1. Necessary and sufficient conditions for a maximum

Module 8 derived the Necessary Conditions for \( x \) to solve the following maximization problem.

\[
\max_{x \geq 0} \{ f(x) \mid \bar{b} - g(x) \geq 0 \}.
\]

The key insight is that these are also the Necessary Conditions for a maximum when the resource is not fixed. Instead it can be purchased or sold. In this section we extend this approach to obtain simple necessary conditions for a maximum.

With multiple constraints, it is necessary to use much more sophisticated mathematics. However the basic insights remain true. To keep the exposition as simple as possible, we will consider the case of two resource constraints.

\[ g_1(x) \leq \bar{b}_1 \text{ and } g_2(x) \leq \bar{b}_2. \]

\[ X = \{ x \mid x \geq 0, ~ \bar{b}_i - g_i(x) \geq 0 \} . \]

Then the maximization problem is

\[
\max_{x \in X} \{ f(x) \} , \text{ equivalently, } \max_{x \geq 0} \{ f(x) \mid \bar{b}_i - g_i(x) \geq 0, \ i = 1, 2 \}
\]

The weakest possible Constraint Qualification is not especially intuitive. In almost all economic applications in an economics course, the following stronger Constraint Qualification will be satisfied:\(^1\)

**Constraint Qualification**

(i) If the \( i \)-th constraint is binding at \( \bar{x} \) then \( \frac{\partial g_i}{\partial \bar{x}}(\bar{x}) \neq 0 \).

(ii) The feasible set \( X \) is convex and has a non-empty interior.

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\(^1\) It is most unlikely that you will be asked to show that the Constraint Qualification holds. Simply trust that it is satisfied.
Suppose that $\vec{x}$ solves the maximization problem,

$$\max_{x \geq 0} \{ f(x) | \vec{b}_i - g_i(x) \geq 0, \ i = 1, 2 \}.$$

There are three cases to consider:

Case (i) No resource constraint is binding at $\vec{x}$.
Case (ii) One resource constraint is binding at $\vec{x}$.
Case (iii) Both resource constraints are binding at $\vec{x}$.

We follow the approach with one resource constraint and introduce a vector of “shadow prices” $\vec{\lambda} = (\lambda_1, \lambda_2) \geq 0$. We then consider the “relaxed problem” in which it is possible to purchase or sell units of each resource at these prices.

Note that $\vec{b}_i - g_i(x)$ is the number of units of resource $i$ sold at the shadow price $\lambda_i \geq 0$. The relaxed profit is then

$$\Sigma(x, \vec{\lambda}) = f(x) + \lambda_1(\vec{b}_1 - g_1(x)) + \lambda_2(\vec{b}_2 - g_2(x)).$$

The maximization problem with two resource markets is therefore as follows:

$$\max_{x \geq 0} \{ \Sigma(x, \vec{\lambda}) = f(x) + \lambda_1(\vec{b}_1 - g_1(x)) + \lambda_2(\vec{b}_2 - g_2(x)) \} \tag{1-1}$$

Case (i) No resource constraint is binding at $\vec{x}$.

In this case $\vec{x}$ solves the unconstrained problem $\max_{x \geq 0} \{ f(x) \}$.

This is problem (1-1) if the shadow prices are both zero.

Case (ii) One resource constraint is binding at $\vec{x}$.

Suppose that only the first resource constraint is binding, i.e. $g_1(\vec{x}) = \vec{b}_1$. Then we can ignore the second resource constraint. So $\vec{x}$ solves

$$\max_{x \geq 0} \{ f(x) | \vec{b}_1 - g_1(x) \geq 0 \}.$$
This is the problem examined in Module 8. As we saw, the Necessary Conditions are also the Necessary Conditions for problem (1-1) if $\lambda_2 = 0$.

**Case (iii) Both constraints are binding**

The First Order Necessary Conditions for $\overline{x}$ to solve (1-1) are then

$$\frac{\partial \Omega}{\partial x_j}(\overline{x}) = \frac{\partial f}{\partial x_j}(\overline{x}) - \lambda_1 \frac{\partial g_1}{\partial x_j}(\overline{x}) - \lambda_2 \frac{\partial g_2}{\partial x_j}(\overline{x}) \leq 0, \text{ with equality if } x_j > 0. \quad (1-2)$$

Intuitively, if the shadow prices of both constraints are sufficiently low, $\mathcal{L}(x, \lambda)$ is maximized by purchasing additional units of each resource. And if the shadow prices are sufficiently high, $\mathcal{L}(x, \lambda)$ is maximized by selling some of each resource. It is plausible that at some intermediate shadow price vector $\lambda$, $\overline{x}$ satisfies (1-2) and demand for each resource will be equal to its, i.e.

$$h_i(\overline{x}) = \overline{b}_i - g_i(\overline{x}) = 0.$$

This intuition is correct (although requires a sophisticated proof.) Thus at the prices $(\lambda_1, \lambda_2)$ the First Order Necessary Conditions with resource markets are the First Order Conditions with resources fixed.

We summarize below. Define $X$ to be the set of feasible $x$ vectors, i.e.

$$X = \{x | x \geq 0, \ h_i(x) = \overline{b}_i - g_i(x) \geq 0\}$$

**Proposition 1.1 Necessary conditions for $\overline{x}$ to solve $\max_x\{f(x) | x \in X\}$**

If the Constraint Qualifications hold at $\overline{x}$, there is a shadow price vector $\lambda \geq 0$ such that $\overline{x}$ satisfies the Necessary Conditions when the resources can be purchased and sold at these prices, i.e. (1-2) holds. Moreover, for any constraint that is not binding at $\overline{x}$, the corresponding shadow price is zero.
Paralleling the case on one resource constraint, we have the following additional result.

**Proposition 1.2: Sufficient conditions for a maximum**

Suppose that $\bar{x}$ solves $\max_{x \geq 0} \{ f(x) \mid \bar{b}_i - g_i(x) \geq 0, \ i = 1, 2 \}$, where $f(x)$, $-g_1(x)$ and $-g_2(x)$ are all concave. If the Constraint Qualification holds at $\bar{x}$, then the Necessary Conditions for a maximum (see Proposition 1.1) are also sufficient.

The derivation is essentially identical to that for the cases of one resource constraint.