Principal Agent Problem

Model

Efficient risk spreading

Risk neutral agent

Risk neutral principal

Asymmetric information

Characterizing the efficient contract with asymmetric information

Application: Insurance with moral hazard
PRINCIPAL AGENT PROBLEM

Key ideas: contracting with hidden actions, incentive constraints, insurance with moral hazard

Consider the following 2 person economy. Alex chooses an action \( x \in X = \{x_1, \ldots, x_n\} \), where \( x_1 < \ldots < x_n \). Penny owns the single firm. The set of possible outputs is \( Y = \{y_1, \ldots, y_S\} \) where \( y_1 < \ldots < y_S \). For any action \( x \) the firm’s output is a prospect \((y, \pi(x))\). Higher actions shift probability mass to higher outputs.

If \( x' > x \) then \( \pi(x') \) exhibits FOSD over \( \pi(x) \).

Later we will need to assume that the monotone likelihood ratio property holds. That is, for any \( x' > x \),

\[
\frac{\pi_s(x')}{\pi_s(x)} < \frac{\pi_t(x')}{\pi_t(x)}, \quad \forall s < t
\]

Let \( w = (w_1, \ldots, w_S) \) be the state contingent allocation to Alex and let \( r = (r_1, \ldots, r_S) \) be the state contingent allocation to Penny. Then \( w_s + r_s \leq y_s \). Finally let \( C(x) \) be the utility cost to Alex of taking action \( x \).
Then expected utilities are 

\[ U^A(x, w) = \sum_{s=1}^{S} \pi_s(x)\nu(w_s) - C(x) \quad \text{and} \quad U^P = \sum_{s=1}^{S} \pi_s(x)\nu(r_s) \]

where

\[ r_s + w_s = y_s \]

Substituting for \( r \),

\[ U_P(w, x) = \sum_{s=1}^{S} \pi_s(x)\nu(y_s - w_s) \]

**Efficient allocations**

For any fixed action \( x \) we can in principle solve for

The efficient allocations by solving one of the following problems.

(i) Maximize Penny’s expected utility given that Alex must have an expected utility of at least \( \bar{U}_A \)

(ii) Maximize Alex’s expected utility given that Penny must have an expected utility of at least \( \bar{U}_P \)

Pareto efficient outcomes with 3 actions
**Efficient Allocations**

Step 1:

Fix $\bar{U}_A$ and $x$ and solve for the efficient state contingent payments

$$w(x) \in \arg\max_w \{ U_P(w, x) \mid U_A(w, x) \geq \bar{U}_A \}.$$  

Then Penny’s expected utility is

$$U_P(w(x), x) = \sum_{s=1}^{S} \pi_s(x)u(y_s - w_s(x)).$$

Step 2: Repeat for each $x$ and thus solve for $x^* = \arg\max_x \{ U_P(w(x), x) \}$
Consider Step1:

\[
\max \{ U_p(w, x) \mid U_A(w, x) \geq \overline{U}_A \}
\]

\[
\mathcal{Q} = U_p(w, x) + \lambda(U_A(w, x) - \overline{U}_A)
\]

\[
= \sum_{s=1}^{S} \pi_s(x)u(y_s - w_s) + \lambda \left( \sum_{s=1}^{S} \pi_s(x)v(w_s) - C(x) - \overline{U}_A \right).
\]

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\[
\frac{\partial \mathcal{Q}}{\partial w_s} = -\pi_s(x)u'(y_s - w_s) + \lambda \pi_s(x)v'(w_s) = \pi_s(x)[\lambda v'(w_s) - u'(y_s - w_s)] = 0
\]

Therefore \( u'(y_s - w_s) = \lambda v'(w_s) \)

Rearranging, \( \frac{u'(r_s)}{u'(r_t)} = \frac{v'(w_s)}{v'(r_t)} \) since \( r_s = y_s - w_s \)

If \( r_s < r_t \) concavity implies that the ratio is strictly greater than 1. Then \( w_s < w_t \) and so \((w_s, r_s) < (w_t, r_t)\) .

Therefore \( y_s = r_s + w_s < r_t + w_t = y_t \). Output is higher in higher indexed states so \( s < t \).

Both Alex and Penny have lower payments when output is lower. That is, risk is shared.
Special case: Alex (the agent) is risk neutral

\[
\frac{v'(r_s)}{v'(r_t)} = 1 \text{ therefore } r_s(x) = r_t(x) = \bar{r} . \text{ The risk averse Penny bears no risk.}
\]

Fix Penny’s payoff and solve for the efficient action

\[
\max_x U_A(x) = \sum_{s=1}^{S} \pi_s(x)(y_s - \bar{r}) - C(x)
\]

Equivalently,

\[
\max_x \{ \sum_{s=1}^{S} \pi_s(x)y_s - C(x) \} - \bar{r} . \quad (*)
\]

The Pareto efficient action is

\[
x^* \in \arg \max_x \{ \sum_{s=1}^{S} \pi_s(x)y_s - C(x) \} - \bar{r} .
\]

Asymmetric information (agent’s action is unobserved)

Suppose that Alex (the agent) is offered a rental contract (pay \( \bar{r} \) regardless of the outcome.)

Alex solves (*) thus chooses the efficient outcome.
Special case: Penny (the principal) is risk neutral

\[ 1 = \frac{v'(w_s)}{v'(r_t)} \] therefore \( w_s(x) = \bar{w}(x) \). For efficiency, Alex (who is risk averse) bears no risk.

Note that

\[ U_A(x) = \sum_{s=1}^{S} \pi_s(x) v(\bar{w}(x)) - C(x) = v(\bar{w}(x)) - C(x) = \bar{U}_A \]

Therefore

\[ v(\bar{w}(x)) = C(x) + \bar{U}_A \]

Consider two neighboring actions \( x_i \) and \( x_{i+1} \)

\[ v(\bar{w}(x_i)) = C(x_i) + \bar{U}_A \quad \text{and} \quad v(\bar{w}(x_{i+1})) = C(x_{i+1}) + \bar{U}_A \]

The difference in the fixed payments

\[ w_{i+1} - w_i \] is depicted opposite.

Note that the difference in Penny’s payoffs is

\[ U_P(\bar{w}(x_{i+1})) - U_P(\bar{w}(x_i)) \]

\[ = \sum_{s=1}^{S} \pi_s(x_{i+1})(y_s - \bar{w}(x_{i+1})) - \sum_{s=1}^{S} \pi_s(x_i)(y_s - \bar{w}(x_i)) \]
\[
= \left[ \sum_{s=1}^{S} \pi_s(x_{i+1})y_s - \sum_{s=1}^{S} \pi_s(x_i)y_s \right] - (\bar{w}(x_{i+1}) - \bar{w}(x_i))
\]

= increased expected value of output – marginal cost of higher action

Now consider the difference in payoffs to the principal if the agent has a higher expected payoff \( \bar{U}_A \).

Let the associated wages be \( \bar{w}(x_i) \) and \( \bar{w}(x_{i+1}) \).

These are depicted opposite.

Given the concavity of \( v(\cdot) \) it follows that

\[
\bar{w}(x_{i+1}) - \bar{w}(x_i) > \bar{w}(x_{i+1}) - \bar{w}(x_i)
\]

Thus the higher is the agent’s expected payoff, the higher is the marginal cost to the principal of switching to a higher action.

It follows that either the efficient action is the same for all Points on the Pareto frontier, or, as \( U_A \) increases, the efficient action is a less costly action.
Asymmetric information (agent’s action is unobserved)

We have argued that Alex (the agent) must be paid the same in every state (fixed wage).

His expected payoff is therefore

\[
U_A(x, w) = \sum_{s=1}^{S} \pi_s(x) v(\bar{w}) - C(x) = v(\bar{w}) - C(x).
\]

Note that this is maximized by choosing the lowest action.

Thus unless the optimal action is always the lowest action, providing an incentive to taking a more productive action cannot be achieved without losing Pareto efficiency.
Solving the Principal-Agent problem

Continue with the special case in which the principal is risk neutral and the agent is risk averse.

Three step approach

**Step 1: Incentive constraints to make $\bar{x}$ incentive compatible**

For any action $\bar{x}$, and expected utility for the agent $\bar{U}_A$ characterize the state contingent payments which induce the agent to take action $\bar{x}$ rather than any other $x \in X$

**Step 2: Maximize the principal’s payoff with the action fixed**

Of all these payments schemes choose the one that is best for the Penny the principal. Write her expected payoff as $U_p^*(\bar{x})$

**Step 3: Information constrained Pareto Efficiency**

For each $x \in X$ solve for $U_p^*(x)$ and finally choose $x^* = \arg\max U_p^*(x)$.

Actually we will not take the third step but learn what we can by examining the first two steps.
For each action, we maximize the expected utility of the principal subject to the incentive constraints (no gain to choosing another action) and the participation constraint (lower bound on agent’s expected utility).

**Numerical example (Excel 2010)**
Step 1:

Incentive constraints

For the action $\bar{x}$, the state contingent payment vector $w$ is incentive compatible if the agent has an incentive to take the action $\bar{x}$ rather than any other action, that is,

$$U_A(x, w) \leq U_A(\bar{x}, w), \quad x \in X.$$  \hspace{1cm} \textbf{Incentive Constraints}

Let $\bar{U}_A$ be the contracted expected utility of the agent.

Then $(\bar{x}, w)$ must also satisfy the following constraint

$$U_A(\bar{x}, w) \geq \bar{U}_A$$  \hspace{1cm} \textbf{Participation constraint}

The best contract for the principal, given the selection of action $\bar{x}$, thus solves the following optimization problem:

$$w^{**} = \arg \max_w \{ U_P \mid U_A(\bar{x}, w) \geq \bar{U}_A, \ U_A(\bar{x}, w) \geq U_A(x, w), \ x \in X \}.$$  \hspace{1cm} (**)
Step 2:

Intuitively, the problem is one of designing a contract to deter the agent from taking a less costly action.

We will assume this so that (**) can be rewritten as follows.

\[
w^{**} = \arg \max_w \{ U_p \mid U_A(x, w) \geq \bar{U}_A, \ U_A(x, w) \geq U_A(x, w), \ x < x^* \}.
\]

Suppose that in fact there is a single binding constraint. The optimization problem can then be rewritten more simply as follows.

\[
\max_w \{ U_p \mid U_A(x^*, w) \geq \bar{U}_A, \ U_A(x^*, w) \geq U_A(x, w), \ \text{for some } x < x^* \}.
\]

The Lagrangian of this optimization problem is

\[
\mathcal{L} = U_p + \lambda(U_A(x^*, w) - \bar{U}_A) + \mu(U_A(x^*, w) - U_A(x, w))
\]

\[
= U_p + (\lambda + \mu)(U_A(x^*, w) - U_A(x, w)) + \text{a constant}
\]

\[
= \sum_{s=1}^{S} \pi_s(x^*)(y_s - w_s) + (\lambda + \mu)\sum_{s=1}^{S} \pi_s(x^*)v(w_s) - \mu\sum_{s=1}^{S} \pi_s(x)v(w_s) + \text{a constant}.
\]

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1 It is not important that there be one binding constraint, only that the binding constraints are all associated with lower cost actions. General sufficient conditions to ensure this are quite stringent. However for numerical examples it is typically the “downward” constraints that are binding.
We have seen that

\[ \mathcal{L} = \sum_{s=1}^{S} \pi_s(x^*)(y_s - w_s) + (\lambda + \mu) \sum_{s=1}^{S} \pi_s(x^*)v(w_s) - \mu \sum_{s=1}^{S} \pi_s(x)v(w_s) + \text{a constant}. \]

The first-order conditions are therefore

\[ \frac{\partial \mathcal{L}}{\partial w_s} = -\pi_s(x^*) + (\lambda + \mu)\pi_s(x^*)v'(w_s^*) - \mu \pi_s(x)v'(w_s^*). \]

\[ = \pi_s(x^*)v'(w_s^*) \left[ -\frac{1}{v'(w_s^*)} + (\lambda + \mu)v'(w_s^*) - \mu \frac{\pi_s(x)}{\pi_s(x^*)} \right] = 0, \ s = 1, \ldots, S \]

Hence

\[ \frac{1}{v'(w_s^*)} = \lambda + \mu - \mu \frac{\pi_s(x)}{\pi_s(x^*)} \text{ where } x < x^*. \]

By hypothesis, the likelihood ratio \( \frac{\pi_s(x^*)}{\pi_s(x)} \) is increasing in the output state \( s \). Thus the right hand side of this expression increases with \( s \). Therefore, because \( v(\cdot) \) is concave, \( w_s \) is increasing in \( s \). Thus for efficiency, the higher the output, the higher is the payment to the agent.
Application: Insurance with Moral Hazard

An individual’s house burns down with probability \( \pi_1(x) \), where \( x \) is the effort the owner makes to avoid such a catastrophe. The house can be rebuilt at a cost of \( L \).

If the insurance company does not observe the homeowners action but the home owner is perfectly moral, he tells the truth and the efficient contract is unaffected. However if a homeowner is not so trustworthy, the insurance company faces a “moral hazard problem.” With full coverage the incentive to take care is eliminated so the homeowner’s best choice is the cheapest action \( x_1 \). To give the homeowner the incentive to take appropriate care the homeowner must be offered a contract in which he is sufficiently penalized in the loss state.