**Homework 3R**

Question 1 should be reasonably straightforward. Question 3 examines the model that we analyzed in class. Question 2 is more technical but I have tried to take you through all the steps. In class I will explain why this is important. Please try to get your answers in by Tuesday. If you are struggling with an answer, please call me immediately (310) 459-8818 any time after 6:30 AM and before 11:00 PM. If I am not in, leave your phone number.

1. **Non-Linear Pricing**

There are two types of consumer. The number of consumer of type \( t_j \) is \( f_j \). The unit cost of production is \( c = 60 \). Type \( t_1 \) has a demand price function \( p_1 = 100 - 8q_1 \) while type \( t_2 \) has a demand price function \( p_2 = 120 - 6q_2 \). The monopolist knows only the distribution of types so cannot directly price discriminate.

   (a) Write down the incentive constraints. Explain why, in order to maximize profit, the participation constraint is

   \[
   U(t_1, q_1, R_1) = 100q_1 - 4q_1^2 - R_1 = 0.
   \]

   (b) Show that the informational rent of a type 2 agent is

   \[
   IR_2 = 20q_1 + q_1^2
   \]

   (c) If there were a third type with demand price function \( p_3 = 140 - 4q_3 \), what would be the informational rent of this type?

   (d) Profit from a consumer is total surplus (area under the demand price function) less informational rent. Write down an expression for total profit as a function of outputs.

   (e) Solve for the outputs in the profit maximizing selling scheme.

   (f) Provide the intuition behind the effect of an increase in the fraction of consumers of type \( t_1 \).

2. **Price Discrimination with a Continuum of agents (learning by doing)**

A firm offers a schedule of \((q, R)\) offers, that is, to buy \( q \) units you must pay \( R(q) \). In simple monopoly the schedule is \( R = p^M q \). Type \( t \) has a demand price function \( p(t, q) \)
and hence has a benefit function $B(t,q) = \int_0^q p(t,x)dx$. Types are continuously distributed on $[t,\bar{t}]$ with c.d.f $F(t)$. The total mass of consumers is 1.

Given these assumptions, the utility of a type $t$ consumer, if he chooses $(q,R)$ is $U(t,q,R) = B(t,q) - R$. Let $(q(t),R(t))$ be his choice. This is depicted below.

Note that, at $(q(t),R(t))$, the slope of the revenue function must be equal to the slope of the consumer’s indifference curve. Since this is the demand price (his marginal willingness to pay) we must have

$$\frac{dR}{dq}(q(t)) = p(t,q(t)). \quad (2.1)$$

Putting this more formally, if $(q(t),R(t))$ is type $t$’s optimal choice, then $(q(s),R(s))$ the choice of some other type $s$ must yield a lower payoff. Mathematically,

$$U(t,q(s),R(s)) \leq U(t,q(t),R(t)) \text{ for all } s \neq t.$$

The first order condition for the optimization is

$$\frac{d}{ds} U(t,q(s),R(s)) = 0 \text{ at } s = t.$$

That is

$$U_2q'(t) + U_3R'(t) = p(t,q(t))q'(t) - R'(t) = 0$$

Rearranging, $\frac{dR}{dq} = \frac{R'(t)}{q'(t)} = p(t,q(t))$. 

This is just equation (2.1). The optimization yields something else - - the informational rent of type $t$. If we totally differentiate $U(t,q(t),R(t))$ we get

$$\frac{dU}{dt} = U_1 + U_2 q'(t) + U_3 R'(t).$$

But we have just argued that the second and third terms must sum to zero. Moreover $U_1 = \frac{\partial B}{\partial t}$. Therefore the marginal informational rent is

$$\frac{dU}{dt} = \frac{\partial}{\partial t} B(t,q(t)).$$

The total informational rent over all consumers is

$$IR = \int_\tau U(t)F'(t)dt.$$ 

Since $F'(t) = -\frac{d}{dt}(1 - F(t))$, we can integrate by parts to obtain

$$IR = U(\tau) + \int_\tau U'(t)(1 - F(t))dt.$$ 

Hence

$$IR = U(\tau) + \int_\tau \frac{\partial}{\partial t} B(t,q(t))(1 - F(t))dt.$$ 

It is now easy to compute total profit since it is simply total surplus less total informational rent.

$$\Pi = \int_\tau (B(t,q(t)) - cq(t))F'(t)dt - \int_\tau \frac{\partial}{\partial t} B(t,q(t))(1 - F(t))dt - U(\tau).$$ 

Consider an example. The demand price function is $p(t,q) = t - q$ and types are uniformly distributed on the interval $[0, \beta]$.

(a) Show that total profit can be written as follows.
\[ \Pi = \int_0^\beta \left[ (2t - \beta - c)q(t) - \frac{1}{2}q(t)^2 \right] dt - U(0) \]

(b) Solve point-wise for the profit maximizing quantity and hence show that

\[ q^*(t) = 2t - \beta - c, \quad t \geq \frac{1}{2} (c + \beta). \]

Note that to maximize social surplus price must equal Marginal cost. That is

\[ p(t, q^S(t)) = t - q^S(t) = c, \quad \text{and so} \quad q^S(t) = t - c. \]

Then, except for the highest type \( q^*(t) < q^S(t). \)

Henceforth assume that \( c = 20 \) and \( \beta = 100 \) and solve for the inverse function \( t = T(q) \).

(c) From (2.1),

\[ \frac{dR}{dq} = p(t, q) = t - q = T(q) - q. \]

Substitute for \( T(q) \) and then integrate to solve for the profit-maximizing payment scheme \( R(q) \).

(d) Confirm that the profit-maximizing scheme involves quantity discounts.

(e) Consider the family of two-part pricing schemes

\[ R = \alpha(s) + \beta(s)q, \quad s \in [60, 100], \quad \text{where} \quad \alpha(s) = (t - 60)^2 \quad \text{and} \quad \beta(t) = 120 - t. \]

Confirm that for \( s = 60 \) and \( s = 100 \), these lines touch the profit maximizing \( R(q) \). In fact the curve \( R(q) \) is the lower envelope of these lines. Comment on the feasibility of using a family of 2 part pricing schemes to maximize profit.

3. Collusion

There are 2 identical firms sharing a market with demand price \( p = 28 - q \). Unit cost is 4.

(a) Solve for the Nash Equilibrium and profit-maximizing (symmetric) outputs and profits \( (\Pi_1^N \text{ and } \Pi_2^N) \)

(b) If firm 2 is colluding what is the profit-maximizing deviation by firm 1? What is the maximum profit from deviating \( \Pi_1^D \)?

(c) Show that for the threat to switch to the Nash output for t-1 periods to be credible, an inequality of the following type must hold.
(d) If $\delta = 0.9$, how many periods are necessary for the threat to be credible?
(e) Reconsider the problem with 3 firms and show that the number of periods must be longer.
(f) Suppose there are $n$ firms. Is it the case that with a large enough $n$ collusion is impossible?
(f) Do you find this a plausible explanation for the industry “consolidation” that is typical of maturing industries?