Answers

1. The utility of a type $t_1$ buyer is $u_1(q, R) = 100q - 4q^2 - R$ and of a type $t_2$ buyer is $u_2(q, R) = 120q - 4q^2 - R$. For the first type the local downward constraint (LDC) is the participation constraint. $u_1(q_1, R_1) \geq 0$. For the second type the LDC is $u_2(q_2, R_2) \geq u_2(q_1, R_1)$. Ignoring other constraints, profit maximizing requires that the first constraint must be binding. Otherwise increase both payments an equal amount and the preference rankings are unaffected. The second constraint must also be binding, otherwise increase $R_2$.

Type $t_2$ is willing to pay more for $q_1$ so the participation constraint is satisfied. Since type $t_1$'s demand price is lower, she is willing to pay less for additional units. Therefore, if type 2 is indifferent, type $t_1$ strictly prefers the lower quantity.

$$u_2(q_2, R_2) = u_2(q_1, R_1) = 120q_1 - 4q_1^2 = 100q_1 - 3q_1^2 + (20q_1 + q_1^2) = u_1(q_1, R_1) + 20q_1 + q_1^2 = 20q_1 + q_1^2 = IR_1.$$

“IR” is just the consumer gain (called “informational rent” because the gain would be zero with full information and perfect price discrimination.

$$u_3(q_3, R_3) = u_3(q_2, R_2) = 140q_2 - 2q_2^2 = 120q_2 - 3q_2^2 + (20q_2 + q_2^2) = u_2(q_2, R_2) + 20q_2 + q_2^2 = (20q_1 + q_1^2) + (20q_2 + q_2^2) = IR_2.$$

Total surplus is the area under the demand price functions less cost

$$S = f_1(40q_1 - 4q_1^2) + f_2(60q_2 - 3q_2^2).$$

The seller gets the total surplus less the IR.

$$U_0 = f_1(40q_1 - 4q_1^2 - IR_1) + f_2(60q_2 - 3q_2^2 - IR_2) = f_1(40q_1 - 4q_1^2) + f_2(60q_2 - 3q_2^2) - 20q_1 - q_1^2.$$ Maximizing it follows immediately that $q_2^* = 10$. This is the efficient output.

$$\frac{\partial}{\partial q_1}U_0 = f_1(40 - 8q_1) - f_2(20 + 2q_1) = 40f_1 - 20f_2 - (8f_1 + 2f_2)q_1.$$ Without loss of generality, assume that the total mass is 1.

$$\frac{\partial}{\partial q_1}U_0 = 40f_1 - 20(1 - f_1) - (8f_1 + 2(1 - f_1))q_1.$$
Choose positive \( q_i \) if and only if \( 40f_i - 20(1 - f_i) > 0 \), that is \( f_i > 1/3 \).

The profit maximizing strategy optimally trades off the lost revenue from reducing \( q_i \) (and hence the profit on type \( t_i \) and the revenue gained from making the high types alternative less attractive and so changing more for \( q_z \). As \( f_i \) increases (and \( f_2 \) falls) the cost of lowering \( q_i \) rises and the benefit falls. Thus \( q_i^*(f_i) \) is an increasing function.

2.

\[
B(t, q) = t - \frac{1}{2}q^2
\]

Hence total surplus is

\[
S = \int_0^\beta (tq(t) - \frac{1}{2}q^2(t) - cq(t))dF(t) = \frac{1}{\beta} \int_0^\beta (tq(t) - cq(t) - \frac{1}{2}q^2(t))dt
\]

Hence

\[
u(t, x) = U(t, q(x), R(x)) = tq(x) - \frac{1}{2}q^2(x) - R(x).
\]

For incentive compatibility,

\[
\frac{\partial}{\partial x} u(t, x) = 0 \text{ at } x = t.
\]

Then

\[
\frac{d}{dt} u(t, t) = \frac{\partial}{\partial t} u(t, x) \bigg|_{x=t} = q(t).
\]

Hence

\[
IR = \int_0^\beta U(t)dF(t) = U(0) + \int_0^\beta U'(t)(1 - F(t))dt = \int_0^\beta q(t)(1 - \frac{t}{\beta})dt.
\]

Firm profit is therefore

\[
U_0 = \frac{1}{\beta} \int_0^\beta (tq(t) - cq(t) - \frac{1}{2}q^2(t))dt - \frac{1}{\beta} \int_0^\beta q(t)(\beta - t)dt
\]

\[
U_0 = \frac{1}{\beta} \int_0^\beta [(t - \beta + c)q(t) - \frac{1}{2}q^2(t)]dt
\]

Point-wise maximization yields

\[
q(t) = \begin{cases} 
0, & t < \frac{1}{2}(c + \beta) \\
2t - c - \beta, & t \geq \frac{1}{2}(c + \beta)
\end{cases}
\]
Taking the special case $q(t) = 2t - 120$. Let $R(q)$ be the revenue function generated by the mapping $(q(t), R(t))$. The slope of this curve at $(q(t), R(t))$ must be equal to the slope of the indifference curve

$$p(t, q(t)) = t - q(t).$$

Then

$$\frac{dR}{dq} = t - q(t).$$

Also $q(t) = 2t - 120$.

Then

$$\frac{dR}{dq} = 60 - \frac{1}{2}q$$

Integrating,

$$R(q) = 60q - \frac{1}{4}q^2 = q(60 - \frac{1}{4}q)$$

Thus the price per unit,

$$\frac{R(q)}{q} = 60 - \frac{1}{4}q,$$

decreases with $q$.

The revenue function is quadratic thus is the lower envelope of a continuum of 2-part pricing schemes.

The consumer chooses $s$ and $q$ to maximize

$$u(s, q) = B(t, q) - \alpha(s) - \beta(s)q = tq - \frac{1}{4}q^2 - (s - 60)^2 - (120 - s)q.$$

Differentiating by $q$ yields the FOC

$$\frac{\partial u}{\partial q} = s + t - 120 - q = 0.$$

Hence $q^*(s) = s + t - 120$

Differentiating by $s$,

$$\frac{\partial u}{\partial s} = -2(s - 60) + q^*(s) = 2(t - s) = 0.$$
Thus type $t$ chooses $s = t$ and purchases $2t - 120$ units.

This alternative incentive scheme is more complicated but it has an advantage when the quantity is an action by the agent. The monopolist asks you what sort of wage and bonus scheme you want. You let him know which scheme and he then gives you a wage ($\alpha(s)$) and a piece rate $\beta(s)$. The higher the piece rate the greater the effort incentive.

Hongbin has done work on this topic.

3.

If you defect you get the Nash outcome $\Pi^N_1$ for $t-1$ periods after your period 1 defection. Then if you cooperate after than you get the cooperative outcome. The PV is therefore

$$U^D = \Pi^D_i + (\delta + \ldots + \delta^{T-1})\Pi^N_1 + \delta^T (1 + \delta + \ldots)\Pi^*_1$$

$$= \Pi^D_i + \frac{\delta - \delta^T}{1 - \delta} \Pi^N_1 + \frac{\delta^T}{1 - \delta} \Pi^*_1$$

If you cooperate, you get

$$U^C = \Pi^*_1 + (\delta + \ldots + \delta^{T-1})\Pi^*_1 + \delta^T (1 + \delta + \ldots)\Pi^*_1.$$}

Subtracting the first from the second,

$$U^C - U^D = \Pi^*_1 - \Pi^D_i + \frac{\delta - \delta^T}{1 - \delta} (\Pi^*_1 - \Pi^N_1).$$

Hence this is positive if and only if

$$\frac{\Pi^D_i - \Pi^*_1}{\Pi^*_1 - \Pi^N_1} < \frac{\delta - \delta^T}{1 - \delta}.$$

Let the symmetric equilibrium output be $q^N$. Firm 1 has a profit of

$$U_1 = (24 - (n-1)q^N - q_1)q_1.$$

From the FOC $q^N_i = \frac{24}{n+1}$ and $U^N_1 = \frac{24^2}{(n+1)^2} = 64$ if $n = 2$.

If the firms cooperate they will choose the profit maximizing output (12) and share it among the $n$ firms. Thus $U^*_1 = \frac{144}{n}$. 

The cooperative output for each firm is \( \frac{12}{n} \) thus if the other \( n-1 \) firms cooperate, firm 1’s profit is

\[
U_{i}^{D} = (24 - (n-1)q^* - q_i)q_i = (24 - \frac{n-1}{n} 12 - q_i)q_i.
\]

Maximizing it follows that \( U_{i}^{D} = (\frac{n}{2})^2 = 81 \), when \( n = 2 \).

Then for cooperation to be a best response,

\[
\frac{81-72}{72-64} < \frac{\delta}{1-\delta}(1-\delta^{r-1}).
\]

Setting \( \delta = 9/10 \) it follows that cooperation is best if \( (\frac{9}{10})^{r-1} < \frac{7}{8} \). This holds for all \( T>2 \).

As \( n \) increases the cooperative and Nash profits go to zero while the profit from deviating goes to 12. Thus for high enough \( n \) the IC cannot be satisfied.