Choice under uncertainty

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57 slides
1. Introduction to choice under uncertainty (two states)

Let $X$ be a set of possible outcomes ("states of the world").

An element of $X$ might be a consumption vector, health status, inches of rainfall etc.

Initially, simply think of each element of $X$ as a consumption bundle. Let $\bar{x}$ be the most preferred element of $X$ and let $\underline{x}$ be the least preferred element.
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Initially, simply think of each element of $X$ as a consumption bundle. Let $x_\bar{X}$ be the most preferred element of $X$ and let $x$ be the least preferred element.

Consumption prospects

Suppose that there are only two states of the world. $X = \{x_1, x_2\}$ Let $\pi_1$ be the probability that the state is $x_1$ so that $\pi_2 = 1 - \pi_1$ is the probability that the state is $x_2$.

We write this "consumption prospect" as follows:

$$(x; \pi) = (x_1, x_2; \pi_1, \pi_2)$$

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1. **Introduction to choice under uncertainty (two states)**

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**Consumption prospects**

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If we make the usual assumptions about preferences, but now on prospects, it follows that preferences over prospects can be represented by a continuous utility function

$$U(x_1, x_2, \pi_1, \pi_2).$$
Prospect or “Lottery”

\[ L = (x_1, x_2, \ldots, x_S; \pi_1, \ldots, \pi_S) \]

(outcomes; probabilities)

Consider two prospects or “lotteries”, \( L_A \) and \( L_B \)

\[ L_A = (x_1, x_2, \ldots, x_S; \pi_1^A, \ldots, \pi_S^A) \quad L_B = (c_1, c_2, \ldots, c_S; \pi_1^B, \ldots, \pi_S^B) \]

**Independence Axiom (axiom of complex gambles)**

Suppose that a consumer is indifferent between these two prospects (we write \( L_A \sim L_B \)).

Then for any probabilities \( \pi_1 \) and \( \pi_2 \) summing to 1 and any other lottery \( L_C \)

\[ (L_A, L_C; \pi_1, \pi_2) \sim (L_B, L_C; \pi_1, \pi_2) \]

**Tree representation**

\[ \pi_1 \quad \pi_2 \]

\[ L_A \quad L_C \sim \]

\[ \pi_1 \quad \pi_2 \]

\[ L_B \quad L_C \]
This axiom can be generalized as follows:

Suppose that a consumer is indifferent between the prospects $L_A$ and $L_B$

and is also indifferent between the two prospects $L_C$ and $L_D$, i.e. $L_A \sim L_B$ and $L_C \sim L_D$

Then for any probabilities $\pi_1$ and $\pi_2$ summing to 1,

$$(L_A, L_C; \pi_1, \pi_2) \sim (L_B, L_D; \pi_1, \pi_2)$$

Tree representation

We wish to show that if $L_A \sim L_B$ and $L_C \sim L_D$ then
Proof: $L_A \sim L_B$ and $L_C \sim L_D$

Step 1: By the Independence Axiom, since $L_A \sim L_B$
Proof: $L_A \sim L_B$ and $L_C \sim L_D$

Step 1: By the Independence Axiom, since $L_A \sim L_B$

$$\begin{align*}
\pi_1 & \sim_{IA} \pi_2 \\
L_A & \sim L_B
\end{align*}$$

Step 2: By the Independence Axiom, since $L_C \sim L_D$

$$\begin{align*}
\pi_1 & \sim_{IA} \pi_2 \\
L_A & \sim L_C \\
L_B & \sim L_C \\
L_D & \sim L_D
\end{align*}$$
**Expected utility**

Consider some very good outcome \( \overline{x} \) and very bad outcome \( \underline{x} \) and outcomes \( x_1 \) and \( x_2 \) satisfying

\[ x < x_1 < \overline{x} \text{ and } x < x_2 < \overline{x} \]

**Reference lottery**

\[ L_R(v) = (\overline{w}, w, v, 1 - v) \] so \( v \) is the probability of the very good outcome.

\[ L_R(0) < x_1 < L_R(1) \text{ and } L_R(0) < x_2 < L_R(1) \]

Then for some probabilities \( v(x_1) \) and \( v(x_2) \)

\[ x_1 \sim L_R(v(x_1)) = (\overline{x}, x; v(x_1), 1 - v(x_1)) \text{ and } x_2 \sim L_R(v(x_2)) = (\overline{x}, x; v(x_2), 1 - v(x_2)) \]

Then by the independence axiom

\[ (x_1, x_2; \pi_1, \pi_2) \sim (L_R(v(x_1)), L_R(v(x_2)); \pi_1, \pi_2) \]

**Definition: Expectation of \( v(x) \)**

\[ \mathbb{E}[v(x)] = \pi_1 v(x_1) + \pi_2 v(x_2) \]
Note that in the big tree there are only two outcomes, $\bar{x}$ and $x$. The probability of the very good outcome is $\pi_1 v(x_1) + \pi_2 v(x_2) = \mathbb{E}[v(x)]$

The probability of the very bad outcome is $1 - \mathbb{E}[v(x)]$. Therefore
We showed that

\[ x_1 \sim x_2 \quad \pi_1 \quad L_R(v(x_1)) \sim \pi_2 \quad L_R(v(x_2)) \]

Thus the expected win probability in the reference lottery is a representation of preferences over prospects.

i.e.

\[ (x_1, x_2; \pi_1, \pi_2) \sim (\bar{x}, \bar{x}; \mathbb{E}[v], 1 - \mathbb{E}[v]) \]
An example:

A consumer with wealth $\hat{w}$ is offered a “fair gamble”. With probability $\frac{1}{2}$ his wealth will be $\hat{w} + x$ and with probability $\frac{1}{2}$ his wealth will be $\hat{w} - x$. If he rejects the gamble his wealth remains $\hat{w}$. Note that this is equivalent to a prospect with $x = 0$.

In prospect notation the two alternatives are

\[(w_1, w_2; \pi_1, \pi_2) = (\hat{w}, \hat{w}; \frac{1}{2}, \frac{1}{2})\]

and

\[(w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \frac{1}{2}, \frac{1}{2}).\]

These are depicted in the figure assuming $x > 0$.

**Expected utility**

\[U(w_1, w_2, \pi_1, \pi_2) = \mathbb{E}[v] = \pi_1 v(w_1) + \pi_2 v(w_2)\]

**Class discussion**

MRS if $v(w)$ is a concave function
Convex preferences

The two prospects are depicted opposite.

The level set for $U(w_1, w_2; \frac{1}{2}, \frac{1}{2})$ through the riskless prospect $N$ is depicted.

Note that the superlevel set

$$U(w_1, w_2; \frac{1}{2}, \frac{1}{2}) \geq U(\hat{w}, \hat{w}; \frac{1}{2}, \frac{1}{2})$$

is a convex set.

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Note that the superlevel set

\[
U(w_1, w_2; \frac{1}{2}, \frac{1}{2}) \geq U(\hat{w}, \hat{w}; \frac{1}{2}, \frac{1}{2})
\]

is a convex set.

This is the set of acceptable gambles for the consumer.

As depicted the consumer strictly prefers the riskless prospect \( N \) to the risky prospect \( R \).

Most individuals, when offered such a gamble (say over $5) will not take this gamble.
2. Risk aversion

Class Discussion: Which alternative would you choose?

\[ N: (w_1, w_2; \pi_1, \pi_2) = (\hat{w}, \hat{w}; \pi_1, \pi_2) \quad R: (w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \pi_1, \pi_2) \text{ where } \pi_1 = \frac{50}{100} \]

What if the gamble were “favorable” rather than “fair”

\[ R: (w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \pi_1, \pi_2) \text{ where } (i) \pi_1 = \frac{55}{100}, \ (ii) \pi_1 = \frac{60}{100}, \ (iii) \pi_1 = \frac{75}{100} \]

*
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\[ R: (w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x; \pi_1, \pi_2) \text{ where } (i) \pi_1 = \frac{55}{100} \quad (ii) \pi_1 = \frac{60}{100} \quad (iii) \pi_1 = \frac{75}{100} \]

What is the smallest integer \( n \) such that you would gamble if \( \pi_1 = \frac{n}{100} \)?

Preference elicitation

In an attempt to elicit your preferences write down your number \( n \) (and your first name) on a piece of paper. The two participants with the lowest number \( n \) will be given the riskless opportunity.

Let the three lowest integers be \( n_1, n_2, n_3 \). The win probability will not be \( \frac{n_1}{100} \) or \( \frac{n_2}{100} \). Both will get the higher win probability \( \frac{n_3}{100} \).
2. Risk preferences

\[ U(x, \pi) = \pi_1 v(x_1) + \pi_2 v(x_2) \quad \text{or} \quad U(x, \pi) = \mathbb{E}[v] \]

Risk preferring consumer

Consider the two wealth levels \( x_1 \) and \( x_2 > x_1 \).

\[ v(\pi_1 x_1 + \pi_2 x_2) < \pi_1 v(x_1) + \pi_2 v(x_2) \]

If \( v(x) \) is convex, then the slope of \( v(x) \) is strictly increasing as shown in the top figure.
$U(x, \pi) = \pi_1 v(x_1) + \pi_2 v(x_2)$

**Risk averse consumer**

$\nu(\pi_1 x_1 + \pi_2 x_2) > \pi_1 \nu(x_1) + \pi_2 \nu(x_2)$. 

In the lower figure $u(x)$ is strictly concave so that

$\nu(\pi_1 x_1 + \pi_2 x_2) > \pi_1 \nu(x_1) + \pi_2 \nu(x_2) = \mathbb{E}[\nu]$. 

In practice consumers exhibit aversion to such a risk. Thus we will (almost) always assume that the expected utility function $\nu(x)$ is a strictly increasing strictly concave function.

**Class Discussion:**

**If consumers are risk averse why do they go to Las Vegas?**
3. Acceptable gambles: Improving the odds to make the gamble just acceptable.

New risky alternative: \((w_1, w_2; \pi_1, \pi_2) = (\hat{w} + x, \hat{w} - x, \frac{1}{2} + \alpha, \frac{1}{2} - \alpha)\).

Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

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3. Acceptable gamble: Improving the odds to make the gamble just acceptable.

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Choose \(\alpha\) so that the consumer is indifferent between gambling and not gambling.

For small \(x\) we can use the quadratic approximation of the utility function

**Quadratic approximation of his utility**

As long as \(x\) is small we can approximate his utility as a quadratic. Suppose \(u(w + x) = \ln(w + x)\).

Define \(\bar{u}(x) = \ln(w + x)\).

Then (i) \(\bar{u}(0) = \ln w\) (ii) \(\bar{u}'(0) = \frac{1}{w}\) and (iii) \(\bar{u}''(0) = -\frac{1}{w^2}\)

Consider the quadratic function

\[
q(x) = \ln w + \left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2.
\] (3.1)

If you check you will find that \(\bar{u}(x)\) and \(q(x)\) have the same, value, first derivative and second derivative at \(x = 0\). We then use this quadratic approximation to compute the gambler’s (approximated) expected gain.
With probability \( \frac{1}{2} + \alpha \) his payoff is \( q(x) \) and with probability \( \frac{1}{2} - \alpha \) his payoff is \( q(-x) \). Therefore his expected payoff is

\[
\mathbb{E}[q(x)] = \left( \frac{1}{2} + \alpha \right) q(x) + \left( \frac{1}{2} - \alpha \right) q(-x)
\]

Substituting from (3.1)

\[
\mathbb{E}[q(x)] = \left( \frac{1}{2} + \alpha \right) \left[ \ln w + \left( \frac{1}{w} \right) x - \frac{1}{2} \left( \frac{1}{w^2} \right) x^2 \right] + \left( \frac{1}{2} - \alpha \right) \left[ \ln w + \left( \frac{1}{w} \right) (-x) - \frac{1}{2} \left( \frac{1}{w^2} \right) (-x)^2 \right].
\]

\*
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\mathbb{E}[q(x)] = \left(\frac{1}{2} + \alpha\right)\left[\ln w + \left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2\right]
$$

$$
+ \left(\frac{1}{2} - \alpha\right)\left[\ln w + \left(\frac{1}{w}\right)(-x) - \frac{1}{2}\left(\frac{1}{w^2}\right)(-x)^2\right].
$$

Collecting terms,

$$
\mathbb{E}[q(x)] = \ln w + 2\alpha\left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2.
$$

If the gambler rejects the opportunity his utility is $\ln w$. Thus his expected gain is

$$
\mathbb{E}[q(x)] - \ln w = 2\alpha\left(\frac{1}{w}\right)x - \frac{1}{2}\left(\frac{1}{w^2}\right)x^2 = \frac{2x}{w}\left[\alpha - \frac{1}{4}\left(\frac{1}{w}\right)x\right].
$$

Thus the gambler should take the small gamble if and only if $\alpha > \frac{1}{4}\left(\frac{1}{w}\right)x$. 

The general case: quadratic approximation of his utility

\[ q(x) = v(\hat{w}) + v'(\hat{w})x + \frac{1}{2} v''(\hat{w})x^2 \]

**Class Exercise:** Confirm that the value and the first two derivatives of \( u(\hat{w} + x) \) and \( q(x) \) are equal at \( x = 0 \).

The expected value utility of the risky alternative is

\[ \mathbb{E}[u(\hat{w} + x)] \approx \mathbb{E}[q(x)] = (\frac{1}{2} + \alpha)q(x) + (\frac{1}{2} - \alpha)q(-x) \]
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\]

\[ = (\frac{1}{2} + \alpha)[v(\hat{w}) + v'(\hat{w})x - \frac{1}{2} v''(\hat{w})x^2]
\]

\[ + (\frac{1}{2} - \alpha)[v(\hat{w}) + v'(\hat{w})(-x) - \frac{1}{2} v''(\hat{w})(-x)^2]. \]

Collecting terms,

\[ \mathbb{E}[q(x)] = v(\hat{w}) + 2\alpha v'(\hat{w})x - \frac{1}{2} v''(\hat{w})x^2. \]

*
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\[ = (\frac{1}{2} + \alpha)[v(\hat{w}) + v'(\hat{w})x + \frac{1}{2} v''(\hat{w})x^2]
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\[ + (\frac{1}{2} - \alpha)[v(\hat{w}) + v'(\hat{w})(-x) + \frac{1}{2} v''(\hat{w})(-x)^2]. \]

Collecting terms,

\[ \mathbb{E}[q(x)] = v(\hat{w}) + 2\alpha v'(\hat{w})x + \frac{1}{2} v''(\hat{w})x^2. \]

The gain in expected utility is therefore

\[
\mathbb{E}[q(x)] - v(\hat{w}) = 2\alpha v'(\hat{w})x + \frac{1}{2} v''(\hat{w})x^2
\]

\[ = 2v'(\hat{w})x[\alpha - \frac{1}{4}(-\frac{v''(\hat{w})}{v'(\hat{w})})x] \]

Thus the probability of the good outcome must be increased by \( \alpha = \frac{1}{4}(-\frac{v''(\hat{w})}{v'(\hat{w})})x \).