3. ROBINSON CRUSOE ECONOMY

3.1 Production set and profit maximization.

A firm has a production set \( \mathcal{Y} = \{ y | -128y_1 - y_2^2 \geq 0, y_1 \leq 0, y_2 \geq 0 \} \).

(a) What is the production function of the firm? HINT: \( -y_1 \) is labor demand and \( y_2 \) is output.

(b) Appeal to any propositions you wish to confirm that the production set is convex.

(c) The firm is a price taker. Solve for the profit maximizing input, output and profit given the price vector \( p \).

3.2 Utility-maximizing worker

A consumer has utility function \( U(x) = (24 + x_1)x_2 \). The consumer begins with 24 units of time per day and supplies some of these hours to a firm. (So \( x_1 < 0 \).) He purchases commodity 2.

(a) If the consumer has a dividend \( D \) each day and the price vector is \( p \), what is his budget constraint.

(b) Solve for his utility maximizing choice.

3.3 Robinson Crusoe economy.

If there is just one person on an island with production set and preferences as given above, what will this consumer choose?

3.4 Robinson Crusoe Economy

Rob Crusoe has a utility function \( U(x) = (24 + x_1)x_2 \). The consumer begins with 24 units of time per day and supplies some of these hours to a firm. (So \( x_1 < 0 \).) He purchases commodity 2.

(a) If the consumer has a dividend \( D \) each day and the price vector is \( p \), what is his budget constraint.
(b) Solve for his utility maximizing choice.

Rob owns a firm with production set \( Y = \{ y \mid -100y_1 - y_2^2 \geq 0, y_1 \leq 0, y_2 \geq 0 \} \).

(c) What is the production function of the firm? HINT: \(-y_1\) is labor demand and \(y_2\) is output.

(d) Appeal to any propositions you wish to confirm that the production set is convex.

(e) The firm is a price taker. Solve for the profit maximizing input, output and profit given the price vector \( p \).

(f) If Rob Crusoe is the only person on an island with production set and preferences as given above, what will he choose?

### 3.5 Robinson Crusoe Economy

There is a single firm and a single consumer. The firm uses commodity 1 as an input (so \( y_1 \leq 0 \)) and produces commodities 2 and 3. The production set of the firm is

\[ Y = \{ y \mid y_1^2 - 2y_2^2 - 2y_3^2 \geq 0 \} \]. (This is a convex set.) The utility function of the consumer is

\[ U(x) = \ln x_1 + \ln x_2 + \ln x_3 \]. There is an endowment of 12 units of commodity 1 and no other endowment.

(a) Explain why the best production plan for Robinson Crusoe is the maximizer of the following optimization problem.

\[ \max_y \{\ln(y_1 + 12) + \ln y_2 + \ln y_3 \mid y_1^2 - 2y_2^2 - 2y_3^2 \geq 0\} \]

(b) From the FOC confirm that \( y^* = (-8, 4, 4) \) and hence \( x^* = (4, 4, 4) \) is the maximizer.

(c) What price vector is a WE price vector for this economy?

HINT: You can consider the optimization problem of either the price taking firm or the consumer at some price vector \( p \) and then find prices consistent with \( y^* \) or \( x^* \).

(d) What is the maximized profit of the firm at these prices?

(e) Would the price vector still be a WE price vector if there were many consumers and for each \( h \), \( U^h(x^h) = \ln x_1^h + \ln x_2^h + \ln x_3^h \)? Explain.
(f) What would be the answer to (c) if the initial endowment were instead \( \omega = (\omega_1, 0, 0) \)?

**ANSWERS**

### 3.1 Production set and profit maximization.

Convert to a problem with all non-negative commodities.

Let \( L = \text{labor} \) and \( q = \text{output} \). Then \( L = y_1 \) and \( q = y_2 \).

\[
Y = \{ (L, q) \mid 128L - q^2 \geq 0, \ (L, q) \geq 0 \}
\]

Good idea to draw the production set.

The maximum output that can be produced for any input \( L \) satisfies \( 128L - q^2 = 0 \). That is \( q = (128L)^{1/2} \).

This is the production function of the firm.

Define \( h(L, q) = 128L - q^2 \). This is a concave function since the first term is linear and \(-q^2\) has a decreasing slope.

It is therefore quasi-concave and so the upper contour set \( h(L, q) \geq 0 \) bounds a convex set.
Profit $\Pi = p_2 q - p_1 L$

Profit maximization: $\text{Max}\{\Pi = p_2 q - p_1 L | 128L - q^2 \geq 0\}$

$\mathcal{L} = p_2 q - p_1 L + \lambda (128L - q^2)$.

Look for a strictly positive solution. Since both $\Pi(L, q)$ and $h(L, q)$ are concave, if we find a solution to the FOC they are both necessary and sufficient for a maximum.

FOC

\[
\frac{\partial \mathcal{L}}{\partial L} = -p_1 + 128\lambda = 0 \\
\frac{\partial \mathcal{L}}{\partial q} = p_2 - 2\lambda q = 0.
\]

$\lambda = p_1 / 128$ then $p_2 - p_1 q / 64 = 0$ and so $q^* = 64p_2 / p_1$ and so $L = q^2 / 128 = 32(p_2 / p_1)^2$ and $\Pi^* = 32p_2^2 / p_1$.

### 3.2 Utility-maximizing worker

Convert to a problem with positive variables. $L =$ labor $q =$ consumption. Then $L = -x_1$ and $q = x_2$.

Budget constraint

Expenditure = $p_2 q$ cannot exceed income = $p_1 L + D$

$D + wL - p_2 q \geq 0$

Utility $U(L, q) = (24 - L)q$

Maximization problem:

$\text{Max}\{U(L, q) = (24 - L)q | D + p_1 L - p_2 q \geq 0\}$.

This is a quasi-concave problem since the logarithm of the maximand is concave.

Seek a strictly positive solution to the FOC.

$\mathcal{L} = (24 - L)q + \lambda (D + p_1 L - p_2 q)$
FOC

\[ \frac{\partial \mathcal{L}}{\partial L} = -q + \lambda p_1 = 0 \quad \Rightarrow q = \lambda p_1 \]

\[ \frac{\partial \mathcal{L}}{\partial q} = 24 - L - \lambda p_2 = 0 \quad \Rightarrow L = 24 - \lambda p_2 \]

\[ \frac{\partial \mathcal{L}}{\partial \lambda} = D + p_1 L - p_2 q = 0 \]

Substituting into the budget constraint,

\[ D + p_1 (24 - \lambda p_2) - p_2 \lambda p_1 = 0 \quad \text{Then} \quad 2 p_1 p_2 \lambda = D + 24 p_1 \quad \text{and so} \quad \lambda = \frac{D + 24 p_1}{2 p_1 p_2} \].

Then \( q = \frac{D + 24 p_1}{2 p_2} \) and \( L = 24 - \frac{D + 24 p_1}{2 p_1} = 12 - \frac{D}{2 p_1} \).

### 3.3 Robinson Crusoe economy.

Maximization problem

Maximize \( U(L, q) \) subject to \( h(L, q) \geq 0 \)

For a change of pace note that for any \( L \) the consumer will maximize \( q \) thus

\( h(L, q) = 100 L - q^2 = 0 \). Hence \( q = (128 L)^{1/2} \)

Substituting into the utility function,

\[ U = (24 - L) 128^{1/2} L^{1/2} = 128^{1/2} (24 L^{1/2} - L^{3/2}) \]

Taking the derivative and solving, \( L^* = 8 \). Then \( q^* = (128 L^*)^{1/2} = 32 \).

Returning to question 1, what prices will lead Robinson Crusoe to this outcome?

\[ q^* = 64 p_2 / p_1 = 32 \].

Therefore

\[ p_2 / p_1 = 1/2 \quad \text{and so} \quad p = (2, 1) \] is a market clearing price vector.
You can check to confirm that demand and supply are equal in both markets. But remember that the dividend $D$ is the distributed profit of the firm.

Note: An example like this is discussed at length in section 5.1 of EM.