Winter 2004 with answers to Riley's questions

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Econ 271: Final Exam

The final exam consists of 4 questions. The exam lasts 3 hours. Optimally use your time. Read each question very carefully, clearly write down each step, and mark your answers. Good luck!

Question 1:

A firm and its customer are exchanging one unit of a good for a fixed price p. This good may or may not work as promised. A working unit is worth v to the buyer. If it fails it will do so immediately (i.e. before providing any value to its buyer). The probability it works is given by the function $w(e,q)$ where q is the quality built into the product by the seller and e the care (or effort) with which the product is used by the buyer. Assume w_e and w_q are both positive, and w_{ee} and w_{qq} are both negative (where the subscripts refer to first and second derivatives). The cost of producing a product with quality q is given by $C(q)$ while the cost of effort for the buyer is given by $G(e)$. Assume $C'(0)=0,\ C''(q)>0$ and $G'(0)=0,\ G''(e)>0$. Finally, the seller offers a warranty: if the product breaks, the seller will compensate the buyer by giving him/her an amount sv where $0<s<1$. Assume this warranty is a direct transfer of cash so that the product itself is just discarded when it breaks. The seller maximizes expected profits and the buyer maximizes his expected surplus.

(a) Derive conditions that describe the first-best levels of e^* and q^*.

(b) Now assume the seller and the buyer make their choices simultaneously and noncooperatively. For any s, derive conditions that describe the Nash equilibrium values of e^{**} and q^{**}.

(c) Explain how the slopes of the reaction functions in part (b) will depend upon the cross-derivative w_{eq}.

(d) How does the equilibrium outcome in (b) changes in s?

(e) Now assume $w(e,q)=\alpha e+\beta q$, $C(q)=0.5q^2$ and $G(e)=0.5e^2$. Solve for the Nash

equilibrium effort and quality. Then derive the optimal warranty contract s that maximizes the total social surplus.

Question 2:

Suppose there are two coffee houses along Mall Street. The street is one km long. Starbucks Coffee House is located at the left end point of Mall Street and Second Cup Coffee House is located halfway between the two end points of the street. Residents are uniformly distributed along this stretch and each resident purchases one cup of coffee per day. Each consumer derives a utility of $v_1 = \$3.00$ from a cup of coffee from the Starbucks. The Second Cup is considered of inferior quality in general, but consumers differ in their tastes about it. Suppose consumer at location $x \in [0,1]$ values a cup of coffee at the Second Cup at $v_2 = 2 + x$.

Each consumer incurs a cost of \$1.00 per km from traveling from his or her location to the location of either coffee house. The price of coffee at Starbucks is p_1 and at Second Cup, it is p_2. The marginal cost of a cup of coffee is zero.

(a) At any pair of prices (p_1, p_2), determine the location of the consumer who is indifferent between purchasing from Starbucks or Second Cup.

(b) Derive the best reply functions to the pricing game in which the coffee houses choose prices simultaneously and determine the equilibrium prices and market shares.

(c) Second Cup is thinking of opening another coffee house at the right-end point of the street. If it did so, what would be the impact on the equilibrium outcome? For what costs of opening the additional store should the Second Cup have the store?

Question 3: Price Discrimination

There are three types of buyer, type t has demand price function $p(a_t, q_t)$ which is increasing in a and decreasing in q. The consumer benefit is for a type t buyer then $B(a_t, q) = \int_0^q p(a_t, x)dx$.

The fraction of each type in the population is strictly positive. Assume that $a_1 < a_2 < a_3$ so that higher types are willing to pay more. The unit cost of production is c.

(a) Write down the monopolist's optimization problem assuming he knows the fractions of each type (f_1, f_2, f_3) but cannot identify an individual's type.

(b) Consider the relaxed problem in which only the participation constraints and local downward constraints are imposed. In order to maximize expected profit, which must be binding? Explain.

(c) Using a neat figure (or otherwise) explain carefully why the solution of the relaxed problem is also the solution of the original problem.

(d) Prove that $q_3^* > q_2^*$.

(e) Show that q_2^* is a function of the ratio f_3 / f_2. Holding this ratio constant, a change in which of the following parameters would affect this output. (f_1, f_2, f_3), (a_1, a_2, a_3)?

(f) If f_2 is sufficiently small, is it necessarily the case that $q_2^* = q_1^*$? Explain carefully by appealing to the first order conditions.

Question 4: Auctions

There are two buyers. Each has a valuation which is an independent random draw from a distribution $\tilde{v}$ with support $[0, \overline{v}]$ and continuously differentiable c.d.f. $F(v)$.

(a) Explain why in the sealed second price auction there is an equilibrium in dominant strategies.
(b) Is it a Nash equilibrium for bidder 1 to bid $\overline{v}$ and buyer 2 to bid zero?
(c) For the uniform case show that in the dominant strategy equilibrium the expected payment by a buyer with valuation v is $\frac{1}{2}v^2$.
(d) Consider next a sealed high bid auction with the following change in the rules. Both the winner and the loser must pay their bids. Characterize the equilibrium bid function in this auction.
(e) For the uniform case, compare the (expected payment) by a buyer with valuation v in this auction and the sealed second price auction.
(f) BONUS (Do not try this unless you have finished with all your answers.) What if instead the rules of the sealed high bid auction are modified as follows. The high bidder is the winner and pays nothing. The losing low bidder pays his bid. Is there a symmetric equilibrium bid function?

ANSWERS

Question 3

(a) Type t's payoff $U_t(q, R) = B(q, a_t) - R$.

Choose (q_t, R_t), $t = 1, 2, 3$ to maximize $\Pi = \sum_{t=1}^{3} f_t(R_t - cq_t)$ subject to the participation constraints $U_t(q_t, R_t) \geq 0$ and incentive constraints $U_t(q_t, R_t) \geq U_t(q_j, R_j)$, $j \neq t$.

While you were not asked to demonstrate monotonicity, consider the following local upward and downward constraints.

$$B_{t+1}(q_{t+1}) - R_{t+1} \geq B_{t+1}(q_t) - R_t \;\Rightarrow \int_{q_t}^{q_{t+1}} p(q, a_{t+1}) \geq R_{t+1} - R_t$$

$$B_t(q_t) - R_t \geq B_t(q_{t+1}) - R_{t+1} \;\Rightarrow \int_{q_t}^{q_{t+1}} p(q, a_t) \leq R_{t+1} - R_t$$

Hence

$$\int_{q_t}^{q_{t+1}} p(q, a_t) \leq R_{t+1} - R_t \leq \int_{q_t}^{q_{t+1}} p(q, a_{t+1}) .$$

This inequality holds if and only if $q_{t+1} \geq q_t$

(b) Relaxed problem

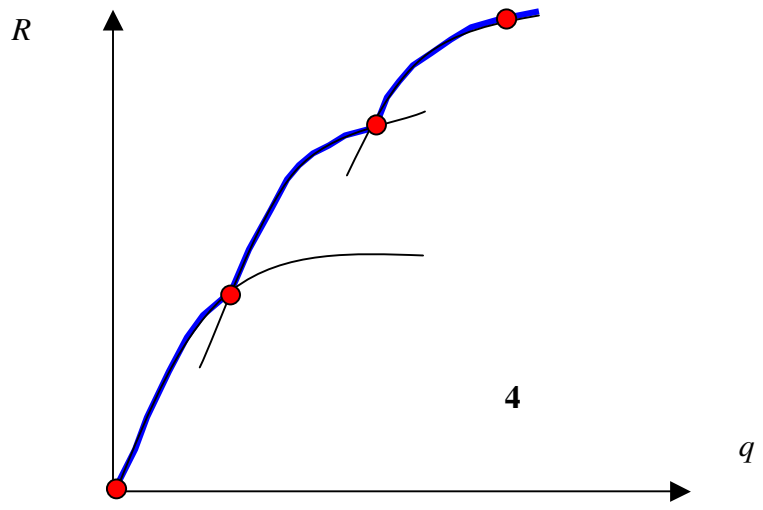
First note that if $U_1(q_1, R_1) = B(q_1, a_1) - R_1 \geq 0$ then $U_t(q_1, R_1) = B(q_1, a_t) - R_1 \geq 0$ since benefit is non-decreasing in a (actually strictly increasing unless $q_1 = 0$.) Thus we can consider only the following constraints.

$$U_1(q_1, R_1) = B_1(q_1) - R_1 \geq 0,$$
$$U_t(q_t, R_t) = B_t(q_t) - R_t \geq B_t(q_{t-1}) - R_{t-1} = U_t(q_{t-1}, R_{t-1}),\ t = 2, 3 .$$

If the PC is not an equality we can increase revenue for all types and not violate any constraint. If a participation constraint is not an equality so we can increase revenue from all higher types without violating any downward constraint.

Thus all 3 constraints must be binding.

(c) The solution of the relaxed problem lies on the upper envelope of the indifference curves thus all other incentive constraints must be satisfied.



(d) Given the binding constraints,

$$U_1 = B_1(q_1) - R_1 = 0$$
$$U_2 = B_2(q_2) - R_2 = B_2(q_1) - R_1 = I_{21}(q_1) + U_1$$

where $I_{21}(q_1) = B_2(q_1) - B_1(q_1)$ is the informational rent of type 2.

$$U_3 = B_3(q_3) - R_3 = B_3(q_2) - R_2 = I_{32}(q_2) + U_2 = I_{32}(q_2) + I_{21}(q_1).$$

Rearranging we can solve for the revenue. Hence the firm's profit is

$$\Pi = f_1[B_1(q_1) - cq_1] + f_2[B_2(q_2) - I_{21}(q_1) - cq_2] + f_3[B_3(q_3) - I_{32}(q_2) - I_{21}(q_1) - cq_3]$$

$$\frac{\partial \Pi}{\partial q_1} = f_1[p_1(q_1) - c] - (f_2 + f_3) I_{21}'(q_1) = f_1[p_1(q_1) - c - \frac{1 - f_1}{f_1} I_{21}'(q_1)].$$

$$\frac{\partial \Pi}{\partial q_2} = f_2[p_2(q_2) - c] - f_3 I_{32}'(q_2).$$

$$\frac{\partial \Pi}{\partial q_3} = f_3[p_3(q_3) - c].$$

Note that $\frac{\partial \Pi}{\partial q_2} \geq 0 \Rightarrow \frac{\partial \Pi}{\partial q_3} > 0$, at $q_3 = q_2$. Thus $q_3^* > q_2^*$.

(e) Assume that $q_2^* > q$. THE FOC can be rewritten as follows.

$$\frac{\partial \Pi}{\partial q_2} = f_2[p_2(q_2) - c - \frac{f_3}{f_2} I_{32}'(q_2)] = 0.$$

Thus the optimal quantity depends only on the ratio f_3 / f_2.

Note that q_3^* is independent of the population and the demand curves of the lower types. We have just seen that q_2^* depends on the ratio f_3 / f_2 and the parameters of the medium and high demands. Thus it does not depend on a_1. Finally

$$\frac{\partial \Pi}{\partial q_1} = f_1[p_1(q_1) - c - (\frac{1-f_1}{f_1}) I_{21}{}'(q_1)].$$

Hence q_1^* depends on $a_1, a_2, \frac{1-f_1}{f_1}$.

(f)

$$\frac{\partial \Pi}{\partial q_2} = f_2[p_2(q_2) - c - \frac{f_3}{f_2} I_{32}{}'(q_2)].$$

Thus, for sufficiently small f_2, $\frac{\partial \Pi}{\partial q_2} < 0$ for all $q_2 > 0$. Thus the firm will reduce q_2 until $q_2^* = q_1^*$.

4.

(a) Let m be the maximum of the other bids. Suppose instead of bidding my value *v* I bid $b < v$. The outcome is the same unless $b < m < v$. In this case I am worse off bidding *b* (and losing) rather than bidding *v* and winning for a profit of $v - m$.

(b) Yes. If bidder 1 bids $\overline{v}$ bidder 2 cannot make a profit so bidding zero is a best response. But then any bid by bidder 1 is a best response since bidder 1 wins at a price of zero.

(c) If bidder 1 bids v and bidder 2 bids her valuation her bid distribution $G_2(b_2) = b_2$. Then the expected payoff is $\int_0^v (v - b_2) dG(b_2) = \int_0^v (v - b_2) db_2 = \frac{1}{2} v^2$.

(d) Suppose the equilibrium is $b(\cdot)$. If bidder 1 bids $b(v_2)$ and bidder 1 bids $b_1 = b(x)$, bidder 1 wins if $v_2 < x$, that is, with probability $F(x)$. Then

$$U_1(x, v_1) = F(x) v_1 - b(x).$$

FOC

$$\frac{\partial U_1}{\partial x}(x, v_1) = F'(x)v_1 - b'(x) = 0 .$$

For equilibrium, bidder1 must bid $b_1 = b(v)$ if his valuation is v, that is $x^* = v$. Hence

$$F'(v)v - b'(v) = 0$$

Integrating, $b(v) = \int_0^v yf(y)dy = \frac{1}{2}v^2$.

(e) We have seen that the expected buyer payoffs are the same thus the expected revenue is the same.

(f) Let $b(v)$ be the equilibrium if there is one. Bidder 1 bids $b(x)$. She pays $b(x)$ if v_2 is in the interval $[x,1]$. Her payoff is v_1 if $v_2 \in [0, x]$. Her expected payoff is

$$U = F(x)v_1 - \int_x^1 b(x)dF_2(v_2) .$$

FOC

$$\frac{\partial U}{\partial x} = F'(x)v_1 - b'(x)\int_x^1 dF_2(v_2) + b(x)F_2{}'(x) = 0 .$$

For equilibrium $x = v_1$. Then

$$-b(x)F'(x) + b'(x)(1 - F_2(x)) = F'(x)x .$$

Integrating,

$$b(v)(1 - F(v) = \int_0^v xF'(x)dx .$$

(No one attempted this question!)