hence the constrained maximization problem) are therefore as follows:

$$\frac{\partial \mathfrak{L}}{\partial z_1} = \frac{\partial f}{\partial z_1}(z) - \lambda r_1 = 0 \text{, since } \bar{z}_1 > 0 \ . \qquad (3\text{-}1)$$

$$\frac{\partial \mathfrak{L}}{\partial z_1} = \frac{\partial f}{\partial z_2}(z) - \lambda r_2 = 0 \text{, since } \bar{z}_1 > 0 \ . \qquad (3\text{-}2)$$

Therefore

$$\alpha_1 z_1^{\alpha_1 - 1} z_2^{\alpha_2} = \lambda r_1$$

and

$$\alpha_2 z_1^{\alpha_1} z_2^{\alpha_2 - 1} = \lambda r_2$$

There are three unknowns (including the shadow price).

We reduce these to two by eliminating λ

$$\frac{\alpha_1 z_1^{\alpha_1 - 1} z_2^{\alpha_2}}{\alpha_2 z_1^{\alpha_1} z_2^{\alpha_2 - 1}} = \frac{\lambda r_1}{\lambda r_2} .$$

This simplifies as follows:

$$\frac{\alpha_1 z_2}{\alpha_2 z_1} = \frac{r_1}{r_2} .$$

Therefore $r_2 z_2 = \dfrac{\alpha_2}{\alpha_1} r_1 z_1$.

Substitute this in the budget constraint.

$$r_1 z_1 + r_2 z_2 = r_1 z_1 + \frac{\alpha_2}{\alpha_1} r_1 z_1 = \frac{\alpha_1 + \alpha_2}{\alpha_1} r_1 z_1 = B \ .$$

Therefore the expenditure on input 1 is a constant fraction of the budget.

$$r_1 \bar{z}_1 = \frac{\alpha_1}{\alpha_1 + \alpha_2} B \ .$$

The rest of the budget is spent on input 2 so

$$r_2 \bar{z}_2 = \frac{\alpha_2}{\alpha_1 + \alpha_2} B .$$

Method 2: Equate the marginal output per dollar of input.

The rate at which output goes up with spending on each input is

$$\frac{1}{r_1} \frac{\partial f}{\partial z_1} \ \text{ and } \ \frac{1}{r_2} \frac{\partial f}{\partial z_2} .$$

Remark: To prove this appeal to the Necessary conditions, (3-1) and (3-2)

They can be rewritten as follows:

$$\frac{1}{r_1}\frac{\partial f}{\partial z_1}(z)=\lambda \text{ and } \frac{1}{r_{21}}\frac{\partial f}{\partial z_2}(z)=\lambda .$$

More informally (but just as appropriately) consider the increase on output if an additional amount ΔB is spent on input 1. The increase in input 1 is $\Delta z_1 = \frac{1}{r_1}\Delta B$. The "marginal product" of input 1 is the rate at which output rises with z_1 , i.e. $\frac{\partial f}{\partial z_1}(z)$. Thus the increases in output is

$$\Delta q = \frac{\partial f}{\partial z_1}\Delta z_1 = \frac{\partial f}{\partial z_1}\frac{1}{r_1}\Delta B \ .$$

Therefore the rate at which output increases is

$$\frac{\Delta q}{\Delta B}=\frac{1}{r_1}\frac{\partial f}{\partial z_1}.$$

By the same argument if ΔB is spent on input 2 the rate at which output increases is $\frac{1}{r_2}\frac{\partial f}{\partial z_2}$

If $\frac{1}{r_i}\frac{\partial f}{\partial z_i}(z) > \frac{1}{r_j}\frac{\partial f}{\partial z_j}(z)$, then output can be increased by increasing spending on input i and less on input j . We know that $\overline{z} >> 0$, thus for $\overline{z}$ to be maximizing,

$$\frac{1}{r_1}\frac{\partial f}{\partial z_1}(\overline{z})=\frac{1}{r_2}\frac{\partial f}{\partial z_2}(\overline{z}).$$

For the example

$$\frac{1}{r_1}\alpha_1 z_1^{\alpha_1-1}z_2^{\alpha_2} = \frac{1}{r_2}\alpha_2 z_1^{\alpha_1}z_2^{\alpha_2-1}$$

This can be simplified as follows:

$$\frac{\alpha_1}{r_1 z_1} = \frac{\alpha_2}{r_2 z_2} . \tag{3-3}$$

Then proceed as in the first method.

I find the following short-cut helpful.[2]

The Ratio Rule

If $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}$

Proof: Define $k = \frac{a_1}{b_1} = \frac{a_2}{b_2}$. Then $a_1 = kb_1$ and $a_2 = kb_2$. Hence $a_1 + a_2 = k(b_1 + b_2)$ and so

$$\frac{a_1 + a_2}{b_1 + b_2} = k .$$

QED

Appealing to the Ratio Rule, it follows from (3-3) that

$$\frac{\alpha_1}{r_1 z_1} = \frac{\alpha_2}{r_2 z_2} = \frac{\alpha_1 + \alpha_2}{r_1 z_1 + r_2 z_2} .$$

The solution must satisfy the budget constraint with equality so $r_1 z_1 + r_2 z_2 = B$. Therefore

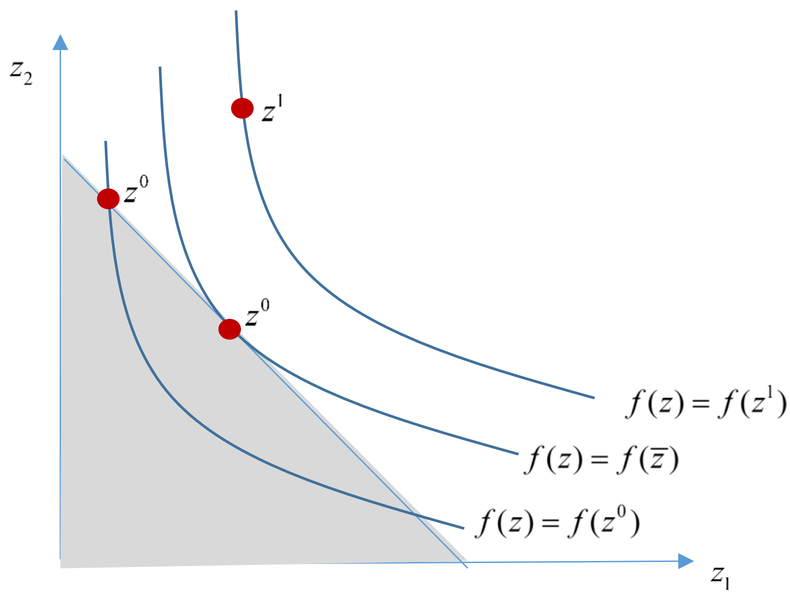
$$\frac{\alpha_1}{r_1 z_1} = \frac{\alpha_2}{r_2 z_2} = \frac{\alpha_1 + \alpha_2}{B} .$$

It is then a simple matter to solve for $\bar{z}$.

[2] Lots of students do not find this particularly helpful, presumably because it only works in special cases.

Method 3: Graphical approach

In the figure below the budget constraint and level sets of the production function are depicted.



Along the budget line $r_1 z_1 + r_2 z_2 = B$ so

$$z_2 = \frac{B}{r_2} - \frac{r_1}{r_2} z_1$$

Therefore the slope of the budget line is $-r_1 / r_2$. At the maximum, this must be equal to the slope of the level set.

`To compute the slope of the level set we note that it implicitly defines the strictly decreasing function $z_2 = \phi(z_1)$. i.e. on a level set

$$f(z_1, z_2) = f(x_1, \phi(z_1)) = k$$

Along the level set the derivative of the function $q = f(z_1, \phi(z_1))$ is zero. i.e.

$$\frac{dq}{dz_1} = \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2} \phi'(z_1) = 0 \ .$$

Therefore the slope of the level set is

$$\phi'(z_1) = -\frac{\partial f}{\partial z_1} / \frac{\partial f}{\partial z_2} \ .$$

At the maximum $\bar{z}$ the two slopes are equal, i.e.

$$r_1 / r_2 = \frac{\partial f}{\partial z_1}(\bar{z}) / \frac{\partial f}{\partial z_2}(\bar{z}) .$$

Therefore

$$\frac{1}{r_1}\frac{\partial f}{\partial z_1}(\bar{z}) = \frac{1}{r_2}\frac{\partial f}{\partial z_2}(\bar{z}) \ .$$

Then proceed as in Method 1 or Method 2.

Remark: A simplification

Consider any strictly increasing function $g(q)$. If $\overline{z}$ solves

$$\underset{x\geq 0}{Max}\{f(z)\mid B-r\cdot z\geq 0\}$$

Then $\overline{z}$ must also solve

$$\underset{x\geq 0}{Max}\{h(z)=g(f(z))\mid B-r\cdot z\geq 0\}.$$

If we can find a function that separates the variables it makes taking derivatives easier. In this example the natural logarithm helps since

$$h(z)=\ln(f(z))=\ln z_1^{\alpha_1}z_2^{\alpha_2}=\ln z_1^{\alpha_1}+\ln z_2^{\alpha_2}=\alpha_1\ln z_1+\alpha_2\ln z_2$$

Then solve

$$\underset{x\geq 0}{Max}\{h(z)=\alpha_1\ln z_1+\alpha_2\ln z_2\mid B-r\cdot z\geq 0\}$$

The Lagrangian is

$$\mathfrak{L}=\alpha_1\ln z_1+\alpha_2\ln z_2+\lambda(B-r_1z_1-r_2z_2)$$

The Necessary Conditions are therefore as follows:

$$\frac{\partial\mathfrak{L}}{\partial z_1}=\frac{\partial f}{\partial z_1}(z)-\lambda r_1=\frac{\alpha_1}{z_1}-\lambda r_1=0\text{, since } \overline{z}_1>0 \text{ .}$$

$$\frac{\partial\mathfrak{L}}{\partial z_1}=\frac{\partial f}{\partial z_2}(z)-\lambda r_2=\frac{\alpha_2}{z_1}-\lambda r_2=0\text{, since } \overline{z}_1>0 \text{ .}$$