1. Signaling

The cost of signaling at level \( z \) is \( C(t, z) = \frac{z}{t^2} \). The marginal product of type \( t \) is \( \gamma t \). The parameter \( t \) is private information. The support of the distribution is \([0, \omega]\). Each type of worker has the same outside opportunity to earn a wage \( \alpha \). That is, the reservation utility of type \( t \), \( U_R(t) = \alpha \).

(a) Let \( z(t) \) be the equilibrium signaling function and \( w(z) \) be the equilibrium wage function. Explain why a type’s optimization problem can be written as follows.

\[
Max_s \{w(z(s)) - \frac{z(s)}{t^2}\}
\]

(a) Show that the equilibrium wage function must satisfy \( w^2 \frac{dw}{dz} = \gamma^2 \).

(b) Solve for the equilibrium signaling function \( z(t) \) and equilibrium utility \( U_E(t) \), of each type.

Henceforth assume that the reservation utility \( U_R(t) = \alpha + \beta t \).

(c) With full information, explain why those types less than \( \frac{\alpha}{\gamma - \beta} \) will choose their outside opportunity.

(d) With asymmetric information suppose that the minimum signal is \( z \) and the smallest type who signals is \( t \). The equilibrium payoffs must be as depicted. Obtain an expression for the equilibrium payoffs in terms of these minimum levels and \( t \).

\[
\begin{align*}
U_E(t) \\
\alpha + \beta t
\end{align*}
\]

Then \( U_E(t) = U_R(t) \) and \( U_E'(t) \geq U_R'(t) \).

(e) Obtain an expression for \( U_E'(t) \) (this simplifies by the Envelope Theorem). Use the latter condition to show that \( \frac{z}{t} \geq \frac{1}{2} \beta t^3 \). Henceforth consider the smallest education for the lowest type, so this is an equality.

(f) Use the first condition to solve for the lowest type who signals.
(g) Hence obtain a restriction on parameter values that must be satisfied if there is to be a signaling equilibrium.

2. Sad Loser Auction

There are two bidders. Each has a valuation that is an independent draw from a distribution with c.d.f \( F(v) \) and support \([0, \beta]\). Each submits a cash bid. The person paying the most wins the item and gets his cash back. The low bidder goes home without his cash or the item.

(a) Obtain an expression for buyer 1’s utility in the form \( U(x, v_1) \). Hence obtain a necessary condition for equilibrium.

\[
\int_0^v x F'(x) dx
\]

(b) Hence show that the equilibrium bid function is

\[
b(v) = \frac{\int_0^v x F'(x) dx}{1 - F(v)}.
\]

(c) What is the total derivative of \( U(v, v) \)? Hence or otherwise compare the payoffs in this auction to those in the sealed high bid auction.

(d) The auctioneer announces that he will only accept bids greater than \( r \). Assuming that values are uniformly distributed, what will be the lowest type who enters the auction?

(e) What is the new equilibrium bid function?

3. Auction with finite types

There are two bidders. Each has a valuation of 2 with probability 0.75 and 4 with probability 0.25. Valuations are independent.

(a) What is the expected payoff of each type of bidder in the open ascending bid auction? In a sealed high bid auction the low value bidders bid their valuations and the high value buyer’s equilibrium bidding strategy is a mixed strategy \( b \) with continuous c.d.f \( G(b) \) and support \([2, \bar{b}]\).

(b) Obtain an expression for the expected payoff of a high value type if he bids \( b \).

(c) By bidding just above 2, a high value buyer beats a low value buyer and thus wins with probability 0.75. Hence his expected payoff is 1.5. Hence solve for the equilibrium bid function.

(d) Compare expected payoffs and hence expected revenue in the two auctions.

(e) The auctioneer announces that he will only accept bids of 2 or 3 (He starts the bidding at 2 and jumps it immediately to 3.) Show that the equilibrium best response are pure strategies. Compute expected payoffs.

(f) Could the auctioneer increase expected revenue further?