Notes on Non-Linear Pricing

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1. Pricing with Full Information

A monopolist sells to $T$ different types of buyer. If he sells $q$ units to a type $t$ buyer his cost of production is $C_t(q) = \lambda q$. The demand curve of a type $t$ customer is $q_t(p)$. Inverting, the demand price function is $p_t(q)$. Let $B_t(q)$ be the area under the demand curve for a type $t$ consumer. If the consumer has to pay $R$ for the $q$ units his payoff or “utility” is

$$U_t(q, R) = B_t(q) - R = \int_0^q p_t(x)dx - R.$$

Indifference curves for this consumer type are drawn below. Note that the marginal rate of substitution is

$$\frac{dR}{dq}igg|_{U_t=U_t} = -\frac{\partial U_t}{\partial q} = p_t(q).$$

Thus the slope of the indifference curve is the demand price. If the monopolist sets a price $p_0$ the consumer pays $p_0 q$ for $q$ units. That is, the consumer can choose any point on the line $R = p_0 q$. In the diagram, the consumer chooses $q_t(p_0)$. The profit is then

![Indifference Curves](image-url)
the vertical distance between $R_0 = p_0 q_0(p_0)$ and $\lambda q_0(p_0)$.

If consumer types can be distinguished and resale is prohibitively costly, it is easy to see that the firm can increase its profit with a non-linear pricing scheme. In fact it can extract all surplus with a two-part pricing scheme. Total surplus from the sale of $q$ units is $B_1(q) - \lambda q$. This is maximized where $B_1'(q) = \lambda$, that is $q = q_1(\lambda)$.

Consider a pricing plan in which there is a fixed “access” fee $k$ and a per unit “user” fee $p$. Then if the consumer purchases $q$ units, his total payment is $R = k + pq$.

Suppose that the user fee is set equal to the marginal cost $\lambda$. Then as long as the access fee is not too high, the consumer will purchase the surplus maximizing amount $q_1(\lambda)$. The firm can then set the access fee so that the consumer is (almost) indifferent between making the purchase and purchasing nothing. In terms of the diagram, the indifference curve through his optimal purchase must also go through the origin.

**Exercise 1.1**: For any linear pricing plan, show that there is a two part pricing plan which is preferred by both the consumer and the firm.

![Figure 2: Two part pricing with full surplus extraction](image)

*Figure 2: Two part pricing with full surplus extraction*
2. Two Part Pricing with asymmetric Information

Under the assumptions that (1) the seller can distinguish different types of consumer and direct price discrimination is legal, and (2) resale is prohibitively costly, the monopolist can extract all surplus from each type of consumer. The solution is very simple. Charge each the same user fee but charge different access fees for each type.

In this section we replace assumption (1) with the assumption that the seller knows only the distribution of consumer types. We will also assume that higher indexed types have higher demands. That is,

\[ p_s(q) > 0 \Rightarrow p_t(q) > p_s(q), \quad t > s. \]

A two-part pricing plan \((k, p)\) is an access fee \(k\) and a unit price (user fee) \(p\). For full surplus extraction all customers are offered the same user fee \(p = \lambda\) and different access fees. Clearly this is impossible if the seller cannot distinguish types. However, as we shall see, it is still profitable for the firm to offer multiple two-part pricing plans and so indirectly price discriminate.

Preferences of type \(s\) and \(t > s\) are depicted below. Since the slope of an indifference curve is equal to a type’s demand price, the higher indexed type has indifference curves which are everywhere steeper.

![Figure 3: Single-crossing property](image)

Technically, the preference maps of types \(s\) and \(t > s\) satisfy the single-crossing property.
Consider preferences over alternative two part pricing plans. The indirect utility function \( u_t(p,k) \) is the utility of type \( t \) when he makes his optimal purchase, that is,

\[
 u_t(p,k) = \max_q \{ B_t(q) - pq - k \} = CS_t(p) - k.
\]

Note that \( \frac{\partial u_t}{\partial k} = -1 \) and \( \frac{\partial u_t}{\partial p} = \frac{\partial}{\partial p} CS_t = q_t(p) \). Then the marginal rate of substitution,

\[
 MRS_t = \left. \frac{dk}{dp} \right|_{u_t} = -\frac{\partial p}{\partial u} = q_t(p).
\]

Thus the indifference map is everywhere steeper for a type with a higher demand.

Suppose that the seller offers a pair of two-part pricing plans \((p',k')\) and \((p'',k'')\) where \(p' > p''\) and that type \( t \) (weakly) prefers the plan \((p',k')\) with the higher user fee. (Then it must have a lower access fee.) Intuitively any type \( s \) with \( s < t \) must strictly prefer the plan with the higher user fee since he is willing to pay less for additional units.

**Lemma 1: Single Crossing Property**

Suppose that type \( t \) weakly prefers plan \((p',k')\) over \((p'',k'')\) where \( p' > p'' > 0 \) and \( p' < p_t(0) \). Then any type \( s \) with \( s < t \) must strictly prefer \((p',k')\).

![Figure 4: Single Crossing Property for two-part plans](image-url)
Proof: Type $t$ has an indirect utility function

$$u_t(\hat{p}, \hat{k}) = \text{Max}_q \{B_t(q) - \hat{p}q - \hat{k}\}.$$  

Type $t$ chooses to consume until his demand price $p_t(q)$ is equal to the user fee $\hat{p}$, that is, he consumes $q_t(\hat{p})$ units. Thus his utility is

$$u_t(\hat{p}, \hat{k}) = \int_0^{q_t(\hat{p})} (p_t(q) - \hat{p})dq - \hat{k}.$$  

The first term on the right hand side is the shaded area to the left of the demand curve. This is depicted below.

![Figure 5: Consumer Surplus before paying access fee](image)

Thus we can rewrite consumer utility as follows.

$$u_t(\hat{p}, \hat{k}) = \int_{\hat{p}}^{p_t(0)} q_t(p)dp - \hat{k} \tag{2.1}$$

Then the net gain to choosing the plan with the higher user fee is
\[
\begin{align*}
\quad & u_t(p',k') - u_t(p'',k'') = \left( \int_{p'}^p q_t(p)dp - k' \right) - \left( \int_{p'}^p q_t(p)dp - k'' \right) \\
& = k'' - k' - \int_{p'}^p q_t(p)dp.
\end{align*}
\]

The integral is strictly smaller for a lower indexed type thus
\[
\quad u_t(p',k') - u_t(p'',k'') > u_t(p',k') - u_t(p'',k'')
\]

Q.E.D.

By an almost identical argument we have the following additional Lemma.

**Lemma 2:**

Suppose that type \( t \) is indifferent between plans \((p',k')\) and \((p'',k'')\) where \( p' > p'' > 0 \) and suppose that \( p' < p_t(0) \). Then any type \( s \) with \( s > t \) must strictly prefer \((p'',k'')\).

Suppose that the monopolist offers a schedule of plans. The consumer also has the (outside) option of purchasing nothing. We can include this in the schedule of feasible plans by adding the plan \((p_0,k_0) = (p_t(0)+\delta,0)\). Since the user fee exceeds the demand price of each type, a buyer who chooses “plan zero” pays no access fee and purchases nothing.

Let \((p_t,k_t)\) be the plan chosen by type \( t = 1,\ldots,T \) from the schedule of plans and plan zero. Type \( t \) is indifferent between plan zero and any plan with a user fee exceeding his maximum willingness to pay, \( p_t(0) \), since in each case he purchases nothing. Thus without loss of generality we may assume that either type \( t \) chooses plan 0 or a plan with a price \( p_t < p_t(0) \).

Given that plans \( \{(p_t,k_t)\}_{t=1}^T \), are selected by the \( T \) types, the monopolist loses nothing by dropping all other plans. By construction, plan \( t \) is optimal for type \( t \) thus if the auctioneer simply asks each type to indicate a plan by number, truthful revelation is a best response for each type. This simple point is known as the **Revelation Principle**. Working in type space rather than in the original space (in this example the 2-dimensional space of access and user fees) is sometimes very useful. This is especially so when the type-space is continuous rather than finite.
Participation constraints

Since each has the (“outside”) option of purchasing nothing, the schedule of choices \{ (p_t, k_s) \}_{s=1}^T must satisfy the following \( T \) participation constraints.

\[
    u_t(p_t, k_t) \geq u_t(p_0, k_0) = 0, \quad t = 1, \ldots, T
\]

However, since higher indexed types have higher demand prices,

\[
    u_t(p_t, k_t) = CS_t(p_t) - k_t \leq CS_s(p_t) - k_t \leq u_t(p_t, k_t), \quad t > 1.
\]

Thus if the participation constraint is satisfied for the lowest indexed type it is necessarily satisfied for all types.

Suppose \( u_t(p_t, k_t) = CS_t(p_t) - k_t = \theta > 0 \). Since utility is linear in income raising every access fee by \( \frac{1}{2} \theta \) does not affect any incentive constraint. Moreover the new utility of type 1 is \( u_t(p_t, k_t + \frac{1}{2} \theta) = u_t(p_t, k_t) - \frac{1}{2} \theta = \frac{1}{2} \theta > 0 \) so the participation constraint is also satisfied. Thus \( (p_t, k_t) \) does not maximize revenue. It follows that a necessary condition for profit maximization is that the participation constraint should be binding for type 1, that is

\[
    u_t(p_t, k_t) = u_t(p_0, k_0) = 0
\]

Incentive Constraints

Since \{ (p_t, k_t) \}_{t=1}^T is the schedule of choices by each type, the following \( T \times (T - 1) \) incentive constraints must be satisfied.

\[
    u_t(p_t, k_t) \geq u_s(p_s, k_s), \quad t = 1, \ldots, T, \quad s \neq t.
\]

We will argue that higher indexed types will never choose a plan with a higher user fee.

Lemma 3: Necessary condition for Incentive Compatibility

For \( s < t \), \( p_s \geq p_t \)

Proof: Suppose that for some \( s < t \) \( p_s < p_t \). To satisfy the incentive constraints,

\[
    u_t(p_t, k_t) \geq u_s(p_s, k_s).
\]

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Then by Lemma 1, \( u_s(p_t, k_t) > u_s(p_s, k_s) \), since \( p_t > p_s \). But this violates the incentive constraints for \( s \) Q.E.D.

We will begin by looking at a relaxed mathematical problem where the necessary condition holds and only the “local downward constraints” are imposed, that is

\[ u_t(p_t, k_t) \geq u_t(p_{t-1}, k_{t-1}), \quad t = 2, ..., T \]

We will show that the solution to this relaxed problem is incentive compatible. Suppose that for some type \( s \), \( u_s(p_s, k_s) - u_s(p_{s-1}, k_{s-1}) = \theta > 0 \). Raise the access fee of type \( s \) and all higher indexed types by \( \frac{1}{2} \theta \). Clearly the local downward incentive constraints are still satisfied and revenue is higher. Thus a necessary condition for profit maximization under the relaxed constraints is that is that the local downward constraints should be binding.

**Lemma 4**: If the local downward constraints hold with equality then all the upward constraints hold.

**Proof**:

\[ u_T(p_T, k_T) = u_T(p_{T-1}, k_{T-1}) \quad \text{(local downward constraint is binding)} \]

Then by Lemma 1,

\[ u_t(p_t, k_t) \leq u_t(p_{t-1}, k_{t-1}), \quad t < T \quad \text{since, by Lemma 3, } p_T \leq p_{T-1}. \]

Thus no type below \( T \) has an incentive to claim to be type \( T \).

Similarly

\[ u_{T-1}(p_{T-1}, k_{T-1}) = u_{T-1}(p_{T-2}, k_{T-2}) \quad \text{(local downward constraint is binding)} \]

Then by Lemma 1,

\[ u_t(p_{t-1}, k_{t-1}) \leq u_t(p_{t-2}, k_{t-2}), \quad t < T - 1 \quad \text{since, by Lemma 3, } p_{T-1} \leq p_{T-2}. \]

Thus no type below \( T-1 \) has an incentive to claim to be type \( T-1 \).

Repeating this argument \( T \) times yields all the upward incentive constraints.

Q.E.D.
Lemma 5: If the local downward incentive constraints hold, all the downward incentive constraints are satisfied.

Exercise 2.1: Prove this Lemma. (The proof follows closely the proof of Lemma 4.)

Combining all these results yields the following proposition.

Proposition 1: Necessary conditions for profit maximization

To maximize profit the monopolist chooses a schedule \(\{(p_t, k_t)\}_{t=1}^{T}\) such that (1) \(p_1 \geq \ldots \geq p_T\) (2) the participation constraint is binding for the lowest type and (3) the local downward constraint holds for all other types.

Corollary 2: The user fee for the highest type is set equal to marginal cost.

Note that the only constraint on the highest type is that it should have a utility level \(u_T(p_{T-1}, k_{T-1})\). The maximization problem is depicted below.

The surplus that can be potentially extracted from the highest type is the vertical distance between the indifference curve and the cost line. Thus the possible profit is as depicted, that is, at the output \(q_T(\lambda)\) where the slope of the indifference curve and the cost line are the same. This is achieved by choosing the user fee \(p_T = \lambda\) and choosing the access fee so that the local downward constraint is satisfied with equality.

Figure 6: The highest indexed type
Example: Two types

The type 1 (low) demanders have a utility of zero. The type 2 (high) demanders are indifferent between their choice and that of the low demanders. Therefore

\[ u_2 = CS_2(p_2) - k_2 = CS_2(p_1) - k_1 \]
\[ = [CS_2(p_1) - CS_1(p_1)] + CS_1(p_1) - k_1 = I_2 + u_1 \]

where

\[ I_2 = CS_2(p_1) - CS_1(p_1) \]

Since the utility of the low demanders is zero, the high demanders get an “informational rent” \( I_2(p_1) \) equal to the difference between the high and low valuations of plan 1.

With full information, the revenue to the monopolist from a given type is equal to the total consumer benefit (the area under the demand price function.) With asymmetric information this is reduced by the consumer payoff (the informational rent.) Thus

\[ R_1(p_1) = B_1(q_1(p_1)) \quad \text{and} \quad R_2(p_1, p_2) = B_2(q_2(p_2)) - I_2(p_1) \].

Let \( f_t \) be the number of type \( t \) consumers. Total profit is then

\[ \Pi = f_1R_1(p_1) + f_2R_2(p_1, p_2) - \lambda(f_1q_1(p_1) + f_2q_2(p_2)) \]
\[ = f_1[B_1(q_1(p_1)) - \lambda q_1(p_1)] + f_2[B_2(q_2(p_2)) - \lambda q_2(p_2) - I_2(p_1)] \]

Exercise 2.2: Write down the first order conditions for a profit maximum. Hence show that \( p_1^* > p_2^* = \lambda \)

Exercise 2.3: Solve for the optimal plans if the demand curves are \( p_1(q) = 1 - q \) and \( p_2(q) = 2 - q \)

A pair of two-part pricing plans satisfying Proposition 1 and Corollary 2 is depicted below.

Consider alternative plans on the zero payoff indifference curve for type 1. Note that a higher user fee and a lower access fee for type 1 lowers the profit from a type 1 consumer. However the new plan for type 1 is less attractive for a type 2 buyer. It is therefore possible to raise the access fee on type 2 consumers. Intuitively, the smaller the number of type 1 consumers and the more different they are from type 2 consumers, the greater will be the incentive to raise the user fee for type 1 in order to further squeeze
type 2 customers. Indeed if the type 1 consumers are sufficiently unimportant, the profit
maximizing plan is to squeeze type 1 out of the market completely. In this case the firm only offers one plan and this extracts all the surplus from the high demanders.

The General Problem

We can easily extend the analysis to the $T$ type case. From the necessary conditions, type $t$ is indifferent between plan $t$ and plan $t-1$. This gives type $t$ a higher informational rent than type $t-1$. We have

$$u_t = CS_t(p_t) - k_t = [CS_t(p_t) - CS_t(p_{t-1})] + CS_t(p_{t-1}) - k_t - 1 = I_t(p_{t-1}) + u_{t-1}$$

Since this condition holds for each type the total informational rent of type $t$ is

$$u_t = \sum_{s=2}^{t} I_s(p_{s-1}) .$$

The revenue earned by the firm from a type $t$ buyer is the total consumer benefit less the informational rent,
\[ R_t = B_t(q_t(p_t)) - \sum_{s=1}^{T} I_s(p_{s-1}) \]

We can then compute the profit from all \( T \) types.

The seller maximizes profit subject to the necessary conditions \( p_t \geq p_{t+1}, \ t = 1, \ldots, T - 1 \)

The Lagrangian is then

\[ L = \sum_{t=1}^{T} f_t R_t - \lambda \sum_{t=1}^{T} f_t q_t(p_t) + \sum_{t=1}^{T-1} \mu_t(p_t - p_{t+1}) \]

**Extension to General Cost functions**

We solve in two stages. First we ignore any costs and add an aggregate supply constraint

\[ \sum_{t=1}^{T} f_t q_t(p_t) \leq Q \]

The Lagrangian for this problem is

\[ L = \sum_{t=1}^{T} f_t R_t + \lambda(Q - \sum_{t=1}^{T} f_t q_t(p_t)) + \sum_{t=1}^{T-1} \mu_t(p_t - p_{t+1}) \]

Note that the first order conditions must be exactly the same as in the problem when costs are linear. The only difference is that \( \lambda \) is now the shadow price of the aggregate supply constraint. By the Envelope Theorem

\[ MR = \frac{\partial R^*(Q)}{\partial Q} = \frac{\partial L}{\partial Q} = \lambda. \]

This suggests a simple method of solution. Pick a value for \( \lambda \) and solve for the profit maximizing user fees assuming a marginal cost of production equal to \( \lambda \). Then compute total output and hence actual marginal cost \( C'(Q) \). If \( \lambda > C'(Q) \) then choose a smaller value for \( \lambda \) and compute the solution again. For the full optimum, the shadow price and actual \( MC \) must be equal.
3. Designing an optimal mechanism

Suppose that the plans shown below are the optimal two-part pricing plans in the two type case. It is easy to see that the monopolist can do better. Suppose that he eliminates the second plan and replaces it by the take-it-or-leave-it offer \((q_2(\lambda), \hat{R})\), that is, the offer to deliver \(q_2(\lambda)\) units if paid a total of \(\hat{R}\). This has no effect on type 1 but further squeezes type 2.

We can easily build on the analysis of the previous section to design an optimal mechanism. Given the mechanism that the monopolist designs, there must be some resulting delivery of units to each type and some total payment. For type \(t\), let this be the pair \((q_t, R_t)\). Arguing almost exactly as in the previous section, a necessary condition for incentive compatibility is

\[q_t \leq q_{t+1}, \quad t = 1, ..., T - 1\]
Moreover for profit maximization the participation constraint and the local downward constraints are binding. That is

\[ U_1(q_1, R_t) = U_1(0, 0) = 0 \]
and

\[ U_t(q_t, R_t) = U_t(q_{t-1}, R_{t-1}), \quad t = 2, \ldots, T \]

Exercise 3.1: Write down Lemmas comparable to those of Section 2 and sketch the proofs.

Similarly, since type \( t \) is indifferent between his plan and that for type \( t-1 \), for type \( t \) we can compute the informational rent.

\[ U_t = CS_t(q_t) - R_t = CS_t(q_{t-1}) - R_{t-1} = I_t(q_{t-1}) + U_{t-1} \]

where

\[ I_t = CS_t(q_{t-1}) - CS_{t-1}(q_{t-1}) \]  

The total informational rent for type \( t \) is therefore

\[ U_t = \sum_{s=1}^{t} I_s(q_{s-1}) \]

Arguing exactly as in the previous section, the revenue earned by the monopolist on each type \( t \) consumer is the total benefit to type \( t \) less the informational rent, that is

\[ R_t = B_t(q_t) - \sum_{s=2}^{t} I_s(q_{s-1}) \]  \hspace{1cm} (3.1)

For the two type case, total profit is therefore

\[ \Pi = f_1(B_1(q_1) - \lambda q_1) + f_2(B_2(q_2) - \{CS_2(q_2) - CS_1(q_1)\}) \]

Example: Linear Demands

Suppose that the unit cost of production is \( \lambda \) and that \( p_t(q) = a_t - 2q \) so that

\[ U_t(q, R) = a_t q - q^2 - R. \]
Binding Participation constraint:

\[ U_1(q_i, R_i) = a_i q_i - q_i^2 - R_i = 0 \]

hence

\[ R_i = a_i q_i - q_i^2 \]  
(3.2)

Binding Local Downward Constraint:

\[ U_2(q_2, R_2) = U_2(q_1, R_1) \]

\[ \Rightarrow U_2 = I_2 = CS_2(q_i) - CS_1(q_i) = a_2 q_i - q_i^2 - (a_i q_i - q_i^2) = (a_2 - a_i) q_i \]

Thus \( R_2 = B_2(q_2) - I_2 = a_2 q_2 - q_2^2 - (a_2 - a_i) q_i \)  
(3.3)

Total revenue is therefore

\[ R = f_1 R_1 + f_2 R_2 = f_1(a_i q_i - q_i^2) + f_2(a_2 q_2 - q_2^2 - (a_2 - a_i) q_i). \]

Given linear costs \( C = \lambda (f_1 q_i + f_2 q_2) \) it is a straightforward matter to solve for the optimal quantities and payments.

**Choosing the optimal quantities**

Total revenue is

\[ R(q) = \sum_{t=1}^{T} f_t R_t \]

where, from equation (3.1),

\[ R_t = B_t(q_t) - \sum_{s=2}^{t} I_s(q_{s-1}) \]

The firm then maximizes revenue less cost subject to the incentive constraints

\[ q_t \geq q_{t-1} \]

To compute the marginal revenue of increasing the quantity in plan \( t \) we collect all the \( q_t \) terms.

\[ f_t B_t(q_t) + f_{t+1} I_{t+1}(q_t) + f_{t+2} I_{t+2}(q_t) + ... \]

\[ = f_t B_t(q_t) - (I - F_t) I_{t+1}(q_t) = f_t B_t(q_t) - (I - F_t)(B_{t+1}(q_t) - B_t(q_t)) \]
where \( F_t = \sum_{s=1}^{t} f_s \) is the cumulative probability.

The marginal revenue is therefore

\[
\frac{dR}{dq_t} = f_t \cdot p_t(q_t) - (1 - F_t)(p_{t+1}(q_t) - p_t(q_t))
\]

To understand this better, consider the figure below.

Consider the effect of increasing the quantity sold to type \( t \) from \( q_t \) to \( q_t + \Delta q_t \). This has no effects on the local downward constraints for lower types. Since type \( t \)'s local downward constraint is binding,

\[
U_t = B_t(q_t + \Delta q_t) - R - \Delta R_t = B_t(q_{t-1}) - R_{t-1}
\]

Then taking the first order approximation,

\[
\Delta R_t = \frac{\partial}{\partial q_t} B_t(q_t) \Delta q_t
\]
However, to satisfy the local downward constraint for type $t+1$ it is necessary to increase his utility by some amount $\Delta B$. This is shown below.

The increased utility of type $t+1$ is just the difference in the slopes of the two indifference curves through $(q_t, R_t)$ multiplied by the change in quantity $\Delta q_t$. That is,

$$\Delta B = (\frac{\partial B_{t+1}(q)}{\partial q} - \frac{\partial B_t(q)}{\partial q})\Delta q$$

Since indifference curves are vertically parallel, the fact that local downward constraints are binding implies that all will have a utility increase of $\Delta B$. Thus the net change in revenue is

$$f_i\Delta R = f_i\Delta R_t - (1 - F_i)\Delta B$$

where $F_i = \sum_{s=1}^{i} f_s$

Substituting and dividing by $\Delta q$,

$$f_i MR_i = \frac{\Delta R}{\Delta q} = f_i \frac{\partial B_t(q)}{\partial q} - (1 - F_i)\left(\frac{\partial B_{t+1}(q)}{\partial q} - \frac{\partial B_t(q)}{\partial q}\right).$$

Hence

$$MR_i(q) = \frac{p_i(q) - (1 - F_i)(p_{t+1}(q_t) - p_t(q_t))}{f_i}$$

**Exercise 3.2: Comparative statics with two types**

Suppose that the unit cost of production is $\lambda$ and that $p_t(q) = a_t - 2q$

(a) Show that the greater the proportion of high demanders and the greater the difference in demand price functions, the smaller is the quantity offered to low demanders.

(b) Under what conditions will the small demanders be squeezed out completely?

**Exercise 3.3: Three types with pooling**

Extend the previous example to three types. Show that if $n_1$ is sufficiently large and $n_2$ is sufficiently small, it will be optimal only to offer two packages and that both types 1 and 2 will choose the smaller package.
4. Continuously distributed types

We will consider the continuous case as the limit of the discrete model. We begin we parametrize the model and write the demand price function as \( p(q, \theta) \). Then the utility of type \( t \) if he accepts the offer \((q, R)\) is

\[
U(q, R, \theta) = B(q, \theta_t) - R = \int_0^q p(x, \theta_t) dx - R
\]

Suppose that types are continuously distributed and that the number of buyers of type \( \theta \) or lower is \( F(\theta) \). Define \( f(\theta) = F'(\theta) \). We can approximate any continuous distribution by a finite type distribution as shown below with \( \theta_{i+1} - \theta_i = \Delta \theta \) and \( f_i = f(\theta_i) \Delta \theta \).

From the previous section, the extra revenue from increasing \( q_i \) is

\[
f_iMR(q_i, \theta_i) = p(q_i, \theta_i) f_i - (1-F_i)(p(q_i, \theta_{i+1}) - p(q_i, \theta_i))
\]

\[
= p(q_i, \theta_i) f_i - (1-F_i) \left( \frac{p(q_i, \theta_{i+1}) - p(q_i, \theta_i)}{\theta_{i+1} - \theta_i} \right) \Delta \theta.
\]

Since \( f_i = f(\theta_i) \Delta \theta \) we obtain

\[
MR(q_i, \theta_i) = p(q_i, \theta_i) - \frac{(1-F_i)}{f_i} \left( \frac{p(q_i, \theta_{i+1}) - p(q_i, \theta_i)}{\theta_{i+1} - \theta_i} \right) \Delta \theta
\]

Taking the limit,
\[ MR(q(\theta), \theta) = p(q, \theta) - \frac{(1 - F(\theta)) \frac{\partial p}{\partial \theta}}{f(\theta)} (q, \theta) \] 

(4.1)

**Example:** \( p(q, \theta) = 5 + 10\theta - q \), \( MC = 5 \) and \( F(\theta) = \theta \).

Substituting from (4.1),

\[ MR(q(\theta), \theta) = 5 + 10\theta - q - (1 - \theta)10 = 20\theta - 5 - q. \]

Then

\[ MR(q(\theta), \theta) - MC(q(\theta), \theta) = 20\theta - 10 - q \]

Note that marginal profit is increasing in \( \theta \) so we can ignore the monotonicity constraints. For \( \theta < \frac{1}{2} \) the optimal quantity is zero. For larger \( \theta \).

\[ q(\theta) = 20\theta - 10. \] 

(4.2)

To compute the payments \( R(\theta) \) we note that all those types \( \theta \leq \frac{1}{2} \) receive nothing so must pay nothing. Then \( R(\frac{1}{2}) = 0 \). Also, the optimal allocation and payment scheme \( (q(\theta), R(\theta)) \) implicitly defines the payment function \( R^*(q) \). This is depicted below.

Since the slope of the indifference curve for type \( \theta \) is his demand price \( p(q, \theta) \) it follows that
\[
\frac{dR^*}{dq} = p(q, \theta) = 5 + 10\theta - q
\]

Substituting for \( \theta \) from (4.2),

\[
\frac{dR^*}{dq} = 10 - \frac{1}{2}q .
\]

Then \( R^*(q) = 10q - \frac{1}{4}q^2 \)

**Exercise 4.1 Two-part pricing schedules**

(a) For the example above show that \( R'(q(\theta)) = 10(1 - \theta) \) and that \( R(q(\theta)) = -25/4 + 15\theta - 5\theta^2 \).

(b) Draw the line tangential to the curve \( R^*(q) \) in the Figure above. Using your answer to part (a) or otherwise, show that this line can be written in the form \( R = \alpha(\theta) + \beta(\theta)q \).

(c) Suppose that instead of offering the non-linear pricing schedule \( R^*(q) \), the firm offers a continuum of two-part pricing schedules \( R = \alpha(\theta) + \beta(\theta)q, \quad \theta \in [\frac{1}{2}, 1] \). Which schedule will type \( \theta \) choose? (Answer using the Figure.)

**Exercise 4.2** Suppose \( p(q, \theta) = 40 + 20\theta - q \), \( MC = 10 \) and types are uniformly distributed on \([0, 1]\).

(a) Solve for the optimal supply to each type.

(b) Hence confirm that no types will be excluded.

(c) Solve for the optimal non-linear pricing scheme \( R^*(q) \). HINT: What must be the payment by the lowest type if the participation constraint is binding?

### 5. Applying the Revelation Principle

In this final section we examine the continuous case directly. For any set of alternatives offered by the firm, let \((q(\theta), R(\theta))\) be the choice of type \( \theta \). Then for all \( \theta \) and all \( x \neq \theta \),

\[
U(q(\theta), R(\theta), \theta) \geq U(q(x), R(x), \theta) .
\]

Thus for all \( \theta \) the following condition is necessary for Incentive Compatibility.
\[
\frac{d}{dx} \left. U(q(x), R(x), \theta) \right|_{x=\theta} = 0 .
\]

Thus the total derivative of buyer utility,

\[
\frac{dU}{d\theta} = \frac{d}{dx} \left. U(q(x), R(x), \theta) \right|_{x=\theta} + \frac{\partial}{\partial x} \left. U(q(x), R(x), \theta) \right|_{x=\theta}
\]

\[
= \frac{\partial}{\partial x} U(q(x), R(x), \theta) \bigg|_{x=\theta} = \frac{\partial B}{\partial \theta} (q(\theta), \theta)
\]

(5.1)

The revenue from a type \( t \) buyer is the consumer benefit of this type less his informational rent, that

\[
R(\theta) = B(q(\theta), \theta) - U(\theta).
\]

Summing over all types, total revenue is

\[
R = \int_a^\beta R(\theta)F'(\theta)d\theta = \int_a^\beta B(q(\theta), \theta)F'(\theta)d\theta - \int_a^\beta F'(\theta)U(\theta)d\theta
\]

Integrating the last term by parts and noting that \( U(\alpha) = 0 \),

\[
R = \int_a^\beta B(q(\theta), \theta)F'(\theta)d\theta - \int_a^\beta (1 - F(\theta)) \frac{dU}{d\theta} d\theta.
\]

Substituting from (5.1)

\[
R = \int_a^\beta B(q(\theta), \theta)F'(\theta)d\theta - \int_a^\beta (1 - F(\theta)) \frac{\partial B}{\partial \theta} (q(\theta), \theta)d\theta
\]

\[
R = \int_a^\beta \hat{R}(q(\theta), \theta)F'(\theta)d\theta
\]

where \( \hat{R}(q, \theta) = B(q, \theta) - \frac{1 - F(\theta)}{F'(\theta)} \frac{\partial B}{\partial \theta} (q, \theta) \).

Hence marginal revenue is

\[
\frac{\partial \hat{R}}{\partial q} = p(q, \theta) - \frac{1 - F(\theta)}{F'(\theta)} \frac{\partial p}{\partial \theta} (q, \theta).
\]