Profit maximization: An example

A plant has a production function $F(z)=z_1^{1/4}z_2^{1/2}$. The input price vector is $p=(p_1,p_2)=(2,16)$. The output price is $p_3=32$ The plant manager is given a budget of B. Her instructions are to maximize profit. She therefore solves the following problem

$$\underset{z\geq 0}{Max}\{\pi = p_3F(z)-p_1z_1-p_2z_2\}\ .$$

Preliminary observation: Since $F(z)>0$ if and only if $z>>0$ the solution $\bar{z}>>0$. Thus the Necessary conditions for a maximum must be satisfied with equality.

Necessary conditions:

$$\frac{\partial\pi}{\partial z_1}=32\frac{\partial F}{\partial z_1}(z)-2=8z_1^{-3/4}z_2^{1/2}-2=0$$

$$\frac{\partial\pi}{\partial z_2}=32\frac{\partial F}{\partial z_2}(z)-16=16z_1^{1/4}z_2^{-1/2}-16=0.$$

Therefore if $\frac{\partial\pi}{\partial z_1}=0$.

$$z_2^{1/2}=\tfrac{1}{4}z_1^{3/4}\ \text{ Hence }\ z_2=\tfrac{1}{16}z_1^{3/2}. \qquad (1.1)$$

And if $\frac{\partial\pi}{\partial z_2}=0$

$$z_1^{1/4}=z_2^{1/2}.\ \text{Hence }\ z_2=z_1^{1/2} \qquad (1.2)$$

Hence

$$\tfrac{1}{16}z_1^{3/2}=z_2=z_1^{1/2}$$

Hence $\bar{z}_1=16$ and so $\bar{z}_2=4$.

But is this the maximizer?

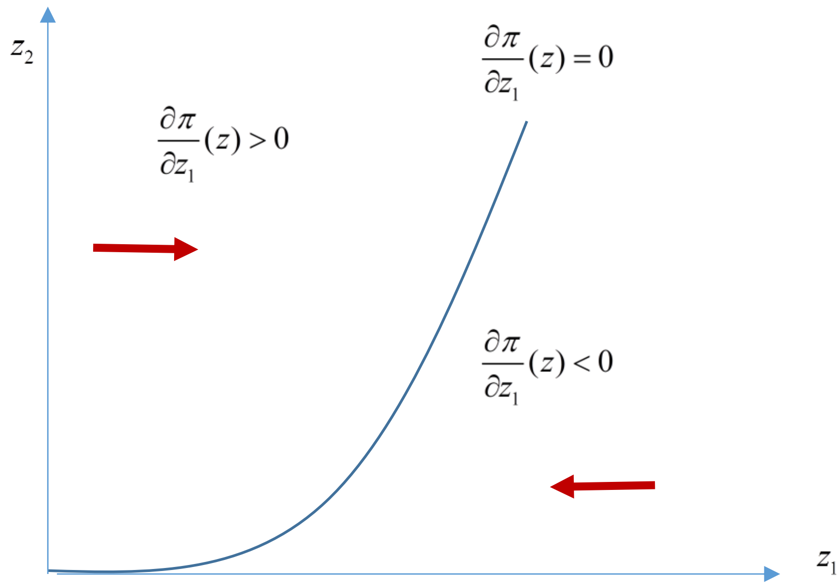
Consider the necessary condition for z_1 .

$$\frac{\partial\pi}{\partial z_1}=8z_1^{-3/4}z_2^{1/2}-2=8\frac{z_2^{1/2}}{z_1^{3/4}}-2. \qquad (1.3)$$

Note that marginal profit $\frac{\partial \pi}{\partial z_1}$ declines as z_1 increases. The set of points for which $\frac{\partial \pi}{\partial z_1} = 0$ is given by equation (1.1).

This is depicted below. Fix z_2 and consider marginal profit as z_1 increases.

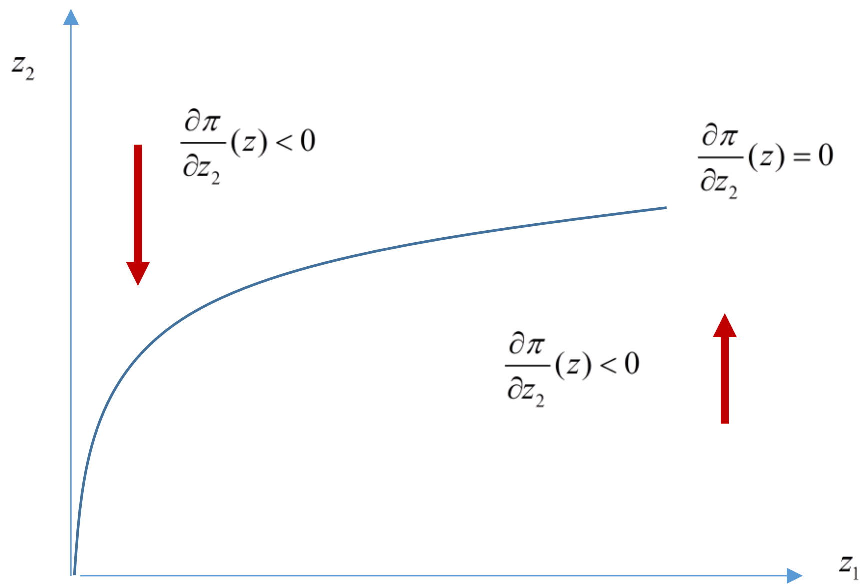
SInce marginal profit, $\frac{\partial \pi}{\partial z_1}$ declines as z_1 increases, marginal profit is positive for small z_1 and negative for large z_1. This is depicted below.



Regions of positive and negative marginal profit, $\frac{\partial \pi}{\partial z_1}$

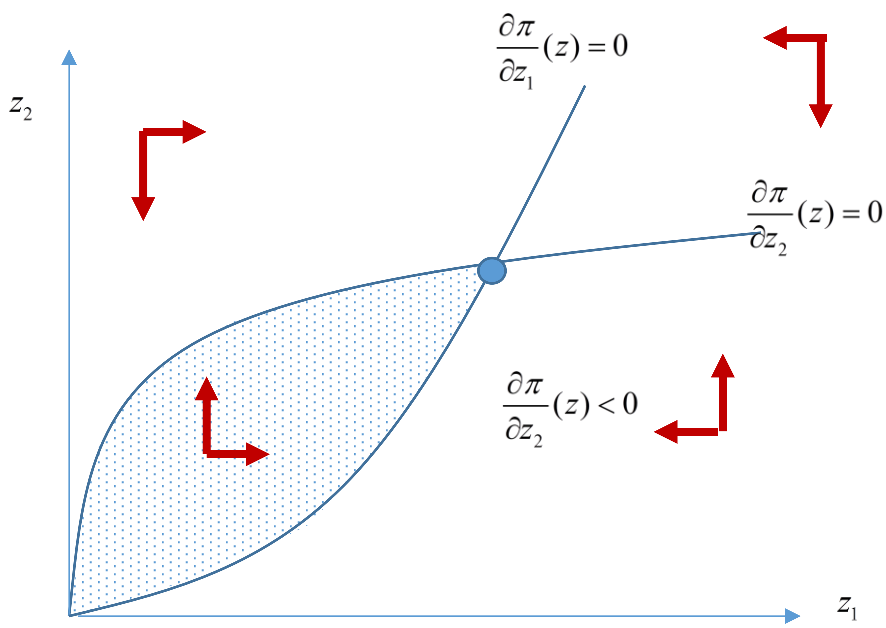
Note that the horizontal arrows indicate the direction in which profit is increasing.

If you do a similar analysis for the marginal profit of input 2 you will end up with a figure like the one below.



Regions of positive and negative marginal profit, $\frac{\partial \pi}{\partial z_2}$

We the combine the two figures below. At the intersection point, the marginal profit is zero for both inputs. So this is the critical point.

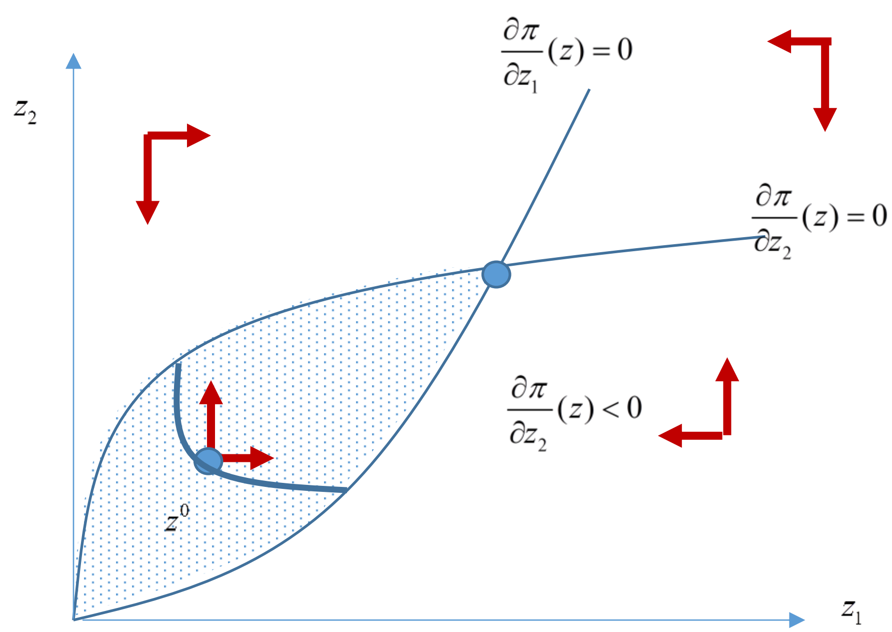


Regions of positive and negative marginal profit

Pick any initial input vector z^0 . The arrows indicate that $\pi(z)$ is increasing in the direction of the critical point thus changing the input vector to increase profit leads to the critical point.

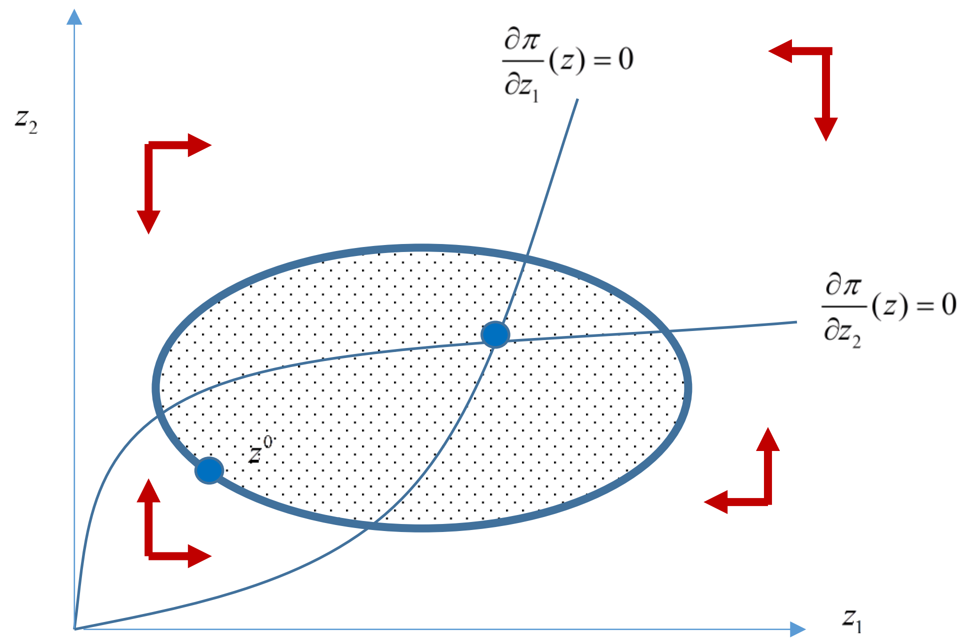
The critical point is therefore the maximizer.

Another way to explain this is to consider what the level sets of the profit hill must look like. Consider z^0 in the dotted region. Profit is increasing to the North-East and decreasing to the South-West so the level set through z^0 must be as indicated below.



Regions of positive and negative marginal profit

Making a similar argument in the other three regions, it follows that the level set is as depicted below.



Level set through z^0. The dotted region is the

super-level set $\pi(z) \geq \pi(z^0)$

All the points at which the profit hill is higher are in the dotted region. This is the superlevel set $\pi(z) \geq \pi(z^0)$. Thus the top of the profit mountain must be the critical point.

Exercise: Suppose instead that $F(z) = z_1^{1/2} z_2^{3/4}$.

(a) Show that there is a unique critical point.
(b) Draw the level sets and you will see that the critical point is not the maximizer.
(c) Is there a maximizer? Hint: Consider the profit with input $z = (\theta, \theta)$. For what values of θ does the profit increase with θ ?