4. 2×2 EXCHANGE ECONOMY

4.1 Equilibrium in an exchange economy

There are two consumers A and B with the same utility function $U^h(x^h) = \ln x_1^h + x_2^h$
$h \in H = \{A, B\}$ The aggregate endowment is $\omega = (1,1)$. Consider price vectors normalized so that prices sum to 1. That is $(p_1, p_2) = (\pi, 1-\pi)$.

(a) Show that if each consumer purchases both commodities, market demand for commodity 1 is $x_1(p) = \frac{2(1-\pi)}{\pi}$.

(b) Hence solve for the equilibrium price vector.

(c) What is each consumer’s demand for commodity 2?

(d) Use your answer to show that (b) is not the equilibrium price vector if $\omega_A^A + \frac{1}{2} \omega_B^A \leq \frac{1}{2}$.

(e) Are there any other restrictions on $\omega_A^A$ that must be satisfied?

Let $x^A(p, \omega_A^A)$ be A’s demand. Let $p^E(\omega_A^A)$ be the equilibrium value of $p$ when consumer A’s endowment is $\omega_A^A$. A’s equilibrium consumption is therefore $x^A(p^E(\omega_A^A), \omega_A^A)$. Note that as A’s endowment approaches zero, her demand must approach zero. That is, if there is a equilibrium,

$$\lim_{\omega_A^A \to 0} x^A(p^E(\omega_A^A), \omega_A^A) = 0.$$ 

In equilibrium markets must clear. Therefore $\lim_{\omega_A^A \to 0} x^A(p^E(\omega_A^A), \omega_B^A) = \omega = (1,1)$

(f) Use this observation to show that, in the limit, the equilibrium price vector $p(\omega_A^A)$ approaches $(\frac{1}{2}, \frac{1}{2})$. That is

$$\lim_{\omega_A^A \to 0} p^E(\omega_A^A) = (\frac{1}{2}, \frac{1}{2}).$$

(g) BONUS: Is it also true that $\lim_{\omega_A^A \to 0} p^E(\omega_A^A) = (\frac{1}{2}, \frac{1}{2})$?
4.2 Pareto Efficiency and Walrasian Equilibrium

Consumer A has utility function \( U^A(x^A) = (x_1^A + 4)(x_2^A + 2) \). Consumer B has utility function \( U^B(x^B) = x_1^B x_2^B \). The aggregate endowment is \( \omega = (16, 38) \).

Consider a PE allocation \( \{x^A, x^B\} \) where \( x^h \gg 0, \ h \in \mathcal{H} = \{A, B\} \). Show that the set of such allocation is a line through one of the corners of the Edgeworth Box.

Characterize the set of endowments for which the WE is on this line. Is the WE price ratio the same for all these endowments?

What are the other PE allocations? Explain and provide a formal derivation by appealing to the K-T conditions.

What are the possible Walrasian Equilibrium price ratios in this economy?

4.3 Pareto Efficiency and Walrasian Equilibrium when consumers have identical CES preferences

Consumer \( h \) has utility function \( U^h(x^h) = \left(\frac{\alpha_1}{x_1^h} + \frac{\alpha_2}{x_2^h}\right)^{-1} \), \( h \in \mathcal{H} = \{A, B\} \) and \( \alpha = (\alpha_1, \alpha_2) \gg 0 \).

The aggregate endowment is \( \omega = (100, 100) \).

(a) Solve for the PE allocations \( \{x^A, x^B\} \) where \( x^h \gg 0, \ h \in \mathcal{H} = \{A, B\} \).

(b) Are there also PE allocations where consumption by one consumer is not strictly positive?

(c) As the endowment of consumer A changes, how does the WE price ratio change?

(d) How do the answers change if the set of consumers is \( \mathcal{H} = \{A, B, C\} \)?

4.4 Pareto Efficiency and Walrasian Equilibrium

Consumer A has utility function \( U^A(x^A) = (x_1^A + \gamma_1)(x_2^A + \gamma_2) \). Where \( \gamma = (\gamma_1, \gamma_2) \gg 0 \).

Consumer B has utility function \( U^B(x^B) = x_1^B x_2^B \). The aggregate endowment is \( \omega \).
(a) Consider a PE allocation \( \{ x^A, x^B \} \) where \( x^h >> 0 \), \( h \in H = \{ A, B \} \). Show that the set of such allocation is a line must go through at least one of the corners of the Edgeworth Box. Under what condition does it go through both corners?

Henceforth assume that \( \gamma = (4,2) \) and that the aggregate endowment is \( \omega = (16,38) \)

(b) Characterize the set of endowments for which the WE allocation is in the interior of the Edgeworth Box. Is the WE price ratio the same for all these endowments?

(c) What are the other PE allocations? Explain and provide a formal derivation by appealing to the K-T conditions.

(d) What are the possible Walrasian Equilibrium price ratios in this economy?

### 4.5 Pareto Efficiency and Walrasian Equilibrium

Consumer \( h \) has utility function \( U^h(x^h) = (x_1^h + \gamma_1^h)(x_2^h + \gamma_2^h) \), \( h \in H = \{ A, B \} \) where \( \gamma^h > 0 \). The aggregate endowment is \( \omega \).

(a) Define \( \hat{x}^h = x^h + \gamma^h \). Then preferences can be written as \( U^h(\hat{x}^h) = \hat{x}_1^h \hat{x}_2^h \). Confirm that for aggregate feasibility \( \hat{x}_1^h + \hat{x}_2^h \leq \omega \equiv \omega^A + \omega^B \).

(b) Consider a PE allocation \( \{ x^A, x^B \} \) where \( x^h >> \gamma^h \), \( h \in H = \{ A, B \} \). Show that the set of such allocation is a line.

(c) For each such PE allocation, solve for the supporting price ratio.

### 4.6 Pareto Efficiency and Walrasian Equilibrium in a linear economy

Consumer A has utility function \( U^A(x^A) = x_1^A + x_2^A \). Consumer B has utility function \( U^B(x^B) = x_1^B + \frac{1}{2} x_2^B \). The aggregate endowment is \( \omega = (100, 200) \).

(a) Using an Edgeworth Box diagram explain why no PE allocation can lie in the interior of the box.

(b) Characterize the PE allocations and show that for all PE allocations sufficiently favorable to consumer A, the supporting price ratio \( \pi^A = p_1 / p_2 \) is one of the following (i) 1 (ii) 1.5 (iii) 2.

(c) What is the supporting price ratio \( \pi^B \) for PE allocations that are sufficiently favorable to consumer B?

(d) For what endowments \( \omega^A \) is the WE price ratio equal to (i) \( \pi^A \) (ii) \( \pi^B \). Hint: By the First Welfare Theorem a WE is Pareto Efficient.
(e) For what endowments $\omega^i$ is the WE price ratio neither $\pi^A$ nor $\pi^B$. In such cases what are the WE allocations?

### 4.7 Pareto Efficient Allocations

Consumer $h$, $h \in H = \{1, ..., H\}$ has utility function

\[ U^h(x^h) = (\beta^h + x^h_1)^\alpha + (x^h_2)^\beta, \quad 0 < \alpha < 1, \quad \beta^h > 0 \quad \text{and} \quad \sum_{h \in H} \beta^h = \beta \]

The aggregate endowment is $\omega = (\gamma - \beta, \gamma)$.

For this question assume that $H = 2$, $\beta^1 = \beta^2 = 1$ and $\gamma = 20$.

An allocation is Pareto Efficient if there is no other feasible allocation in which no consumer is worse off and at least one is strictly better off. Thus the above two allocations are PE allocations.

(a) Prove that the allocation $\{\vec{x}^h\}_{h \in H}$ that maximizes the utility of consumer 2 (Bev) given that consumer 1 (Alex) must have a utility of at least 1 is $\{\vec{x}^h\}_{h \in H} = \{(0,1),(18,19)\}$. Solve also for the allocation $\{\vec{x}^h\}_{h \in H}$ that maximizes the utility of Alex given that consumer Bev must have a utility of at least 2.

(Be precise about why the Constraint Qualifications hold and why the necessary conditions are also sufficient.)

The first figure below depicts the allocations to Alex and his indifference curves through the two allocations.
Bev’s allocation is \( x^2 = \omega - x^1 \). Thus we can depict Bev’s allocations and indifference curves as shown below.

Superimposing the two diagrams we have the Edgeworth Box diagram. (see EM Chapter 3)

(b) Show that any convex combination of \( \{ \bar{x}^h \}_{h \in H} \) and \( \{ \overline{x}^h \}_{h \in H} \) is also a PE allocation.

Graphically, the convex combinations are the points on the line joining \( \bar{E} \) and \( \overline{E} \).
(c) Are there any other PE allocations. Explain using the Edgeworth Box diagram.

(d) Consider the PE allocation \( \{ \hat{x}^h \}_{h \in H} \) for any \( \beta, \gamma \). Explain why \( \{ \hat{x}^h \}_{h \in H} \) uniquely solves

\[
\max \{ U^1(x^1) \mid U^2(x^2) \geq U^2(\hat{x}^2), \sum_{h \in H} x^h \leq \omega \}
\]

(e) Let \( \mu_j \) be the shadow price associated with the constraint \( \sum_{h \in H} x^h_j \leq \omega_j \). Show that \( \mu \gg 0 \).

(f) Suppose that the initial endowments \( \{ \omega^h \}_{h \in H} \) are Pareto Efficient. Write down the necessary and sufficient conditions for utility maximization if the price vector is \( p = \mu \).

(g) Compare these with the necessary and sufficient conditions for a PE allocation. Hence show that no consumer will wish to trade at these prices.

Since no consumer wishes to trade, \( p = \mu \) is an equilibrium (market clearing) price vector.

4.8 Pareto Efficient Allocations*

Consider the same economy as the previous question for any \( H, \beta, \gamma \).

(a) Consider the PE allocation \( \{ \hat{x}^h \}_{h \in H} \). Explain why \( \{ \hat{x}^h \}_{h \in H} \) uniquely solves

\[
\max \{ U^1(x^1) \mid U^2(x^2) \geq U^2(\hat{x}^2), h > 1 \sum_{h \in H} x^h \leq \omega \}
\]

(b) Let \( \mu_j \) be the shadow price associated with the constraint \( \sum_{h \in H} x^h_j \leq \omega_j \). Show that \( \mu_1 = \mu_2 > 0 \).

(c) Suppose that the initial endowments \( \{ \omega^h \}_{h \in H} \) are Pareto Efficient. Write down the necessary and sufficient conditions for utility maximization if the price vector is \( p = \mu \).

(d) Compare these with the necessary and sufficient conditions for a PE allocation. Hence show that no consumer will wish to trade at these prices. That is,

\[ U(x^h) > U(\hat{x}^h) \Rightarrow \mu \cdot x^h > \mu \cdot \hat{x}^h. \]

(e) Suppose that the utility function is \( U^h(x^h) = u(\beta^h + x^1_h) + u(x^h_2) \) where \( u(x) = \frac{1}{x^{1/\delta}}, \delta > 0 \).
(This is the same function if $\sigma > 1$.) Show that all the arguments above continue to hold.

4.9 Walrasian Equilibrium and Pareto efficiency when all consumers have a strictly positive consumption

(a) Alex has utility function $U^A = x_1^A x_2^A$. Solve for his demand for commodity 1 if his endowment is $\omega^A$.

(b) Bev has utility function $U^B = x_1^B x_2^B$. Solve for her demand for commodity 1 if her endowment is $\omega^B$.

(c) Solve for the WE price ratio and show that it depends only on the ratio of aggregate endowments.

(d) An allocation is Pareto efficient if it is not possible to increase the utility of one consumer without lowering the utility of another. In this 2 person economy explain why the PE allocation $\{\hat{x}^A, \hat{x}^B\}$ must have the following property.

$$\hat{x}^A \in \arg \max \{U^A(x^A) | U^B(x^B) \geq U^B(\hat{x}^B) | x^A + x^B \leq \omega\}.$$ 

(e) Show that the PE allocations satisfy $x_1^A = x_1^B = \frac{\omega_1}{\omega_2}$.

HINT: What does efficiency imply about marginal rates of substitution?

(f) Is the WE allocation a PE allocation?

4.10 Walrasian Equilibrium and Pareto efficiency

$U^A = x_1^A (x_2^A + 22)$ and $U^B = (7 + x_1^B) x_2^B$. The endowments are $\omega^A = (8, 8)$ and $\omega^B = (2, 2)$.

(a) Consider the price vector $p = (5, 2)$. Solve for the demands for the two commodities at these prices and hence determine whether or not markets clear.

HINT: Be careful when solving for Bev’s demand.
(b) What must be true if there is a PE allocation \( \{ \hat{x}^A, \hat{x}^B \} \) in this economy in which \( \hat{x}^A >> 0 \) and \( \hat{x}^B >> 0 \)?

(c) Use your answer to show that there can be no such PE allocation.

(d) Try to draw the Edgeworth Box diagram for this economy showing all the PE allocations.

4.11 Edgeworth Box with linear utility

Alex has utility function \( U^A(x) = a \cdot x \) and Bev has utility function \( U^B(x) = b \cdot x \) where \( x \in \mathbb{R}^2_+ \).

Suppose that \( \frac{a_1}{a_2} > \frac{b_1}{b_2} \).

(a) Write down the optimization problem that must be solved to obtain a PE allocation. Hence show that for an allocation to be PE it cannot be the case that \( x^A >> 0 \) and \( x^B >> 0 \).

(b) Use the Lagrange method or a graphical argument to characterize the set of PE allocations.

(c) Depict these in an Edgeworth Box diagram.

(d) If Alex has a large endowment relative to Bev and the endowments are not PE, do both gain from trade in a WE? If so explain. If not, why not.

(e) For every PE allocation what is the no-trade price ratio.

(f) Hence or otherwise draw a conclusion as to the possible WE price ratios in this economy as the endowments vary.

(g) For what endowments if any do both consumers consume just one commodity in a WE?

4.12 Equilibrium and efficiency with L-shaped preferences.

Alex and Bev have the following utility functions \( U^A(x^A) = \text{Min} \{ x_1^A, 2x_2^A \} \), \( U^B(x^B) = \text{Min} \{ x_1^B, x_2^B \} \). The aggregate endowment is \( \omega = (6,5) \). Throughout normalize so that the sum of the prices is 1.

(a) Draw the Edgeworth Box showing with dotted lines the “kinks” in the indifference curves. Suppose that Alex has an endowment of \((2,1)\) so that Bev’s endowment is \((4,4)\). Explain why this is a PE allocation. What price ratios are supporting? That is, what are the WE price vectors given this endowment?
(b) For what endowments is this allocation a WE allocation? HINT: There are two areas of the Edgeworth Box.

(c) What are all the PE allocations in the Edgeworth Box?

(d) What prices are supporting prices for all the PE allocations other than the one in (a)?

Extra: (Do not hand in.) What are the possible equilibrium prices if \( \omega = (5, 6) \)?

4.13 Pareto Efficient Allocations

\[ U^A = \alpha^A \ln x^A_1 + \ln x^A_2, \quad U^B = \alpha^B \ln x^B_1 + \ln x^B_2. \]
Suppose \( \gamma = \alpha^B / \alpha^A < 1 \). The aggregate endowment is \( \omega = (\omega_1, \omega_2) \).

(a) Explain why the PE allocations in the interior of the Edgeworth Box must lie below the diagonal.

(b) What condition must hold for an allocation to be a PE allocation in the interior of the Edgeworth Box?

(c) Appealing to the ratio rule show that at an interior PE allocation

\[
\frac{x^A_2}{x^A_1} = \frac{\gamma x^B_2}{x^B_1} = \frac{(1-\gamma)x^A_1 + \gamma \omega_2}{\omega_1}.
\]

(d) Hence show that that \( x^A_2 / x^A_1 \) rises along the map of the PE allocations as \( x^A_1 \) increases.

(e) Let \((p_1, p_2)\) be a supporting price vector of a PE allocation. Normalize so that the prices sum to 1. (A mathematician would say that the price vector lies on the unit simplex.) Show that \( p_1 \) rises along the map of PE allocations as \( x^A_1 \) increases.

(f) What is the range of possible PE prices of commodity 1?

(g) What is the range of possible WE prices of commodity 1 (for all possible initial endowments)? Explain.

(h) As Alex’s endowment approaches zero (holding \( \omega \) constant) what will be the limiting WE price of commodity 1? What if Alex’s endowment approaches \( \omega \)?
4.14 WE with different preferences

Consumer A (Alex) has utility function $U^A(x^A) = \alpha \ln x^A_1 + (1-\alpha) \ln x^A_2$ and endowment $\omega^A = (a,a)$.

Consumer B (Bev) has utility function $U^B(x^B) = \beta \ln x^B_1 + (1-\beta) \ln x^B_2$ and endowment $\omega^B = (b,b)$.

where $\alpha > \beta > 0$.

Throughout normalize by considering price vectors on the unit simplex so that $p_1 + p_2 = 1$

(a) What is the WE price vector if (i) $0 = b < a$ (ii) $0 = a < b$

(b) Explain why the answer to (i) would be the same if there were a group of consumers with the same utility function as Alex and a total endowment of $(a,a)$.

Henceforth suppose that $a$ and $b$ are both strictly positive.

(c) Solve for Alex and Bev’s demand for commodity 1.

(d) Hence solve for the WE price of commodity 1. Hint: What is the definition of a WE allocation?

(e) What is the WE price of commodity 2?

(f) Show that if $a$ rises the WE price of commodity 1 rises and the WE price of commodity 2 falls.

(g) Provide the intuition.

(h) Draw an Edgeworth box diagram and carefully depict the indifference curves through the endowment $\omega^A = (a,a)$.

(i) Does it follow that any PE allocation in the interior of the Edgeworth box lies below the diagonal?

(j) Does it follow that any WE allocation with $a > 0$ and $b > 0$ in the interior of the Edgeworth box lies below the diagonal?
5. WALRASIAN EQUILIBRIUM

5.1 No trade economy

(a) Suppose at the price vector $p$ consumer $h$ does not wish to trade. Write down the FOC and hence show that if utility is differentiable and strictly increasing, then for some $\lambda$

$$\frac{\partial U^h}{\partial \omega^h}(\omega^h) = \lambda p.$$ 

(b) Suppose that in an endowment economy no individual wishes to trade at price vector $p$. Then for each $h$ demand $x^h(p) = \omega^h$ and so aggregate demand is equal to total supply. The price vector is therefore market clearing. If $U^h(x^h) = (1 + x^h)^\alpha + (x^h)^\alpha$, $0 < \alpha < 1$ and $\omega^h = (\gamma^h, 1 + \gamma^h)$, $h \in \mathcal{H}$ show that there is a no trade equilibrium. What is the market clearing price?

(c) Suppose instead that $\omega^h = (\gamma^h, \delta^h)$, $h \in \mathcal{H}$. Show that there is again a no-trade equilibrium and solve for the equilibrium price.

5.2 Local Non-satiation

A utility function satisfies the local non satiation property over the consumer’s consumption set $X$ if for any $\hat{x} \in X \subset \mathbb{R}^n$ and any $\delta > 0$ there exists $\hat{x} \in N(\hat{x}, \delta)$ such that $U(\hat{x}) > U(\hat{x})$.

Define $e = (1, ..., 1)$. The set $H^n = \{x \mid \hat{x} - \delta e \leq x \leq \hat{x} + \delta e\}$ is a hypercube with center $\hat{x}$.

(a) $H^2$ is a square. Depict the square and the neighborhood $N(\hat{x}, \delta)$. Explain why $N(\hat{x}, \delta) \subset H^2$.

(b) For all $n$ show that $N(\hat{x}, \delta) \subset H^n$.

Suppose that $p \cdot \hat{x} < I$. Prove that for all sufficiently small $\delta > 0$,

$$x \in N(\hat{x}, \delta) \Rightarrow p \cdot x < I$$

(c) Use (b) to show that if utility satisfies the LNS property, $\hat{x} \notin \arg \max_{x \geq 0} \{U(x) \mid p \cdot x \leq I\}$

5.3 Price taking consumers.

Consumer $h \in \mathcal{H} = \{1, ..., H\}$ has continuous utility function $U^h(x^h)$ defined on $\mathbb{R}_+^n$ and endowment $\omega^h$. The LNS property holds on $\mathbb{R}_+^n$. The price vector is $p > 0$. Suppose that
\( \bar{x}^h \in \arg \max \{ U^h(x^h) \mid p \cdot x \leq p \cdot \omega^h \} . \)

(a) Use your answer to exercise 3 to explain why \( p \cdot \bar{x}^h = p \cdot \omega^h . \)

(b) Explain why
\[
U^h(x^h) > U^h(\bar{x}^h) \implies p \cdot x^h > p \cdot \bar{x}^h
\]

(c) Prove that
\[
U^h(x^h) \geq U^h(\bar{x}^h) \implies p \cdot x^h \geq p \cdot \bar{x}^h.
\]

(d) Let \( \{ x^h \}_{h \in \mathbb{R}} \) be an allocation that is Pareto Preferred to \( \{ \bar{x}^h \}_{h \in \mathbb{R}} . \) (No consumer worse off and at least one consumer strictly better off.) Use your answers to show that
\[
\sum_{h \in \mathbb{R}} p \cdot x^h > \sum_{h \in \mathbb{R}} p \cdot \bar{x}^h.
\]

(e) Appealing to (a) and (d)
\[
\sum_{h \in \mathbb{R}} p \cdot (x^h - \omega^h) = p \cdot \left( \sum_{h \in \mathbb{R}} (x^h - \omega^h) \right) > 0.
\]

Explain why it follows that the Pareto preferred allocation \( \{ x^h \}_{h \in \mathbb{R}} \) does not satisfy the feasibility constraint
\[
\sum_{h \in \mathbb{R}} x^h \leq \sum_{h \in \mathbb{R}} \omega^h.
\]

5.4 Second Welfare Theorem in an exchange economy

Consumer \( h = A, B \) has a strictly increasing quasi-concave utility function \( U^h(x) \) and an endowment \( \omega^h \in \mathbb{R}^n_+ . \)

(a) If the price vector is \( \mu \) . Confirm that the Kuhn-Tucker conditions for \( \bar{x}^h \) to be a most preferred choice can be written as follows.
\[
\frac{\partial U^h}{\partial x}(\bar{x}^h) - \nu^h \mu \leq 0 \quad \text{and} \quad \bar{x}^h \cdot \left( \frac{\partial U^h}{\partial x}(\bar{x}^h) - \nu^h \mu \right) = 0.
\]

(b) The Kuhn-Tucker conditions are sufficient for a maximum if certain conditions are satisfied. Confirm that these conditions are satisfied.

(c) Show that the following conditions are necessary conditions for \( \{\hat{x}^h\}_{h=1}^H \) to be Pareto efficient.

\[
\lambda^h \frac{\partial U^h}{\partial x}(\hat{x}^h) - \mu \leq 0 \quad \text{and} \quad \hat{x}^h \cdot \left( \frac{\partial U^h}{\partial x}(\hat{x}^h) - \mu \right) = 0, \quad \text{where} \quad \lambda^A = 1.
\]

(d) Appeal to (a) and (b) to show that the allocation \( \{\hat{x}^h\}_{h=1}^H \) is a WE if the endowment is \( \{\omega^h\}_{h=1}^H = \{\hat{x}^h\}_{h=1}^H \).

(e) To achieve this final allocation given some other endowments \( \{\omega^h\}_{h=1}^H \), what lump sum tax/subsidy is needed?

### 5.5 Equilibrium and efficiency

Individual \( h, \ h=1,...,H \) has a utility function \( U(\chi^h) = \sum_{j=1}^{2} \alpha_j v^h(\chi^h) \). (Note that the utility functions may be different.) The aggregate endowment of commodity 1 and of commodity 2 is \( \omega_1 = \omega_2 = k \).

(a) Show that for Pareto efficiency \( \frac{x^h_2}{x^h_1} = 1 \).

(b) Solve for the WE price ratio on this economy and show that it is independent of individual endowments.

(c) If \( \alpha = (\alpha_1, \alpha_2) = (\frac{1}{4}, \frac{3}{4}) \) confirm that the price vector \( p = (\frac{1}{4}, \frac{3}{4}) \) is a WE price vector.

(d) In a 2 person economy, if \( \omega^A = (5,1) \), \( \omega^2 = (5,1) \) and \( p = (\frac{1}{4}, \frac{3}{4}) \), how much would Alex need to be subsidized for the final allocation to be equal?

### 5.6 Walrasian equilibrium
A consumer has a consumption set \( X = \{(x_1, x_2) \mid (x_1, x_2) \geq (2, 2)\} \) and utility function \( U(x) = \alpha \ln(x_1 - 2) + \ln(x_2 - 2) \). His endowment is \( \omega \).

(a) If the price vector is \( p \) show that his consumption bundle must satisfy the following condition.

\[
\frac{p_1(x_1 - 2)}{\alpha} = \frac{p_2(x_2 - 2)}{1}.
\]

Appealing to the ratio rule it follows that

\[
\frac{p_1(x_1 - 2)}{\alpha} = \frac{p_2(x_2 - 2)}{1} = \frac{p \cdot x - 2p_1 - 2p_2}{1 + \gamma}.
\]

(b) Hence show that demand is

\[
x_i(p) = 2 + \frac{\gamma}{1 + \gamma} ((\omega_i - 2) + \frac{p_2}{p_1} (\omega_2 - 2)).
\]

(c) Suppose that there are two consumers. Alex has utility function \( U(x^A) = 7 \ln(x_1^A - 2) + \ln(x_2^A - 2) \) and endowment \( \omega^A = (9 + \alpha, 1 - \alpha) \) where \( \alpha \in [-9, 1] \). Bev has utility function \( U(x^B) = \frac{1}{7} \ln(x_1^B - 2) + \ln(x_2^B - 2) \) and endowment \( \omega^B = (1 - \alpha, 9 + \alpha) \).

Show that the aggregate demand function is \( x_i(p) = 10 - \frac{3\alpha}{4} (1 - \frac{p_2}{p_1}) \).

Hence show that \( p = (1, 1) \) is a WE price vector for all \( \alpha \).

(d) Depict the WE in an Edgeworth Box diagram.

(e) Note that the aggregate demand for commodity 1 is a decreasing function of \( p_1 \) if \( \alpha < 0 \) and an increasing function if \( \alpha > 0 \). What can you say about WE if \( \alpha = 0 \)?

### 5.7 Walrasian Tatonnement

A consumer has a strictly increasing and strictly concave utility function \( U \) and consumption set \( X(a) = \{x \mid x \geq a\} \).

\[
\bar{x}(p, I, a) \equiv \arg \max_{x \in X(a)} \{U(x - a) \mid p \cdot x \leq I\}.
\]
Note that that there is a solution if and only if \( I - p \cdot a \geq 0 \). Note also that if \( a = 0 \),

\[
\bar{x}(p, I, 0) \equiv \arg \max_{x \in \mathbb{R}^2_+} \{ U(x) \mid p \cdot x \leq I \}
\]

(a) Show that if \( I - p \cdot a \geq 0 \) then \( \bar{x}(p, I, a) \equiv a + \bar{x}(p, I - p \cdot a, 0) \).

HINT: Define the new vectors \( X = x - a \) and \( \Omega = \omega - a \) and rewrite the maximization problem in terms of \( X \) and \( \Omega \).

(b) Hence show that if \( U(x) = \ln(x_1) + \beta \ln(x_2) \) and \( I - p \cdot a \geq 0 \) then

\[
\bar{x}_1(p, I, a) \equiv a_1 + \frac{1}{1+\beta} \left( I - \frac{p \cdot a}{p_1} \right) \quad \text{and} \quad \bar{x}_2(p, I, a) \equiv a_2 + \frac{\beta}{1+\beta} \left( I - \frac{p \cdot a}{p_1} \right)
\]

Alex has a utility function \( U^A(x^A) = \ln(x_1^A - 10) + \frac{1}{5} \ln(x_2^A - 10) \) and endowment \( \omega^A(\alpha) = (30 + \alpha, 10 - \alpha) \). The aggregate endowment is \( \omega = (40, 40) \). Bev has a utility \( U^B(x^B) = 5 \ln(x_1^B - 10) + \ln(x_2^B - 10) \).

(c) Solve for their demands for commodity 1 if \( \alpha = 0 \). (Appeal to (b).)

Define \( z(p, \alpha) \) to be excess market demand with parameter \( \alpha \).

(d) Hence show that \( z(p, 0) = \frac{20}{6} \left( \frac{p_2}{p_1} - 1 \right) \).

(e) Discuss how the tatonnement (trial and error) process might work.

(f) If \( \alpha = 7 \) show that for the intersection of each consumer’s budget set and consumption set to be nonempty, the price ratio lies in some interval \( P = \{ p \mid \beta \leq \frac{p_2}{p_1} \leq \bar{p} \} \).

(g) Solve for \( z(p, \alpha) \) if \( \alpha = 7 \) and \( p \in P \)

(h) Depict the market excess demand function \( z(p, 7) \) for all price ratios such that \( p \in P \). You should put the price ratio \( p_1 / p_2 \) on the vertical axis and \( z_1 \) on the horizontal axis.

(i) Comment on the tatonnement process in this case.

EXTRA CREDIT:
Finally solve for \( z(p, \alpha) \) if \( \alpha = 5 \) and characterize any equilibria in this case.

Remark: The result is striking.

5. WALRASIAN EQUILIBRIUM

4.26 Two person economy with production.

Commodity 1 is leisure/labor and commodity 2 is food. Each consumer has an endowment of 6 units of leisure. There is a single price taking firm with production set

\[
Y = \{ (z, y_2) \mid y_2 \geq 0, \ y_2 \leq 6z \}, \ 	ext{equivalently} \ Y = \{ (y_1, y_2) \mid y_2 \geq 0, \ 6y_1 + y_2 \leq 0 \}.
\]

Alex and Bev each have the same Cobb-Douglas utility function \( U(h) = h_1h_2 \), equivalently \( h = (h_1, h_2) \in \mathbb{R}_{++}^2 \). Each owns has a 50% share of the firm.

(a) Show that preferences are homothetic and then analyze this as a representative agent model. Show that each has a WE allocation \( x^A = x^B = (3, 18) \). Note that \( p_1 \) is the wage.

We next modify the production set as follows.

\[
Y = \{ (z, y_2) \mid y_2 \leq 6z, \ 0 \leq z \leq 2, y_2 \leq 12 + \alpha(z - 2), \ z > 2 \},
\]

equivalently

\[
Y = \{ (y_1, y_2) \mid y_2 + 6y_1 \leq 0, \ -2 \leq y_1 \leq 0, \ y_2 \leq 12 - \alpha(y_1 + 2), \ y_1 < -2 \}.
\]

(b) Initially assume that \( \alpha = 1 \). Depict the firm’s production set.

(c) What is the firm’s supply correspondence?

(d) Continue to use the representative agent model to solve for the WE allocation.

HINT: Normalize by setting the wage \( p_1 = 1 \). Since food is more scarce with the new production set, the WE price of food must rise. If \( p_2 < 1 \) the firm will use 2 units of input to produce 12 units of output. The aggregate consumption vector is therefore \( (10, 12) \). Depict this in a neat figure. Explain why the WE price of food must be \( p_2 = 5/6 \).

(c) What is the WE profit?

(d) Suppose that Alex has a 100% ownership of the firm. At the WE prices from (b) show that Alex’s demand for leisure is not feasible.
(The representative consumer approach ignores the upper bound on leisure so works only if the solution to the relaxed problem satisfies this constraint.)

(e) What then would be the new WE price of food and the equilibrium allocations?

HINT: If Alex is not working and Bev has no dividend, what is the aggregate supply of labor $z(p_1, p_2)$.

BONUS

(a) How does the answer to (e) change if $\alpha = 0$?

5. TIME

5.1 Choice over time

Commodity $t$ is period $t$ consumption. There are two periods. Alex and Bev have linear preferences. Both discount the future but Bev discounts the future more.

$U^A = x_1^A + \frac{1}{2} x_2^A$, $U^B = x_1^B + \frac{1}{3} x_2^B$. The aggregate endowment is $\omega = (20,30)$.

(a) Depict the PE allocations in an Edgeworth Box diagram.

(b) What is the unique supporting price ratio $p_1 / p_2$ for a PE allocations that are close to

(i) $O^A$? (ii) $O^B$?

(c) Consumers can lend and borrow at the interest rate of $r$. Explain why the life-time budget constraint of consumer $h$ can be written as follows.

$x_1^h + \frac{1}{1+r} x_2^h \leq \omega_1^h + \frac{1}{1+r} \omega_2^h$.

(d) What is the WE price ratio and hence interest rate if the endowment is close to (i) $O^A$? (ii) $O^B$?

What is the WE price ratio and hence interest rate if the endowments are $\omega^A = (12,3)$ and $\omega^B = (8,27)$?

5.2 Economy with production
Commodity 1 and 2 are apple and coconut consumption in period 1 while commodity 3 and 4 are apple and coconut consumption in period 2. The period 1 utility is \( u(x_1^h, x_2^h) \) and the period 2 utility is \( u(x_3^h, x_4^h) \). The aggregate endowment is \( \omega \).

The discount factor is \( \delta \) so lifetime utility is

\[
U(x^h) = u(x_1^h, x_2^h) + \delta u(x_3^h, x_4^h)
\]

Suppose that all consumers have the same homothetic utility function

\[
U(x) = \ln x_1 + 4 \ln x_2 + \delta(\ln x_3 + 4 \ln x_4) \quad \omega = (\omega_1, 80, 30, 90), \quad \delta = 3/4 .
\]

Coconuts can be stored but not planted. Apples can be planted. Each apple planted in period bears \( \alpha > 1 \) apples in period 2.

(a) Solve for the WE of this economy if \( \omega_1 = 20 \) and \( \alpha = 2 \). You should normalize so that the period 1 price of apples is 1.

Henceforth suppose that \( \omega_1 = 40 \).

(b) Solve for the efficient number of apples planted for all \( \alpha > 1 \).

(c) Solve for the WE of this economy if \( \alpha = 8 \). You should normalize so that the period 1 price of apples is 1.

(d) Continuing with the data of part (c), suppose that there are no futures markets. Consumers can, however, borrow or lend at an interest rate \( r \). Solve for the WE spot prices of apples and coconuts and the period 2 WE spot prices of these commodities

(i) \( r = 0 \)  
(ii) \( r = 1 \).

5.3 Time and a production function

As in the previous question, we define \( x_3 \) to be the period 2 consumption of commodity 1 and \( x_4 \) to be the period 2 consumption of commodity 2. There is no endowment of commodity 4. It is produced using commodity 1 and commodity 2 as inputs. Let \( z = (z_1, z_2) \) be the period 1 input vector. Output is produced according to the constant returns to scale production function

\[
x_4 = 8z_1^{1/2}z_2^{1/2} .
\]

All consumers have the same homothetic utility function

\[
u^h(x) = 4 \ln x_1 + 2 \ln x_2 + \frac{3}{4}(4 \ln x_3 + 2 \ln x_4)
\]
The aggregate endowment is $\mathbf{\omega} = (19, 44, 24, 0)$.

(a) Show that it is efficient to produce 48 units of commodity 4. (Note that $z_1^{1/2}z_2^{1/2} = (z_1z_2)^{1/2}$)

(b) Solve for a WE price vector.

Henceforth suppose that $x_4 = z_1^{\beta_1}z_2^{\beta_2}$ where $\beta >> 0$.

(c) Discuss briefly (i) how to solve for the efficient production plan and (ii) whether there is a WE price vector for all $\beta >> 0$. 
6. **UNCERTAINTY**

6.1 Uncertainty on a South Pacific Island

Without a volcanic eruption Alex would have a plantation with 500 coconut palm trees and Bev 1100. However every year a volcano erupts and one or the other plantations suffers damage. “State 1” is the event that Alex’s plantation is damaged and “state 2” is the event that Bev’s plantation is damaged. Thus Alex has a state contingent endowment of $\omega^A = (100, 500)$ while Bev has a state contingent endowment of $\omega^B = (1100, 700)$. The probability of state 1 is $\frac{1}{4}$ and the probability of state 2 is $\frac{3}{4}$. Alex and Bev have VNM utility functions.

(a) What are the PE allocation in this economy? (Appeal to your answer to question 2.)

(b) If Alex and Bev can trade in “state claims” markets (i.e. insurance markets) what is the ratio of the WE state claims prices? (Again appeal to your answer to question 2.)

(c) Depict the WE and the PE allocations in an Edgeworth-Box diagram.

(d) What would be the total value of each plantation if the sum of the WE prices is 1?

(e) Suppose that instead of trading in state claims, Alex sells two thirds of his plantation to Alex. What fraction of Bev’s plantation could he purchase? (Use the prices computed in (d).)

(f) Given the new ownership shares what would be Alex’s total dividend in each state?

6.2 More uncertainty on a South Pacific Island
Without a volcanic eruption Alex would have a plantation with 5 coconut palm trees and Bev 11. However every year a volcano erupts and one or the other plantations suffers damage. “State 1” is the event that Alex’s plantation is damaged and “state 2” is the event that Bev’s plantation is damaged. Thus Alex has a state contingent endowment of $\omega^A = (1, 5)$ while Bev has a state contingent endowment of $\omega^B = (11, 7)$. The probability of state 1 is $\frac{1}{4}$ and the probability of state 2 is $\frac{3}{4}$. Alex and Bev have VNM utility functions.

(a) What are the PE allocations in this economy? (Appeal to your answer to question 2.)

(b) If Alex and Bev can trade in “state claims” markets (i.e. insurance markets) show that $p = (1/4, 3/4)$ is a WE state claims price vector. (Again appeal to your answer to question 2.)

(c) Depict the WE and the PE allocations in an Edgeworth-Box diagram.

(d) Depict Alex’s budget set in a neat figure and mark in his maximum feasible consumption of state 1 claims and state 2 claims.

(d) Show that Bev’s plantation is twice as valuable as Alex’s at the WE prices.

(e) Suppose that instead of trading in state claims Alex sells part of his plantation and purchases part of Bev’s plantation. Let the asset prices be $P = (1, 2)$. Let $(\xi_A, \xi_B)$ be Alex’s asset holdings. Since there is one unit of each plantation Alex has an endowment $(1, 0)$. His portfolio constraint is therefore

$$P_A \xi_A + P_B \xi_B = \xi_A + 2 \xi_B = 1.$$  

Therefore $\xi_B = \frac{1}{2} \left(1 - \xi_A\right)$

(f) Let $D$ be the matrix of dividends, that is $D = \begin{bmatrix} 1 & 11 \\ 5 & 7 \end{bmatrix}$. Then with portfolio holding $(\xi_A, \xi_B)$

his final consumption is $x = D \xi = \begin{bmatrix} 1 & 11 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} \xi_A \\ \xi_B \end{bmatrix} = \begin{bmatrix} 1 & 11 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} \xi_A \\ \frac{1}{2} \left(1 - \xi_A\right) \end{bmatrix}$.

Write out expression for $x_1$ and $x_2$ as functions of $\xi_A$. Eliminate $\xi_A$ and so obtain the feasible set of final consumption bundles with asset trading. Compare this with your answer to (d).
Remark: The point of this example is that if there are many as linearly independent dividend vectors as states, then trade in asset markets can perfectly substitute for trade in state claims markets.

6.3 Acceptable gambles

There are two states. Charles is a risk-averse consumer with wealth \( w \). If he chooses to gamble he gains \( x_i \) in state \( s \). (Of course this may be negative or positive.) The probability of state \( s \) is \( \pi_s \). A gamble \( (x_1, x_2) \) is just acceptable to Charles if

\[
\pi_1 v_c(w + x_1) + \pi_2 v_c(w + x_2) = v_c(w)
\]

(a) Suppose that \( v(x) = -e^{-Ax} \). Show that the set of acceptable gambles is independent of the consumer’s wealth.

Define \( v_1 \equiv v_c(w + x_1) \) and \( v_2 \equiv v_c(w + x_2) \) and \( \bar{v} \equiv \pi_1 v_1 + \pi_2 v_2 = v_c(w) \).

(b) Suppose that Denis has a concave utility function \( v_d(w + x) = f(v(x)) \) where \( f \) is strictly increasing and strictly concave. Explain carefully why

\[
\pi_1 v_d(w + x_1) + \pi_2 v_d(w + x_2) = \pi_1 f(v_1) + \pi_2 f(v_2) < f(\bar{v}) = v_d(w).
\]

Thus a gamble just acceptable to Charles is not acceptable to Denis.

For every \( x \) we can map out \( v_c(w + x) \) and \( v_d(w + x) \) as depicted below.
Note that in the figure the implied mapping $f$: from $v_c$ to $v_d$ is strictly increasing and strictly concave.

\[ v_d(w+x) = f(v_c(w+x)). \]

(c) Differentiate with respect to $x$, then take the logarithm of both sides and differentiate again and hence show that

\[ \frac{-v''_d}{v'_d} = -\frac{v''_c}{v'_c} - \frac{f''}{f'} \quad (*) \]

The ratio of $A(w+x) = -v''(w+x)/v'(w+x)$ is called the degree of absolute risk aversion. So we have shown that

\[ A_d(\omega+x) = A_c(\omega+x) - \frac{f''}{f'} \]

Then if Denis has an everywhere higher degree of absolute risk aversion, the implied mapping $f$ is strictly concave and so his set of acceptable gambles is smaller.

Suppose that the only difference between Denis and Charles is that Denis has a wealth $w_d = w + \delta$. Then

\[ v_d(w_d+x) = v_c(w+\delta+x). \]

Suppose that Charles is indifferent between $(w, w)$ and the gamble $(w+x_1, w+x_2)$.

Note that

\[ A_d(w_d+x) = -\frac{v''_d(w_d+x)}{v'_d(w_d+x)} = -\frac{v''_c(w+\delta+x)}{v'_c(w+\delta+x)} = A_c(w+\delta+x) \]

Thus if absolute aversion to risk is independent of wealth, it follows from (*) that Denis will also be indifferent. Typically however, more wealthy individuals are more willing to accept a gamble of fixed size. Thus economists usually assume that absolute aversion to risk is lower for individuals with a higher wealth. Then Denis will strictly prefer the gamble $(w+x_1, w+x_2)$. Thus the set of acceptable gambles rises with wealth.

\section*{6.4 Purchasing insurance}
In state 1 a consumer with wealth $a$ has a loss $L$. The probability of the loss state is $\pi_1$. He can purchase claims in the loss state for claims in the no loss state 2 at the exchange rate $p_1/p_2$ so that his budget constraint is
\[
p_1x_1 + p_2x_2 \leq p_1(a - L) + p_2a.
\]

(a) If insurance is “fair” (zero expected profit) explain why
\[
\pi_1x_1 + \pi_2x_2 = \pi_1(a - L) + \pi_2a.
\]

(b) If this is the case show that the consumer will purchase fill coverage, that is $x_1 = x_1 = \pi_1(a - L) + \pi_2a$.

(b) Suppose that it costs more to purchase insurance so that the optimal choice $x^*$ satisfies $x_1^* < x_2^*$ as depicted below. Show that at the optimum,

\[
M(x^*) \equiv MRS(x_1^*, x_2^*) = \frac{\pi_1y'(x_1^*)}{\pi_2y'(x_2^*)} = \frac{p_1}{p_2}.
\]
A second individual has wealth $a + b$ and faces the same loss. If he purchases the same amount of coverage his final consumption is $b + x^*$ and so his MRS is

$$M(b + x^*) = \frac{\pi_1v'(b + x_1^*)}{\pi_2v'(b + x_2^*)}.$$ 

Note that if $M(b + x^*) = M(x^*)$ the choice $b + x^*$ is optimal.

(c) Define $m(b + x^*) = \ln M(b + x^*)$. Use the rules of logarithms to get a simple expression for $m$ and then differentiate by $b$ and show that

$$m'(b + x^*) = -A(b + x_1^*) + A(b + x_2^*).$$

Under the assumption of decreasing absolute risk aversion this is negative for all $b \geq 0$ since $x_1^* < x_2^*$. Therefore $m(b + x^*) < m(x^*)$ and so $M(b + x^*) < M(x^*)$.

(d) What does this imply about the difference between the slope of the indifference curve at $(b + x_1^*, b + x_2^*)$ and the slope of the “budget line”?

(e) Does it follow that the more wealthy individual will purchase more or less insurance?

6.5 Insurance market equilibrium

Alex is risk averse with VNM utility function $v(x) = \ln x$. Bev is risk neutral. There are 2 equally likely states. The aggregate endowment is $\omega = (200, 50)$.

(a) Show that if Alex has an endowment $\omega^A = (60, 20)$ the WE state claims price ratio will be 1 and that Alex has a WE allocation $\bar{x}^A = (40, 40)$. Thus the risk neutral Bev bears all the risk.

(b) What is the WE price ratio and allocation if Alex has an endowment $\omega^A = (120, 20)$. Does Bev bear all the risk?

(c) Depict both equilibria in an Edgeworth-Box diagram.

More generally, suppose Alex and Bev are both risk averse with individual endowments $\omega^A, \omega^B$ and the probability vector is $(\pi_1, \pi_2)$. The aggregate endowment is $\omega = (\omega_1, \omega_2)$ where $\omega_1 > \omega_2$. 
(d) Suppose that $\frac{p_1}{p_2} \geq \frac{\pi_1}{\pi_2}$. Explain carefully why both consumers will demand more state 2 claims.

(e) Does it follow that for a WE $\frac{p_1}{p_2} < \frac{\pi_1}{\pi_2}$? Explain.