## A useful family of utility functions

Modelling requires making strong simplifying assumptions. One such assumption is that the marginal rate of substitution between pairs of commodities depends only on the ratio of the two commodities.

$$
\operatorname{MRS}\left(x_{i}, x_{j}\right)=\frac{\frac{\partial U}{\partial x_{i}}}{\frac{\partial U}{\partial x_{j}}}=f\left(\frac{x_{j}}{x_{i}}\right) .
$$

## Example 1: Cobb-Douglas utility

$$
U(x)=x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}}, \quad \operatorname{MRS}\left(x_{1}, x_{2}\right)=\frac{\alpha_{1}}{\alpha_{2}} \frac{x_{2}}{x_{1}} .
$$



A superlevel set of $U(x)$

The set $\{x \mid U(x) \geq U(1,1)\}$ is called a superlevel set. This set never touches the axes but approaches an axis as consumption of one of the commodities gets very large.

## Example 2:

$$
U(x)=2 a_{1} x_{1}^{1 / 2}+2 a_{2} x_{2}^{1 / 2}, \operatorname{MRS}\left(x_{1}, x_{2}\right)=\frac{a_{1}}{a_{2}}\left(\frac{x_{2}}{x_{1}}\right)^{2},\left(a_{1}, a_{2}\right) \gg 0
$$

Exercise 1: Depict the level set through $(1,1)$ and solve for the point on each axis that is in this level set. What is the limiting marginal rate of substitution (i) as $x_{2}$ approaches zero (ii) $x_{1}$ approaches zero?

## Example 3:

$$
U(x)=-a_{1} x_{1}^{-1}-a_{2} x_{2}^{-1}, \operatorname{MRS}\left(x_{1}, x_{2}\right)=\frac{a_{1}}{a_{2}}\left(\frac{x_{2}}{x_{1}}\right)^{1 / 2}
$$

Exercise 2: Depict the level set through $(1,1)$ and solve for the point on each axis that is in this level set. As $x_{i}$ grows large, what is the limiting value of $x_{j}$ ? What are the limiting marginal rates of substitution as consumption of one of the commodities grows large?

## CES utility functions

The CES family generalizes these examples. The marginal rate of substitution is a power function of the consumption ratio.

$$
\operatorname{MRS}\left(x_{i}, x_{j}\right)=\frac{\frac{\partial U}{\partial x_{i}}}{\frac{\partial U}{\partial x_{j}}}=\frac{a_{i}}{a_{j}}\left(\frac{x_{j}}{x_{i}}\right)^{1 / \sigma}, \sigma>0
$$

In the first example $\sigma=1$, in the second $\sigma=2$ and in the third $\sigma=\frac{1}{2}$

## Utility maximization with a budget constraint

Note that the MRS declines to zero as $x_{j} / x_{i}$ declines to zero and increases without bound as $x_{2} / x_{1}$ increases. Therefore the FOC for the consumer problem cannot have a corner solution and so

$$
\operatorname{MRS}\left(\bar{x}_{i}, \bar{x}_{j}\right)=\frac{a_{i}}{a_{j}}\left(\frac{x_{j}}{x_{i}}\right)^{1 / \sigma}=\frac{p_{i}}{p_{j}} .
$$

It follows that

$$
\frac{x_{j}}{x_{i}}=\left(\frac{a_{j}}{a_{i}}\right)^{\sigma}\left(\frac{p_{i}}{p_{j}}\right)^{\sigma}
$$

Let the price ratio be $r$ so that $\frac{x_{j}}{x_{i}}=\left(\frac{a_{j}}{a_{i}}\right)^{\sigma} r^{\sigma}$. Then, as you may check, the elasticity of the consumption ratio with respect to the price ratio is

$$
\mathcal{E}\left(\frac{x_{j}}{x_{i}}, r\right)=\frac{r}{\left(\frac{x_{j}}{x_{i}}\right)} \frac{\partial}{\partial r}\left(\frac{x_{j}}{x_{i}}\right)=\sigma .
$$

This elasticity of the consumption ratio is called the elasticity of substitution. Hence the family of such utility functions is called the CES family.

We have defined preferences in terms of the marginal rate of substitution. There many utility representations of such preferences. One simple representation if $\sigma \neq 1$ is

$$
U(x)=\frac{1}{\left(1-\frac{1}{\sigma}\right)}\left(a_{1} x_{1}^{1-\frac{1}{\sigma}}+\ldots+a_{n} x_{n}^{1-\frac{1}{\sigma}}\right) .
$$

To confirm that this is indeed a representation, note that

$$
M U_{i}(x)=\frac{\partial U}{\partial x_{i}}=a_{i} x_{i}^{-1 / \sigma} .
$$

Therefore

$$
\operatorname{MRS}\left(\bar{x}_{i}, \bar{x}_{j}\right)=\frac{M U_{i}}{M U_{j}}=\frac{a_{i} x_{i}^{-1 / \sigma}}{a_{j} x_{j}-1 / \sigma}=\frac{a_{i}}{a_{j}}\left(\frac{x_{j}}{x_{i}}\right)^{1 / \sigma} .
$$

Exercise 3: Show that this representation of the CES utility function is homothetic.

## Exercise 4:

(a) If $U(x)=x_{1}^{1 / 2}+x_{2}^{1 / 2}$ solve for the utility maximizing demand if income is $I$ and the price vector is $p$.
(b) Hence show that maximized utility is

$$
V(p, I)=U(x(p, I))=\left(\frac{1}{p_{1}}+\frac{1}{p_{2}}\right)^{1 / 2} I^{1 / 2} .
$$

HINT: Begin by showing that

$$
\frac{1}{p_{1}{ }^{2} x_{1}}=\frac{1}{p_{2}{ }^{2} x_{2}}=\frac{\frac{1}{p_{1}}}{p_{1} x_{1}}=\frac{\frac{1}{p_{2}}}{p_{2} x_{2}} .
$$

## Exercise 5:

(a) If $U(x)=-\frac{1}{x_{1}}-\frac{1}{x_{2}}$ solve for the utility maximizing demand if income is $I$ and the price vector is $p$.
(b) Solve for maximized utility

## Exercise 6: Walrasian equilibrium with identical homothetic preferences

(a) If the aggregate endowment is $\omega=\left(\omega_{1}, \omega_{2}\right)$, where $\omega_{2}>\omega_{1}$ solve for the unique WE price ratio $p_{1} / p_{2}$ is (i) $U(x)=\ln x_{1}+\ln x_{2}$ (ii) $U(x)=x_{1}^{1 / 2}+x_{2}^{1 / 2}$ (iii) $U(x)=\frac{x_{1} x_{2}}{x_{1}+x_{2}}$.
(b) In each case what is the elasticity of substitution?
(c) Why is the case that the higher is the elasticity of substitution the lower is the WE price ratio?

