Step 1: Modelling bidding with risk aversion

With risk neutrality the payoff to winning is the net gain $\theta - b$. With risk aversion it is natural to write the gain in utility as $V(\theta - b)$ where $V(\cdot)$ is a concave function.

Consider a two buyer sealed high bid auction.

If the win probability is $w$ when a buyer bids $b$, his expected utility gain is

$$E[U] = wV(\theta - b)$$

It is important to note that $V(0) = 0$. Moreover, since buyers are risk averse, $V(\cdot)$ is strictly concave.

Let $B(\theta)$ be the equilibrium bid function. If buyer 1 deviates and bids $B(x)$ he wins if buyer 2 has a value below $x$. Buyer 1’s win probability is therefore $W(x) = F(x)$ and so his expected utility gain is

$$u_1(\theta, x) = F(x)V(\theta - B(x))$$

Buyer 1's change in expected utility gain if he increases his bid from his equilibrium bid from $B(x)$ to $B(x + \Delta x)$ is

$$u_1(\theta, x + \Delta x) - u_1(\theta, x)$$

The marginal expected utility gain is

$$\frac{\partial}{\partial x}u_1(\theta, x) = \lim_{\Delta x \to 0} \frac{u_1(\theta, x + \Delta x) - u_1(\theta, x)}{\Delta x}$$

It is a tiny bit faster to consider instead $\ln u_1$

$$\frac{\partial}{\partial b} \ln u_1(\theta, x) = \frac{\partial}{\partial b} \ln F(x) + \frac{\partial}{\partial b} \ln V((\theta - B(x)) = \frac{f(x)}{F(x)} - \frac{V'(\theta - B(x))}{V'(\theta - B(x))} B'(x)$$

By hypotheses, $B(\theta)$ is buyer 1’s best response. Therefore $u_1(\theta, x)$ and so $\ln u_1(\theta, x)$ takes on its maximum at $x = \theta$.

Thus the FOC for this maximization problem is

$$\frac{\partial}{\partial b} \ln u_1(\theta, x) = 0 \text{ at } x = \theta$$

Substituting for $\theta$,
\[
\frac{f(x)}{F(x)} = \frac{V'(x-B(x))}{V(x-B(x))} \frac{B'(x)}{x-B(x)} = 0 \text{ for all } x \geq \theta. 
\]

We rewrite this ODE (ordinary differential equation) as follows:

\[
B'(\theta) = \frac{f(\theta)}{F(\theta)} \frac{V(\theta-B(\theta))}{V'(\theta-B(\theta))} 
\]

(A)

With risk neutrality,

\[
B_N'(\theta) = \frac{\theta-B_N(\theta)}{F(\theta)} 
\]

(N)

**Step 2: Comparing equilibrium bids**

How are we going to compare the solutions?

Try an example. \(V(z) = z^\gamma, \ 0 < \gamma < 1\).

Remark: This is not a great example as \(z = \theta - B(\theta)\) is near zero for buyers with values near \(\theta\) and so \(V'(z) = \gamma z^{\gamma-1} \to \infty\). However we proceed anyway as examples often lead to more general insights.

The bid function the satisfies

\[
B'(\theta) = \frac{1}{\gamma} \frac{f(\theta)}{F(\theta)} (\theta-B(\theta)) 
\]

Comparing this equation with equation (N), it certainly appears that bidding will be higher under risk aversion \(\gamma \in (0,1)\).

Consider the left boundary \(\theta\).

Case (i) \(\theta > \alpha\). Then \(B'_N(\theta) = B'_N(\theta) = 0\).

Thus the slopes are equal at the left boundary.

Case (ii) \(\theta = \alpha\) \(B'(\theta) = \frac{1}{\gamma} f(\theta) \left[\frac{\theta-B(\theta)}{F(\theta)}\right]\)

In this case as well if is not immediately clear what the derivative is at \(\alpha\).

We make a simplifying assumption about the distribution of values.

Example of case (ii).

\[F(\theta) = \theta \quad \theta \in [0,1]\]
\[ B'(\theta) = \frac{1}{\gamma \theta} (\theta - B(\theta)) \]

i.e.
\[ \gamma \theta B'(\theta) + B(\theta) = \theta \]

To solve such a differential equation we multiply both sides by the derivative \( H'(\theta) \) of some arbitrary function \( H(\theta) \).
\[ \gamma \theta H'(\theta) B'(\theta) + H'(\theta) B(\theta) = \theta H'(\theta) \]

Suppose we can find \( H(\theta) \) satisfying
\[ H(\theta) = \gamma \theta H'(\theta) \]

(In an exam you would be told the integrating factor.)

Then the equation can be rewritten as follows:
\[
\frac{d}{d\theta} [H(\theta) B(\theta)] = H(\theta) B'(\theta) + H'(\theta) B(\theta) = \theta H'(\theta)
\]

The remaining step is to find a function \( H(\theta) \) for which \( H(\theta) = \gamma \theta H'(\theta) \), i.e.,
\[
\frac{H'(\theta)}{H(\theta)} = \frac{1}{\gamma} = \frac{1}{\theta}
\]

Both sides can be integrated
\[
\frac{d}{d\theta} \ln H(\theta) = \frac{1}{\gamma} \frac{d}{d\theta} \ln \theta \quad \text{and so } \quad H(\theta) = \theta^{\frac{1}{\gamma}}. \quad \text{Hence } \quad H'(\theta) = \frac{1}{\gamma} \theta^{-\frac{1}{\gamma}}.
\]

Then
\[
\frac{d}{d\theta} \left[ \theta^{\frac{1}{\gamma}} B(\theta) \right] = \frac{1}{\gamma} \theta^{-\frac{1}{\gamma}}
\]

Thus
\[
\frac{1}{\theta^{\frac{1}{\gamma}}} B(\theta) = \frac{1}{\gamma} \left[ \theta^{\frac{1}{\gamma} - \frac{1}{\gamma}} + K \right] = \frac{1}{\gamma} \theta^{\frac{1}{\gamma} + \frac{1}{\gamma}} + K
\]
Appealing to the left boundary condition, $K = 0$ and so

$$B(\theta) = \frac{1}{1 + \gamma}.$$ 

**A general approach**

Examples (even unrealistic ones) are very helpful as they suggest the possibility of a general result. Proving a general result often requires some moment of inspiration.

In the figure the slope of the tangent line is

$$m = \frac{BC}{AB}.$$ 

Therefore

$$BC = mAB = V'(z)z.$$ 

Given the concavity of $V(z)$, $BC < BO = V(z)$.

Therefore $zV'(z) < V(z)$.

It follows from (A) that

$$B'(x) = \frac{f(x)}{V'(x-B(x))} > \frac{f(x)}{F(x)}(x-B(x)).$$

(AA)

From (N)

$$B_N'(x) = \frac{f(x)}{F(x)}(x-B_N(x)).$$

(N)
Inspired by the example, we wish to establish the following proposition

**Proposition:** \( B(\theta) > B_N(\theta) \) for all \( \theta > \theta \).

To prove this we suppose that the proposition is false and seek a contradiction.

If the proposition is false, then \( B(y) < B_N(y) \) for some \( y > \theta \).

We know that \( B(\theta) = B_N(\theta) \). It follows that there is some closed interval \([x, y]\) such that

(i) \( B(x) = B_N(x) \),  
(ii) \( B(\theta) > B_N(\theta) \) on \((x, y)\) as depicted

[Diagram showing the relationship between \( B(\theta) \) and \( B_N(\theta) \) with points marked at \( \theta, x, y \).]

Appealing to (AA)

\[
B'(\theta) > \frac{f(\theta)}{F(\theta)} (\theta - B(\theta)) \geq \frac{f(\theta)}{F(\theta)} (\theta - B_N(\theta)) = B'(\theta) \quad \text{for all } \theta \in (x, y].
\]

Therefore \( B(y) - B(x) > B_N(y) - B_N(x) \) and so

\[
B(y) - B_N(y) > B(x) - B_N(x) = 0.
\]

But this contradicts (ii).

QED
**Bonus topic (for those interested)**

**Final remark:** Risk aversion increases equilibrium bids. Does it also follow that the higher the degree of risk aversion the greater will be the equilibrium bids? Suppose type 2 buyers are more risk averse

$$B_1'(x) = \frac{f(x)}{F(x)} V_1'(x-B_1(x))$$

and

$$B_2'(x) = \frac{f(x)}{F(x)} V_2'(x-B_2(x)).$$

Using the method of proof above, we can prove this if we can show that

$$\frac{V_1(z)}{V_1'(z)} < \frac{V_2(z)}{V_2'(z)} \quad \text{for all } z > 0 \quad (1.1)$$

Since $V_j(0) = 0, \ j = 1, 2, \ V_j(z) = \int_0^z V_j'(x)dx$

We can therefore rewrite inequality (1.1) as follows:

$$\int_0^z \frac{V_1'(x)}{V_1'(z)} \ dx < \int_0^z \frac{V_2'(x)}{V_2'(z)} \ dx, \quad \text{for all } z > 0.$$  

This is true if

$$\frac{V_1'(z)}{V_1'(x)} < \frac{V_2'(z)}{V_2'(x)} \quad \text{for all } x < z \quad (1.2)$$

Equivalently, this is true if

$$\frac{V_1'(z)}{V_2'(x)} < \frac{V_1'(z)}{V_2'(x)} \quad \text{for all } x < z$$

i.e.,

$$\frac{d}{dx} \frac{V_1'(x)}{V_2'(x)} > 0.$$

Equivalently, (1.2) is satisfied if

$$0 < \frac{d}{dx} \ln \frac{V_1'(x)}{V_2'(x)} = \frac{d}{dx} \ln V_1'(x) - \frac{d}{dx} \ln V_2'(x)$$
\[ \frac{d}{dx} \ln V'_1(x) - \frac{d}{dx} \ln V'_2(x) = \frac{V''_1(x)}{V'_1(x)} - \frac{V''_2(x)}{V'_2(x)} = ARA_2(x) - ARA_1(x) \]

Therefore if type 2 buyers have higher absolute risk aversion than type 1 buyers, then

\[ B'_1(x) = \frac{f(x)}{F(x)} \frac{V_1(x-B_1(x))}{V'_1(x-B_1(x))} \quad \text{and} \quad B'_2(x) > \frac{f(x)}{F(x)} \frac{V_1(x-B_2(x))}{V'_1(x-B_2(x))}. \]

The proof then follows the proof that risk averse buyers bid higher than risk neutral buyers.