Econ 201B: Exam Solutions

1. Game Theory

You are studying a normal form game $G = (I, A, u)$ where I denotes the set of players, A the compact set of action profiles, and $u : A \to R^I$ the players' payoffs.

For parts (a) – (c), explain what additional assumptions you need to impose on the structure of the game, and what mathematical tools and results you can use in order to:

(a) Establish the existence of a Nash equilibrium.

In order to establish existence, we need to show that the best response correspondence has a fixed point. Thus, we need to appeal to a Fixed Point Problem. There are two possible alternatives.

1. Kakutani's FPT states that a non-empty, upper-hemi continuous, compact-valued, convex-valued correspondence on a compact metric space has a fixed point. Thus, we need to show that the best-response correspondence is non-empty, upper-hemi continuous, compact- and convex-valued.

The theorem of the maximum states that if B and X are compact, and $u : B \times X \to R$ is continuous, then the correspondence $g(x) = \arg\max_{b \in B} u(b, x)$ is non-empty, compact-valued and upper-hemi continuous. (remark: the continuity restriction can be relaxed to upper semi-continuity in b, so that at any point, $u(b, x)$ is defined by the maximum of the right- and left-hand limits w.r.t. b).

Therefore, if we use $B = A_i$ and $X = A_{-i}$, and assume that u is continuous, the THM of the maximum implies that the best-response correspondence is non-empty, compact-valued and upper-hemicontinuous.

We still need convex-valuedness. If we assume that u is also strictly quasi-concave in a_i for all a_{-i}, then the BR correspondence will be single-valued (and hence a continuous function) - implying convex-valuedness.

As an alternative to continuity in A, and quasi-concavity in a_i, we can also assume that each player has a finite number of actions, so that A is finite-valued. In that case, we recast the game's strategies as mixed strategy probabilities. The resulting preferences are continuous in the probability (by construction), so the THM of the maximum gives us non-emptyness, compact-valuedness and upper-hemicontinuity of the BR correspondence. Convex-valuedness follows because if any two actions are best responses, any mixture over those two, i.e. any convex combination of mixing probabilities, is a best response as well.

Summary: to apply Kakutani's FPT, we need either continuity of payoffs in A, and quasi-concavity of payoffs in a_i, or finiteness of A, in which case Kakutani is directly applicable to the space of mixed strategies.

2. Topkis FPT states that an increasing function on a compact metric space has a fixed point.

To apply Topkis' FPT, we need to find conditions, under which the best response correspondence admits a monotone selection. This follows when the payoffs are super-modular, i.e. u_i is continuous in a_{-i}, upper-semi continuous in a_i, and exhibits increasing differences:

$$u_i(a_i, a_{-i}) - u_i(a_i', a_{-i}) \geq u_i\left(a_i, a_{-i}'\right) - u_i\left(a_i', a_{-i}'\right)$$

whenever $a_i \geq a_i'$ and $a_{-i} \geq a_{-i}'$.

Summary: to apply Topkis' FPT, we again need the conditions for the maximum theorem to hold, plus increasing differences in payoffs (super-modularity).

(b) Establish the existence of a Nash equilibrium in pure strategies

Again, there are two approaches to show existence of a pure strategy equilibrium.

1) if we are using Kakutani's FPT, it is sufficient to show that the BR correspondence in pure-strategies is convex-valued. For this it suffices to have quasi-concavity in own actions, for a compact action space.

2) if we are using Topkis' FPT, the conditions for super-modularity immediately imply monotone best responses in pure strategies, implying existence of PSE.

(c) Show that the Nash equilibrium is unique.

To establish uniqueness, we need stronger properties on the best response correspondence - essentially we need to rule out the possibility of multiple fixed points. A sufficient condition for uniqueness is that the best response correspondence is a contraction. Alternatively, if one considers a super-modular game, such a game has a 'highest' and a 'lowest' equilibrium, and one can establish uniqueness by showing that the two coincide. This might be easier to establish than the conditions required for a global contraction property to hold.

(d) Show that, if the game is strictly super-modular and has two players, the set of Nash equilibria can be ordered, that is if (a_1, a_2) and (b_1, b_2) both constitute Nash equilibria and $a_1 \geq b_1$, then $a_2 \geq b_2$.

Suppose to the contrary that $a_2 < b_2$. Since the game is super-modular, it then must be the case that

$$u_2(a_1, b_2) - u_2(a_1, a_2) > u_2(b_1, b_2) - u_2(b_1, a_2)$$

Moreover, since (a_1, a_2) and (b_1, b_2) both constitute a Nash equilibrium, we also have

$$\begin{aligned} u_2(a_1, a_2) &\geq u_2(a_1, b_2) \\ \text{and } u_2(b_1, b_2) &\geq u_2(b_1, a_2). \end{aligned}$$

But then, we have the following contradiction:

$$0 \geq u_2(a_1, b_2) - u_2(a_1, a_2) > u_2(b_1, b_2) - u_2(b_1, a_2) \geq 0$$

4. Common Value Auction II

Consider exactly the same set-up as you studied in question 2, but here we ask you to find the optimal auction mechanism, in which the seller seeks to maximize expected revenue.

2 bidder's "type" θ_i, $i = 1, 2$, is an independent draw from a distribution with support $[0, 1]$, c.d.f. $F(\cdot)$ and p.d.f. $f(\cdot)$. You may assume that F satisfies the monotone hazard rate condition, i.e. $(1 - F(\theta_i)) / f(\theta_i)$ is strictly decreasing. Buyer i's valuation $v_i(\theta) = (1 - \lambda)\theta_i + \lambda\theta_{-i}$, where $\lambda \in [0, 1]$.

(a) Set up the mechanism design problem. What variables are determined by the mechanism? What constraints does the mechanism have to satisfy?

We set this up as a direct revelation mechanism. The mechanism defines a probability that bidder i obtains the object, as well as transfers from each bidder to the seller. Formally, let $q_i(\theta)$ denote the probability that bidder i obtains the object, and $t_i(\theta)$ the transfer from i to the seller, conditional on a profile of announcements θ. The mechanism design problem is:

$$\max_{\{q_i(\cdot), t_i(\cdot)\}, i=1,2} \int_0^1 \int_0^1 \sum_{i=1,2} t_i(\theta) f(\theta_1) f(\theta_2) \, d\theta_1 d\theta_2$$

$$\begin{aligned}
\text{subject to } U_i(\theta_i) &\equiv \int_0^1 \left[v_i(\theta_i, \theta_{-i}) q_i(\theta_i, \theta_{-i}) - t_i(\theta_i, \theta_{-i})\right] f(\theta_{-i}) \, d\theta_{-i} \\
&\geq \int_0^1 \left[v_i(\theta_i, \theta_{-i}) q_i(\theta_i', \theta_{-i}) - t_i(\theta_i', \theta_{-i})\right] f(\theta_{-i}) \, d\theta_{-i} \text{ for all } i, \theta_i, \theta_i' \\
U_i(\theta_i) &\geq 0 \text{ for all } i, \theta_i. \\
\sum_{i=1,2} q_i(\theta) &\leq 1
\end{aligned}$$

The first constraint represents the incentive compatibility constraints for a bidder with type θ_i. The second set of constraints represents the participation constraints. The third constraint imposes the requirement that the object can be sold at most once (i.e. the two probabilities have to add up to at most 1).

(b) Using the usual tricks, find an expression for the seller's expected revenue in an incentive compatible mechanism

Define expected transfers and probabilities:

$$\bar{q}_i(\theta_i) = \int_0^1 q_i(\theta_i, \theta_{-i}) f(\theta_{-i}) \, d\theta_{-i} \text{ and } \bar{t}_i(\theta_i) = \int_0^1 t_i(\theta_i, \theta_{-i}) f(\theta_{-i}) \, d\theta_{-i}$$

In addition, define

$$\bar{\Theta}_{-i}(\theta_i) = \int_0^1 \theta_{-i} q_i(\theta_i, \theta_{-i}) f(\theta_{-i}) \, d\theta_{-i}.$$

Now, the indirect utility function is defined as

$$U_i(\theta_i) = (1-\lambda)\theta_i \bar{q}_i(\theta_i) + \lambda \bar{\Theta}_{-i}(\theta_i) - \bar{t}_i(\theta_i).$$

As usual, incentive compatibility is equivalent to monotonicity and local incentive compatibility. Thus, we have:

$$\begin{aligned}
U_i'(\theta_i) &= (1-\lambda)\bar{q}_i(\theta_i) \\
U_i(\theta_i) &= U_i(0) + (1-\lambda)\int_0^{\theta_i} \bar{q}_i(\theta')\,d\theta' \\
\bar{t}_i(\theta_i) &= \int_0^1 (1-\lambda)\theta_i \bar{q}_i(\theta_i) + \lambda \bar{\Theta}_{-i}(\theta_i) - U_i(0) - (1-\lambda)\int_0^{\theta_i} \bar{q}_i(\theta')\,d\theta' \\
&= \int_0^1 v_i(\theta_i,\theta_{-i})\,q_i(\theta_i,\theta_{-i})\,f(\theta_{-i})\,d\theta_{-i} - U_i(0) - (1-\lambda)\int_0^{\theta_i} \bar{q}_i(\theta')\,d\theta'
\end{aligned}$$

Now, the seller's revenue can be expressed as

$$R = \sum_{i=1,2}\int_0^1 \bar{t}_i(\theta_i)\,f(\theta_i)\,d\theta_i$$

$$= \sum_{i=1,2}\left\{\int_0^1\int_0^1 v_i(\theta_i,\theta_{-i})\,q_i(\theta_i,\theta_{-i})\,f(\theta_{-i})\,d\theta_{-i} f(\theta_i)\,d\theta_i - U_i(0) - (1-\lambda)\int_0^1\int_0^{\theta_i} \bar{q}_i(\theta')\,d\theta' f(\theta_i)\,d\theta_i\right\}.$$

Now, notice that

$$\int_0^1\int_0^{\theta_i} \bar{q}_i(\theta')\,d\theta' f(\theta_i)\,d\theta_i = \int_0^1 \bar{q}_i(\theta_i)(1-F(\theta_i))\,d\theta_i = \int_0^1\int_0^1 q_i(\theta_i,\theta_{-i})\frac{1-F(\theta_i)}{f(\theta_i)} f(\theta_1)\,f(\theta_2)\,d\theta_1 d\theta_2,$$

so $R = \int_0^1\int_0^1 \sum_{i=1,2} q_i(\theta_i,\theta_{-i})\left[v_i(\theta_i,\theta_{-i}) - (1-\lambda)\frac{1-F(\theta_i)}{f(\theta_i)}\right] f(\theta_1)\,f(\theta_2)\,d\theta_1 d\theta_2 - \sum_{i=1,2} U_i(0)$

(c) Does the Revenue Equivalence Theorem apply to this auction with common values?

Yes it does: from the above expression, notice that the seller's revenue only depends on the lowest types utility and the allocation rule.

(d) Derive the optimal allocation rule, transfers and the seller's expected revenue. Show that whenever $\lambda < 1/2$*, the item, if it is sold, always goes to the bidder with the highest valuation. What about* $\lambda = 1/2$, or $\lambda > 1/2$?

The allocation rule allocates the good to the bidder with the highest virtual surplus $v(\theta_i,\theta_{-i}) - (1-\lambda)(1-F(\theta_i))/f(\theta_i)$, provided that this is positive.

When does the bidder with the highest actual surplus also have the highest virtual surplus, and/or is the highest type? For that, notice that the bidder with the higher type, also has the higher actual surplus, if and only if $v(\theta_i, \theta_{-i}) > v(\theta_{-i}, \theta_i)$ whenever $\theta_i > \theta_{-i}$, or equivalently,

$$\begin{aligned} (1-\lambda)\theta_i + \lambda\theta_{-i} &> (1-\lambda)\theta_{-i} + \lambda\theta_i \\ (1-2\lambda)(\theta_i - \theta_{-i}) &> 0. \end{aligned}$$

Thus, whenever $\lambda < 1/2$, the higher type has the higher valuation. Due to the monotone hazard condition, we then also have

$$v(\theta_i, \theta_{-i}) - (1-\lambda)(1-F(\theta_i))/f(\theta_i) > v(\theta_{-i}, \theta_i) - (1-\lambda)(1-F(\theta_{-i}))/f(\theta_{-i})$$

if $\theta_i > \theta_{-i}$, so that the item goes to the higher type, if it gets sold. When $\lambda = 1/2$, both types have the same valuation, in which case the good is sold to the higher type purely because of informational rents, and when $\lambda > 1/2$, the higher type actually has the lower valuation.

(e) Does the high-bid auction you studied in problem 2 implement the optimal allocation rule? If not, how would you have to modify the auction so that it does?

No: In the high bid auction, the object was always sold (ex post efficiency), even when virtual surplus was negative. The high bid auction would have to be augmented by a reserve price in order to implement the optimal allocation rule.

5. Insurance with Moral Hazard

An insurance company proposes to protect a driver against accidents. The probability that the driver has an accident is $p(e)$, which is a decreasing, convex function of the driver's choice e of driving safely. When an accident occurs, the resulting losses x are distributed with a pdf $f(x)$ - notice that only the probability of an accident is affected by e, but not the distribution of losses. The driver's utility is $-d(t) - e$, where e stands for the effort cost, and $d(t)$ is an increasing, convex disutility of the driver's financial losses t, which consists of the driver's payment to the insurance company.

For parts a) - d) you may assume that any accident is automatically reported to the insurance company.

a) Set up and solve the insurer's problem, when the driver's choice of e is observable, and contractual obligations can be made contingent on e. You may assume that if the driver doesn't agree to the contract, he is fully responsible for any damage caused by an accident.

The contract specifies a payment $t(\cdot)$ from the driver to the insurer, which is contingent on the occurrence and size of the accident damage (where we use $x = 0$ to denote the occurrence of no accident), and the effort choice e. The contract also specifies a required effort choice e^*, which must be incentive compatible. In this case, the insurer is a monopolist, who will offer the profit maximizing contract, subject to the driver's participation constraint:

$$\max_{\{e,t(\cdot)\}} (1-p(e))\, t(0,e) + p(e) \int_0^\infty (t(x,e) - x)\, f(x)\, dx$$

$$\begin{aligned} &\text{s.t. } (1-p(e))\, d(t(0,e)) + p(e) \int_0^\infty d(t(x,e))\, f(x)\, dx + e \\ \leq\ & (1-p(e'))\, d(t(0,e')) + p(e') \int_0^\infty d(t(x,e'))\, f(x)\, dx - e' \text{ for all } e' \neq e \\ & (1-p(e))\, d(t(0,e)) + p(e) \int_0^\infty d(t(x,e))\, f(x)\, dx + e \\ \leq\ & \min_e \left\{ p(e) \int_0^\infty d(x,e)\, f(x)\, dx - e \right\} \equiv \bar{U} \end{aligned}$$

By requiring punitive payments for any effort level other than the desired one, one can implement any effort choice, wihtout concern about incentive compatibility. The problem can then be reduced to finding the optimal effort level, and transfers for this effort level, such that the participation constraint is satisfied:

$$\max_{\{e,t(\cdot)\}} (1-p(e))\, t(0) + p(e) \int_0^\infty (t(x) - x)\, f(x)\, dx$$

$$\begin{aligned} & (1-p(e))\, d(t(0)) + p(e) \int_0^\infty d(t(x))\, f(x)\, dx + e \\ \leq\ & \min_e \left\{ p(e) \int_0^\infty d(x,e)\, f(x)\, dx - e \right\} \equiv \bar{U} \end{aligned}$$

The point-wise first order condition for this problem w.r.t. $t(x)$ gives $(1 - \lambda d'(t(x))) = 0$, from which it follows immediately that $d'(t(x))$ must be constant for all x, including $x = 0$.

b) Set up the insurer's problem of designing an optimal contract. You may assume that if the driver doesn't agree to the contract, he is fully responsible for any damage caused by an accident.

The contract specifies a payment $t(\cdot)$ from the driver to the insurer, which is contingent on the occurrence and size of the accident damage (where we use $x = 0$ to denote the occurrence of no accident). The contract also specifies a required effort choice e^*, which must be incentive compatible. In this case, the insurer is a monopolist, who will offer the profit maximizing contract, subject to the driver's participation and incentive compatibility constraints:

$$\max_{\{e,t(\cdot)\}} (1-p(e))\,t(0) + p(e)\int_0^\infty (t(x)-x)\,f(x)\,dx$$

$$\begin{aligned}
&\text{s.t. } (1-p(e))\,d(t(0)) + p(e)\int_0^\infty d(t(x))\,f(x)\,dx + e \\
\leq\ & (1-p(e'))\,d(t(0)) + p(e')\int_0^\infty d(t(x))\,f(x)\,dx - e' \text{ for all } e' \neq e \\
& (1-p(e))\,d(t(0)) + p(e)\int_0^\infty d(t(x))\,f(x)\,dx + e \\
\leq\ & \min_e \left\{ p(e)\int_0^\infty d(x)\,f(x)\,dx - e \right\} \equiv \bar{U}
\end{aligned}$$

The first constraint represents the driver's incentive constraint (the proposed e must be incentive compatible). The second constraint represents the driver's participation constraint - accepting the contract must be at least as good as not having any insurance.

c) Show that in the optimal contract, the insurance company conditions the financial responsibility of the driver only on the occurrence of an accident, but not on the actual damage incurred.

Notice that in the optimal contract, the driver's incentive and participation constraints only depend on $D = \int_0^\infty d(t(x))\,f(x)\,dx$ and $d(t(0))$. For given values of D and $d(t(0))$, such that the incentive and participation constraints are satisfied, we can thus derive the profile of payments that maximizes the insurer's profits:

$$\max \int_0^\infty (t(x)-x)\,f(x)\,dx \text{ s.t. } \int_0^\infty d(t(x))\,f(x)\,dx = D$$

Taking point-wise FOC's for $t(x)$, we find $f(x)(1-\lambda d'(t(x))) = 0$, where λ denotes the multiplier on the constraint $\int_0^\infty d(t(x))\,f(x)\,dx = D$. Therefore, $1 = \lambda d'(t(x))$ for all $x > 0$, implying that $t(x)$ is constant for all $x > 0$.

Intuitively, the accident size is not informative of the driving effort, only the occurrence of an accident is. Therefore, the optimal contract should not make the payments contingent on the damages, only on the occurrence of an accident.

d) Show that if the insurer's optimal contract implements $e > 0$, then $t(0) < \bar{t} \equiv t(x)$ for $x > 0$, i.e. implementing positive effort requires that the driver pays strictly more when an accident occurs.

With two payments, the driver's IC constraint becomes

$$(1-p(e))\,d(t(0)) + p(e)\,d(\bar{t}) + e \leq (1-p(e'))\,d(t(0)) + p(e')\,d(\bar{t}) - e' \text{ for all } e' \neq e$$

If the insurer wishes to implement any $e > 0$, it must be the case that

$$\begin{aligned} (1-p(e))\,d(t(0)) + p(e)\,d(\bar{t}) + e &\leq (1-p(0))\,d(t(0)) + p(0)\,d(\bar{t}) \\ \text{or } \left(d(\bar{t}) - d(t(0))\right)(p(0) - p(e)) &\geq e. \end{aligned}$$

Since $p(0) > p(e)$, this can hold only if $d(\bar{t}) > d(t(0))$, and therefore $\bar{t} > t(0)$.

e) How would you set up the problem, if there were two companies who could simultaneously offer competing insurance contracts?

With two competing insurers, the insurer's profits get competed away, and the contract maximizes the driver's expected utility, subject to incentive compatibility, and the insurer breaking even:

$$\min_{\{e,t(\cdot)\}} (1-p(e))\,d(t(0)) + p(e)\int_0^\infty d(t(x))\,f(x)\,dx + e$$

$$\begin{aligned} &\text{s.t. } (1-p(e))\,d(t(0)) + p(e)\int_0^\infty d(t(x))\,f(x)\,dx + e \\ \leq\;& (1-p(e'))\,d(t(0)) + p(e')\int_0^\infty d(t(x))\,f(x)\,dx - e' \text{ for all } e' \neq e \\ 0 \;=\;& (1-p(e))\,dt(0) + p(e)\int_0^\infty (t(x) - x)\,f(x)\,dx \end{aligned}$$

Suppose now that accidents are not automatically reported to the insurance company. Instead, the driver has the option of declaring the accident after it has occurred, along with a documentation of the realized damages.

f) Suppose that the optimal contract you found under c) and d) implements a positive effort level. Show that this contract is no longer implementable, if the driver self-reports any accident. That is, show that with the contract you found, the driver would not be willing to report all accidents.

Consider the realization of an accident with a damage realization $x < \bar{t} - t(0)$. That is, the accident is smaller in size than the penalty the driver would have to pay by self-reporting the accident. Clearly the driver will prefer not to report such a small accident.

g) How would you introduce this form of voluntary self-reporting into the structure of the optimal contracting problem? How do you think this would affect the design of the contract? (no need to provide a full formal analysis here, just a sketch of your argument).

The problem remains the same as in the set-up in a) (for a monopolistic insurer) or d) for a competitive insurance market, except that the contract has to satisfy some additional ex post incentive constraints, after the accident has occurred, and the driver faces the choice of disclosing the accident. The additional constraint requires that the driver always prefers to disclose - this is without loss of generality, since a contract in which some accidents are not disclosed is equivalent to a contract with full disclosure, followed by full liability (i.e. the agent paying the full amount of the damage) in some states. This additional constraint thus requires that $t(x) \leq x$ for all x.